(1.8

# Derivation of the shallow water equations on a tangent plane
## (Free surface case)

We start from the primitive equations of motion on a tangent plane:

**Momentum equations**

$$\frac{Du}{Dt} = fv - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad (1)$$

$$\frac{Dv}{Dt} = -fu - \frac{1}{\rho}\frac{\partial p}{\partial y} \qquad (2)$$

**Hydrostatic equation**

$$\frac{\partial p}{\partial z} = -\rho g \qquad (3)$$

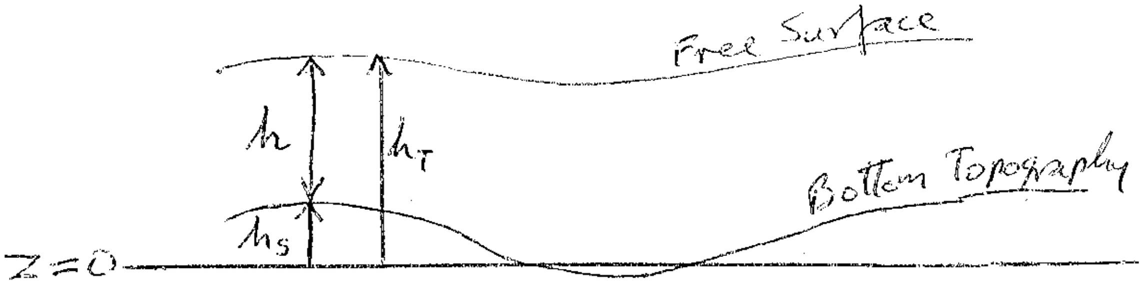
**Continuity equation**

$$\nabla\cdot(\rho \underset{\sim}{u}) \equiv \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = -\frac{\partial \rho}{\partial t} \qquad (4)$$

[These equations are derived in Holton, Chapter 2, and we use the same notation here. The hydrostatic balance assumption follows from the shallowness of the fluid, i.e., the horizontal scale is large by comparison with the vertical scale.]

In addition to the assumption of hydrostatic balance, which applies to all large-scale motions in the atmosphere, the shallow water model involves the following specific assumptions:

a) The fluid is homogeneous and incompressible, i.e., the density ρ is a constant.
b) The pressure at the free surface of the fluid is zero.
c) The horizontal motion is independent of height in the fluid, i.e., ∂u/∂z = ∂v/∂z = 0.
d) A particle which is at the free surface remains at the free surface and a particle which is in contact with the bottom remains in contact with the bottom.



We denote the height of the orography above mean sea level (z = 0) as $h_s$ and the height of the free surface above mean sea level as $h_T$. Thus, the depth of the fluid (h) is

$$h = h_T - h_s \qquad (5)$$

We use the notation

$$\Phi_T = gh_T \text{ (geopotential of the free surface)}$$

$$\Phi_s = gh_s \text{ (geopotential of the orography)}$$

$$\Phi = gh \text{ (geopotential depth of the fluid)}$$

where g is the gravitational acceleration.

From (5) we have

$$\Phi = \Phi_T - \Phi_s \tag{6}$$

**Derivation of the shallow water pressure gradient force**

We integrate the hydrostatic equation (3) from an arbitrary height z within the fluid to the free surface:

$$\int_z^{h_T} \frac{\partial p}{\partial z}\, dz = -\int_z^{h_T} \rho g\, dz$$

As a result of assumptions (a) and (b) this gives

$$p(z) = \rho g\,(h_T - z)$$

Taking $\partial/\partial x$ and $\partial/\partial y$ of this we have

$$\frac{\partial p}{\partial x} = \rho g \frac{\partial h_T}{\partial x}$$

$$\frac{\partial p}{\partial y} = \rho g \frac{\partial h_T}{\partial y}$$

i.e.,

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{\partial \Phi_T}{\partial x} \tag{7}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{\partial \Phi_T}{\partial y} \tag{8}$$

Thus, the pressure gradient force in the shallow water model is independent of height and depends only on variations of the free surface geopotential.

This result is consistent with assumption (c) of the shallow water model, because if we start with (u,v) independent of height the equations of motion (1) and (2) tell us that the accelerations $Du/Dt$ and $Dv/Dt$ are independent of height. Therefore, (u,v) will remain independent of height.

The horizontal equations of motion may now be written

$$\frac{Du}{Dt} = fv - \frac{\partial \Phi_T}{\partial x} \qquad (9)$$

$$\frac{Dv}{Dt} = -fu - \frac{\partial \Phi_T}{\partial y} \qquad (10)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

Note: $w\partial/\partial z = 0$ because $\partial u/\partial z = \partial v/\partial z = 0$.

**Derivation of the shallow water continuity equation**

In view of assumption (a) the continuity equation (4) becomes

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

We integrate this equation throughout the depth of the fluid:

$$\int_{h_S}^{h_T}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)dz = -\int_{h_S}^{h_T}\frac{\partial w}{\partial z}\,dz$$

Using assumption (c) this becomes

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)(h_T - h_S) = -(w_T - w_S) \qquad (11)$$

From assumption (d) we have

$$w_T \equiv \left(\frac{Dz}{Dt}\right)_{z=h_T} = \frac{Dh_T}{Dt}$$

$$w_S \equiv \left(\frac{Dz}{Dt}\right)_{z=h_S} = \frac{Dh_S}{Dt}$$

Therefore

$$w_T - w_S = \frac{Dh_T}{Dt} - \frac{Dh_S}{Dt} = \frac{D}{Dt}(h_T - h_S) = \frac{Dh}{Dt}$$

Thus, equation (11) can be written

1.11

$$\frac{Dh}{Dt} = -h\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

Multiplying by g we have

$$\frac{D\Phi}{Dt} = -\Phi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \qquad (12)$$

**Summary**

The governing equations for the shallow water model are

$$\frac{Du}{Dt} = fv - \frac{\partial \Phi_T}{\partial x}$$

$$\frac{Dv}{Dt} = -fu - \frac{\partial \Phi_T}{\partial y}$$

$$\frac{D\Phi}{Dt} = -\Phi\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$$

$$\Phi_T = \Phi + \Phi_S$$

Two simplifications are used to express the Coriolis parameter on the tangent plane:

1) $f = f_0$ (constant) ["f-plane "]

2) $f = f_0 + \beta y$ ($f_0$ and $\beta$ constants) ["$\beta$-plane"]