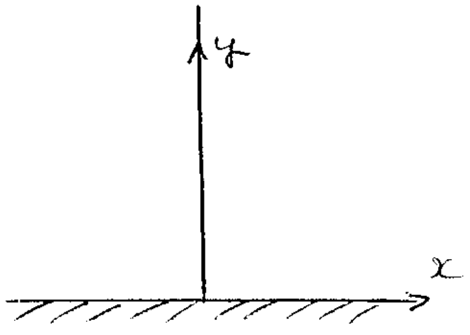
The Kelvin Wave in a (free surface) shallow water model

Up to this point we have been considering the free modes of oscillation of a shallow water model of infinite horizontal extent; thus, no horizontal boundary conditions have been considered. We now consider an infinite half plane, y > 0, with a solid wall at y = 0. Shallow water whose undisturbed depth is H lies in the region y > 0. We take



$$f = f_0,\ \Phi_s = 0,\ \overline{u} = 0$$

and consider waves of small amplitude.

The boundary conditions are:
a) v = 0 at y = 0, i.e., no flow through the wall.
b) (u,v,Φ) finite at y = ∞.

We show that a free wave propagating in the x-direction exists in which <u>v = 0 everywhere</u>. Thus, boundary condition a) is automatically satisfied. The wave in question is known as the Kelvin Wave.

With v = 0 the governing shallow water equations become

$$\frac{Du}{Dt} = -\frac{\partial \Phi}{\partial x} \qquad (1)$$

$$0 = -f_0 u - \frac{\partial \Phi}{\partial y} \qquad (2)$$

$$\frac{D\Phi}{Dt} = -\Phi \frac{\partial u}{\partial x} \qquad (3)$$

The linear perturbation equations are therefore

$$\frac{\partial u'}{\partial t} = -\frac{\partial \Phi'}{\partial x} \qquad (4)$$

$$0 = -f_0 u' - \frac{\partial \Phi'}{\partial y} \qquad (5)$$

$$\frac{\partial \Phi'}{\partial t} = -\overline{\Phi} \frac{\partial u'}{\partial x} \qquad (6)$$

We assume a solution of the form

$$(u', \Phi') = (\hat{u}(y), \hat{\Phi}(y))\, e^{i(kx - \nu t)} \qquad (7)$$

Here, the amplitudes û and $\hat{\Phi}$ are functions of y. We shall see that this is necessary to satisfy the equations and the boundary conditions.

Substituting (7) into (4), (5) and (6) we have

$$-i\nu\hat{u} = -ik\hat{\Phi} \qquad \text{--- (8)}$$

$$0 = f_0\hat{u} + \frac{d\hat{\Phi}}{dy} \qquad \text{--- (9)}$$

$$-i\nu\hat{\Phi} = -\bar{\Phi}\, ik\hat{u} \qquad \text{--- (10)}$$

(8) → $\hat{u} = \frac{k}{\nu}\hat{\Phi}$

(10) → $\hat{u} = \frac{\nu}{k}\frac{1}{\bar{\Phi}}\hat{\Phi}$

Equating these two expressions for û gives

$$\nu = \pm k\sqrt{\bar{\Phi}} \qquad \text{--- (11)}$$

i.e., $c = \frac{\nu}{k} = \pm\sqrt{\bar{\Phi}} = \pm\sqrt{gH}$ --- (12)

i.e., the phase speed is independent of rotation.

Substituting for û from (8) into (9) gives

$$\frac{d\hat{\Phi}}{dy} = -f_0\left(\frac{k}{\nu}\right)\hat{\Phi}$$

The solution to this is

$$\hat{\Phi}(y) = \hat{\Phi}(0)\, e^{-f_0\left(\frac{k}{\nu}\right)y} \qquad \text{--- (13)}$$

In order to satisfy the boundary condition that be finite at y = ∞, we see that only the positive root for ν given by eqn. (11) is allowable. Choosing this, we see that

$$\hat{\Phi}(y) = \hat{\Phi}(0)\, e^{-y/R} \qquad \left(R = \frac{\sqrt{\bar{\Phi}}}{f_0} = \text{Rossby radius}\right.$$

i.e., the amplitude of the wave decreases exponentially away from the boundary, and $c = +c_0$.

Assuming Φ is real, we see that

$$\Phi' = \hat{\Phi}(0)\, e^{-y/R}\cos k(x - c_0 t)$$

$$u' = \frac{1}{c_0}\hat{\Phi}(0)\, e^{-y/R}\cos k(x - c_0 t)$$

(4.3

In summary, the Kelvin Wave has the following properties:

1) The motion is parallel to the wall and is in geostrophic balance, i.e.,

$$v' = 0, \quad u' = -\frac{1}{f_0}\frac{\partial \Phi'}{\partial y}$$

2) The direction of propagation of the wave is such that, for an observer facing in the direction of propagation, the boundary is to the right.

3) The phase speed is the same as for a pure gravity wave in a non-rotating system, i.e., the Kelvin Wave is non-dispersive.

4) The amplitude of the motion falls off exponentially away from the boundary, the characteristic scale being the Rossby radius of deformation.

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Linearization of the shallow water potential vorticity equation on an f-plane about a basic state of rest

We show that the equation

$$\frac{D}{Dt}\left[\frac{\zeta+f_0}{\Phi}\right]=0 \qquad (1)$$

linearizes to

$$\frac{\partial}{\partial t}\left(\zeta'-\frac{f_0}{\bar{\Phi}}\Phi'\right)=0 \qquad (2)$$

<u>Proof:</u>

Written in full, eq. (1) is

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\left[\frac{\zeta+f_0}{\Phi}\right]=0 \qquad (3)$$

For perturbations about a basic state of rest, we have

$$u=u'$$

$$v=v'$$

$$\Phi=\bar{\Phi}+\Phi' \qquad (\bar{\Phi}=\text{constant})$$

Hence

$$\frac{\zeta+f_0}{\Phi}\rightarrow\frac{\zeta'+f_0}{\bar{\Phi}+\Phi'}=\frac{1}{\bar{\Phi}}\left(\frac{\zeta'+f_0}{1+\Phi'/\bar{\Phi}}\right)=\frac{1}{\bar{\Phi}}(\zeta'+f_0)\left(1+\frac{\Phi'}{\bar{\Phi}}\right)^{-1}$$

$$=\frac{1}{\bar{\Phi}}(\zeta'+f_0)\left(1-\frac{\Phi'}{\bar{\Phi}}+O[\Phi']^2\right)$$

Neglecting products of perturbation quantities, we thus have

$$\frac{\zeta+f_0}{\Phi}\rightarrow\frac{1}{\bar{\Phi}}\left(\zeta'+f_0-\frac{f_0}{\bar{\Phi}}\Phi'\right)$$

Hence (3) becomes

$$\left(\frac{\partial}{\partial t}+u'\frac{\partial}{\partial x}+v'\frac{\partial}{\partial y}\right)\left[\frac{1}{\bar{\Phi}}\left(\zeta'+f_0-\frac{f_0}{\bar{\Phi}}\Phi'\right)\right]=0$$

Again neglecting products of perturbation quantities, we have

$$\frac{\partial}{\partial t}\left[\frac{1}{\bar{\Phi}}\left(\zeta'+f_0-\frac{f_0}{\bar{\Phi}}\Phi'\right)\right]=0$$

But $\partial\bar{\Phi}/\partial t=\partial f_0/\partial t=0$; hence the above equation gives

$$\frac{\partial}{\partial t}\left(\zeta'-\frac{f_0}{\bar{\Phi}}\Phi'\right)=0$$

QED