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Gravity waves and gravity-inertia waves in the shallow water (free surface) model

a) Gravity waves

Suppose f=0 (no rotation) and $\Phi_s = 0$ (no bottom topography). Then the shallow water equations become

$$\frac{Du}{Dt} = -\frac{\partial \Phi}{\partial x} \qquad (1)$$

$$\frac{Dv}{Dt} = -\frac{\partial \Phi}{\partial y} \qquad (2)$$

$$\frac{D\Phi}{Dt} = -\Phi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \qquad (3)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

(Note: $\Phi_T = \Phi$ when $\Phi_s = 0$).
We look for a solution for 1D waves:

$$\left.\begin{aligned} u &= u'(x,t) \\ v &= 0 \\ \Phi &= \bar{\Phi} + \Phi'(x,t) \end{aligned}\right\} \qquad (4)$$

where $\bar{\Phi}$ is constant. Linearizing eqns. (1)-(3) using the perturbation method, we then have

$$\frac{\partial u'}{\partial t} = -\frac{\partial \Phi'}{\partial x} \qquad (5)$$

$$0 = 0$$

$$\frac{\partial \Phi'}{\partial t} = -\bar{\Phi}\frac{\partial u'}{\partial x} \qquad (6)$$

Assuming a solution of the form

$$\begin{pmatrix} u' \\ \Phi' \end{pmatrix} = Re\begin{pmatrix} \hat{u} \\ \hat{\Phi} \end{pmatrix} e^{ik(x-ct)} \qquad (7)$$

where $\hat{u}$ and $\hat{\Phi}$ are the amplitudes (in general complex), we find

$$-ikc\,\hat{u} = -ik\hat{\Phi} \tag{8}$$

$$-ikc\,\hat{\Phi} = -ik\bar{\Phi}\hat{u} \tag{9}$$

(8) gives
$$\hat{u} = \hat{\Phi}/c \tag{10}$$

while (9) gives
$$\hat{u} = c\hat{\Phi}/\bar{\Phi} \tag{11}$$

A solution exists only if

$$c^2 = \bar{\Phi}$$

i.e.,

$$c = \pm\sqrt{\bar{\Phi}} = \pm\sqrt{gH} \tag{12}$$

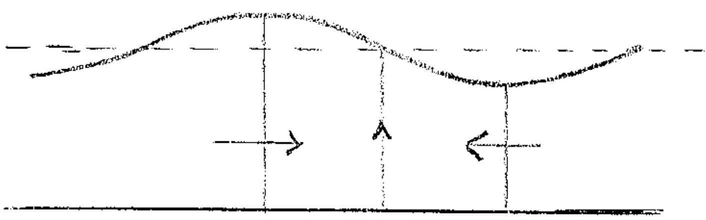
where H is the undisturbed depth of the fluid..

We see that the gravity wave phase speed c is independent of wavelength, i.e., shallow water gravity waves are non-dispersive.
We can assume without loss of generality that $\hat{\Phi}$ is real. Since c is real, eqn. (10) then tells us that $\hat{u}$ is also real. Thus, the solution can be written

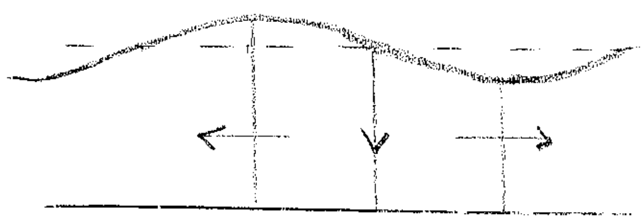
$$\Phi' = \hat{\Phi}\cos k(x-ct) \tag{13}$$

$$u' = \frac{1}{c}\hat{\Phi}\cos k(x-ct) = \frac{\Phi'}{c} \tag{14}$$

Wave propagating in the +ve x-direction $(c>0)$:



Wave propagating in the -ve x-direction $(c<0)$:



The group velocity is given by

$$c_g = \frac{\partial \nu}{\partial k} = \frac{\partial}{\partial k}(kc) = c \tag{15}$$

i.e., the group velocity and the phase speed are the same.

b) Gravity-inertia waves

Now we assume $f = f_0$ (f-plane) and $\Phi_s = 0$ (no bottom topography). The shallow water equations then become

$$\frac{Du}{Dt} = f_0 v - \frac{\partial \Phi}{\partial x} \tag{16}$$

$$\frac{Dv}{Dt} = -f_0 u - \frac{\partial \Phi}{\partial y} \tag{17}$$

$$\frac{D\Phi}{Dt} = -\Phi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{18}$$

We look for a 1D wave solution:

$$\left.\begin{aligned} u &= \bar{u} + u'(x,t) \\ v &= v'(x,t) \\ \Phi &= \bar{\Phi} + \Phi'(x,t) \end{aligned}\right\} \tag{19}$$

Linearizing (16)-(18) using the perturbation method gives, for the case $\bar{u} = 0$

$$\frac{\partial u'}{\partial t} = f_0 v' - \frac{\partial \Phi'}{\partial x} \tag{20}$$

$$\frac{\partial v'}{\partial t} = -f_0 u' \tag{21}$$

$$\frac{\partial \Phi'}{\partial t} = -\bar{\Phi}\frac{\partial u'}{\partial x} \tag{22}$$

Assuming a solution

$$\begin{pmatrix} u' \\ v' \\ \Phi' \end{pmatrix} = \begin{pmatrix} \hat{u} \\ \hat{v} \\ \hat{\Phi} \end{pmatrix} e^{ik(x-ct)} \tag{23}$$

we have

$$-ikc\hat{u} = f_0\hat{v} - ik\hat{\Phi} \tag{24}$$

$$-ikc\hat{v} = -f_0\hat{u} \tag{25}$$

$$-ikc\hat{\Phi} = -\bar{\Phi}ik\hat{u} \tag{26}$$

The condition for the existence of a non-zero solution is

$$\begin{vmatrix} ikc & f_0 & -ik \\ -f_0 & ikc & 0 \\ -\bar{\Phi}ik & 0 & ikc \end{vmatrix} = 0$$

i.e.,

$$ikc\left[k^2c^2 - (f_0^2 + \bar{\Phi}k^2)\right] = 0 \tag{27}$$

Thus, the solutions are

$$c = 0 \tag{28}$$

$$c = \pm\sqrt{\bar{\Phi} + f_0^2/k^2} \tag{29}$$

The solution (28) corresponds to a steady geostrophic flow

$$u' = 0, \quad f_0 v' = \frac{\partial \Phi'}{\partial x} \tag{30}$$

while the solution (29) corresponds to two gravity-inertia waves, one propagating in the positive x-direction and the other in the −ve x-direction.

We note the following features of the gravity-inertia waves:
1) As $f_0 \to 0$ the gravity-inertia waves tend to gravity waves. Rotation always increases the phase speed.
2) The frequency always exceeds the inertial frequency:

$$\nu = kc = \pm\sqrt{f_0^2 + \bar{\Phi}k^2}, \quad |\nu| > f_0$$

3) Rotation makes the waves dispersive. The group velocity is given by

$$c_g = \frac{\partial \nu}{\partial k} = \pm \frac{1}{2} \frac{2k\,\bar{\Phi}}{\sqrt{f_0^2 + \bar{\Phi} k^2}} = \frac{\bar{\Phi}}{c}$$

i.e.,

$$c\, c_g = c_0^2 \qquad (32)$$

where

$$c_0 = \sqrt{\bar{\Phi}} \quad \text{(Gravity wave phase speed)}$$

Thus, whereas rotation increases the phase speed, it decreases the group velocity in such a way that the product of the phase speed and group velocity remains constant.

4) Rotation causes the wave to have a transverse component of motion ($v' \neq 0$).
To examine this further, we assume $\hat{\Phi}$ is real and positive.
Eqn. (26) gives

$$\hat{u} = \frac{c}{\bar{\Phi}} \hat{\Phi} \qquad (33)$$

Using this, eqn. (25) gives

$$\hat{v} = -i \left(\frac{f_0}{k \bar{\Phi}} \right) \hat{\Phi} \qquad (34)$$

Therefore we have

$$u' = \frac{c}{\bar{\Phi}} \hat{\Phi} \cos k(x - ct) \qquad (35)$$

and

$$v' = \frac{f_0}{k \bar{\Phi}} \hat{\Phi} \sin k(x - ct) \qquad (36)$$

[use $-i = e^{-i\frac{\pi}{2}}$

From (35) and (36) we have

$$\frac{(u')^2}{(c\hat{\Phi}/\bar{\Phi})^2} + \frac{(v')^2}{\left(\frac{f_0}{\nu} \frac{c\hat{\Phi}}{\bar{\Phi}}\right)^2} = 1 \qquad (37)$$

The velocity vector traces out an ellipse.
The motion is mainly in the x-direction but has a component in the y-direction.
Note: $|f_0/\nu| < 1$.