ESGI102: Mud filtration at Aughinish
Final Report

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1 Introduction

1.1 Mud filtration via rotating drum filters

Rusal extracts alumina from bauxite via the Bayer process, which produces bauxite residue as a by-product. The residue must be separated from the liquid via filtration. During filtration the mud is washed to reduce the level of caustic before it is sent to be stored.

The aim of this work is to construct a model of this combined process in order to address the following questions:

- *How does the level in the trough depend on the amounts of wash volume flow rate and slurry volume flow rate?*
- *How does the mass flow rate of caustic exiting the drums (via the scrapped cake) depend on the amounts of wash flow rate and slurry flow rate?*
- *How to define a sensible controller of these flow rates so that a system of several drums can operate in an optimal regime?*

For example, ideally one would like to reduce the total amount of mass flow rate of caustic exiting the drums via the scrapped cake, while keeping all drums in an operational regime and maintaining the total flow rate of slurry.

1.2 The model

Our approach to the problem of drum mud filtration is divided in a few sequential steps:

- Modelling the fast transient of initial cake formation (timescale = drum rotation period, about 30 seconds).
- Obtaining the angular profile of the cake thickness after this fast transient has passed. This angular profile will vary on a slow timescale (about 30 drum rotation periods, or 900 seconds).
- Slow variables: conservation of volumes of:
    1. Wash liquid entering the trough.
    2. Slurry liquid & solid entering the trough.
    3. Liquid & solid forming the cake.
    4. Liquid that gets sucked through the drum filter.
- **Partial result:** a system of two nonlinear ODEs for two variables: the volume of liquid in the trough and the volume of solid in the trough.
- **Applications:** Steady-state study of the ODE system, leading to quantitative analysis of:
    1. Thorough understanding of the operational ranges constraining the wash and slurry volume flow rates.
    2. Thorough understanding of how to make savings on the caustic soda lost to the scrap.

1.3 Important parameters

In the final ODE model there are two types of parameters:

- Fixed physical parameters (not really movable):
  1. Dimensions of the drums (radius and length).
  2. Dimensions of the trough.
  3. Pressure difference (vacuum imposed).
  4. Drum rotation frequency.
  5. Ratio between volumes of solid and liquid in the input slurry.
  6. Densities of solid and liquid in the input slurry.
  7. Density of liquid in the input wash.
  8. Cake resistance (sensitive to modelling).
  9. Cake porosity (sensitive to modelling).
  10. Viscosities of liquids in the wash, trough and input slurry. [1]
- Two controllable parameters:
  1. Volume flow rate of input wash liquid (applied downwards from the top of the drum).
  2. Total volume flow rate of slurry towards the trough.

  (These volume flow rates have dimensions of volume per unit time.)

Hypotheses made include:

- Incompressibility of liquids and solids.
- Continuum hypothesis and separation of time scales.
- Darcy's law for flow through cake's pores.
- Prescribed contact area between wash liquid and upper cake.

[1] Trough's liquid viscosities may be made dynamical in a future improvement of the model.

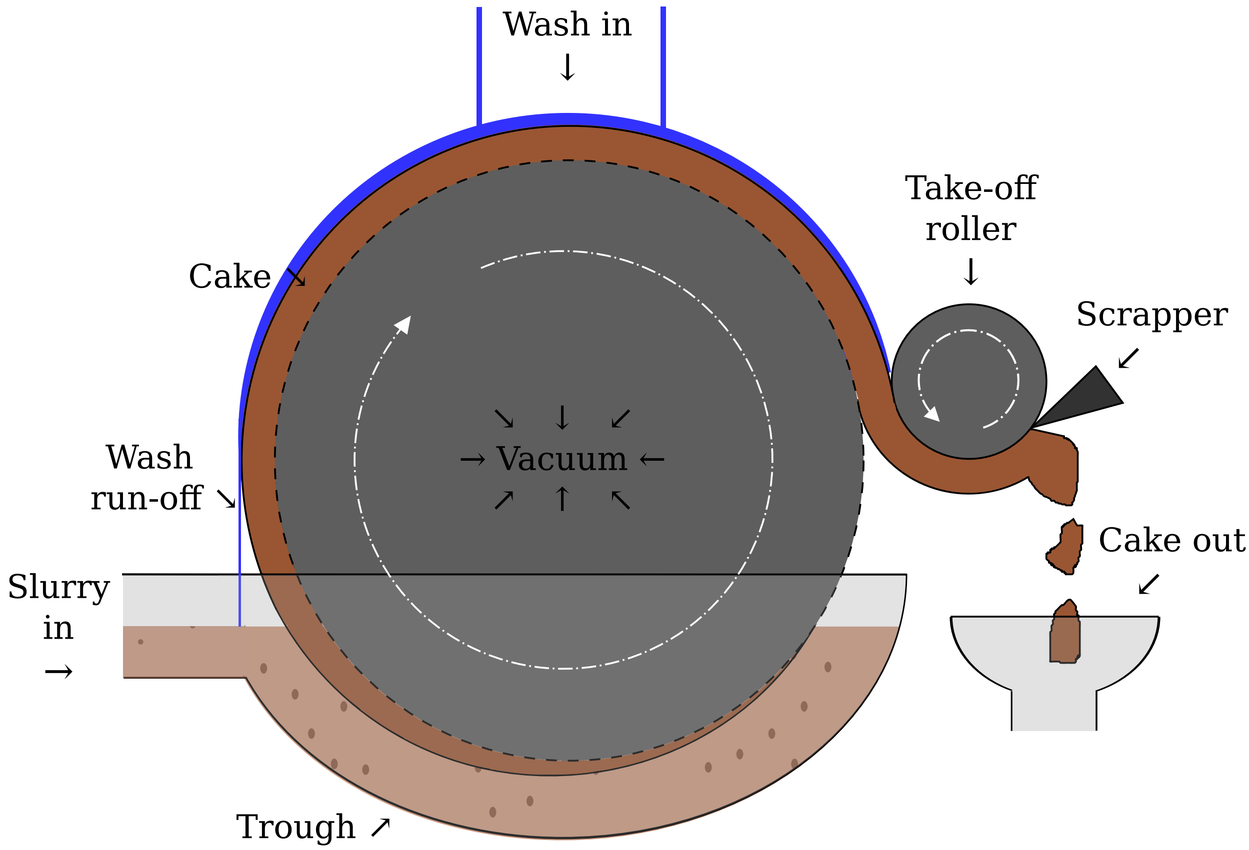

Figure 1: Model of a rotating drum filter in action.

2 Equations

2.1 Cake formation

Formation of cake occurs on the lower surface of the drum filter that is in contact with slurry. The formation of cake is mainly due to the existence of a pressure difference between the slurry and the interior of the rotating drum (vacuum). This pressure difference causes the mixture of solid and liquid in the slurry to become attracted towards the lower drum surface. While all solid that is attracted becomes deposited on the cake, the liquid that is attracted will be distributed in two parts: one part is attached to the deposited solids (i.e., trapped) and the other part goes through the drum filter towards the vacuum. This latter part goes trough the pores of the cake before crossing the filter, therefore Darcy's law applies.

The whole process is best seen in a Lagrangian frame or reference rotating with the drum. Consider an element of cake of area A (fixed) and instantaneous thickness l at an angle θ (measured with respect to the laboratory frame of reference). The angle θ changes with time at the rate

$$\omega = \frac{d\theta}{dt}\,, \tag{1}$$

where ω is the angular frequency of the drum's rotation.

Darcy's law states that, due to a pressure difference $(-\Delta P)$, a volume dV_{liquid} of liquid will cross through this element of thickness l in a time dt, such that

$$\frac{dV_{liquid}}{dt} = \frac{A(-\Delta P)}{r\,\mu\,l}\,, \tag{2}$$

where r is the cake resistance and μ is the viscosity of the liquid.

Assuming perfect mixing of the slurry in the trough, this volume of liquid dV_{liquid} is necessarily accompanied by a proportional volume of solids dV_{solid}. If solid particles were small enough then they would be transported through the pores along with the liquid. However, the solid particles are too big to enter the pores. Instead, they get deposited on the element of cake, adding up to its thickness by an amount dl, which constitutes a volume $dV_{\text{cake}} = Adl$. This added volume is related to the volume of solids in terms of the cake porosity e (which we assume fixed and independent of l) by the following formula:

$$dV_{solid} = (1-e)dV_{\text{cake}} = (1-e)A\,dl.$$

How to determine the relation between dV_{solid} and dV_{liquid}? Perfect mixing in the trough leads to the proportionality

$$\frac{dV_{solid}}{dV_{liquid} + dV'_{liquid}} = \frac{V_S}{V_L}, \quad dV'_{liquid} = e\,dV_{\text{cake}}\,,$$

where V_S and V_L are the instantaneous values of the volumes of solid and liquid in the trough, and the extra liquid volume dV'_{liquid} is the amount of liquid that is trapped in the added volume dV_{cake} (along with the solids).

With these relations, equation (2) becomes the following evolution equation for the thickness: $\left(\frac{(1-e)V_L}{e\,V_S} - 1\right)\frac{dl}{dt} = \frac{(-\Delta P)}{e\,r\,\mu\,l}$ and using equation (1) we derive the equation for the profile of l in terms of the angle θ :

$$l\frac{dl}{d\theta} = \frac{(-\Delta P)}{e\,\omega\,r\,\mu\left(\frac{(1-e)V_L}{e\,V_S} - 1\right)}. \tag{3}$$

The RHS of this equation is a slow function of time via V_L and V_S. The typical time scale in which these volumes change is about 900 seconds. This is 30 times slower than the time scale of the formation of cake, which is completed during one turn of the rotating drum, or in about 30 seconds.

In this way it is a good approximation to simply integrate the above evolution equation in order to obtain an 'instantaneous' profile of the cake thickness. The separation of time scales allows us to neglect the initial transient that occurs when a drum starts its operation with a clean cloth. We obtain

$$l(\theta) = \sqrt{\frac{2\,\theta\,(-\Delta P)}{e\,\omega\,r\,\mu\left(\frac{(1-e)V_L}{e\,V_S} - 1\right)}}\,, \tag{4}$$

where we assume the operational condition that there is no cake attached to the piece of drum that enters the trough. The angle θ is measured from the point where the drum enters the trough, and in the sense of the rotation of the drum.

One of the most important quantities is the thickness of the cake L as it exits the trough. Let the total submerged angle of the drum be denoted by Θ. Evaluating equation (4) at $\theta = \Theta$ we obtain

$$L = \sqrt{\frac{2\,\Theta\,(-\Delta P)}{e\,\omega\,r\,\mu\left(\frac{(1-e)V_L}{e\,V_S} - 1\right)}}\,. \tag{5}$$

In this equation, the quantities V_L, V_S and Θ are slowly time dependent, therefore L is itself slowly time dependent.

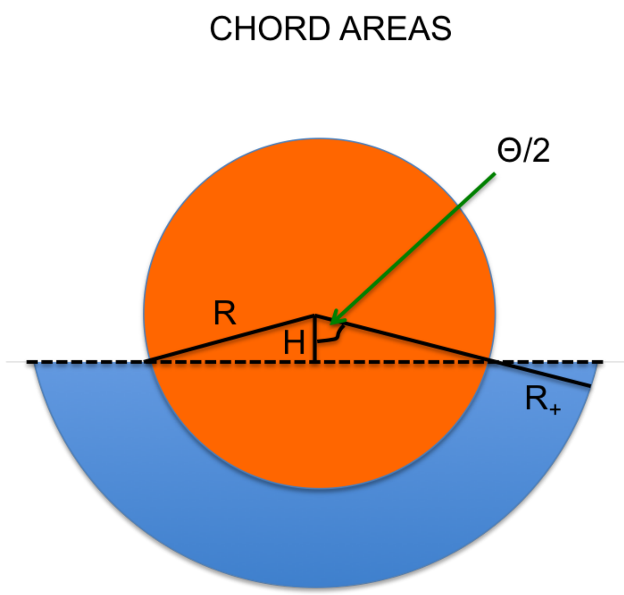

Figure 2: Geometry for computations of chord areas.

Once the cake exits the trough, its thickness remains constant (equal to L) during the time it is exposed to the air. Eventually the cake is scrapped out using a roller and a scraper, at the other side, just before the drum enters the trough again.

2.2 Modelling the evolution of the slow variables

In our analysis, the relevant slow variables are the total volume of solids in the trough V_S and the total volume of liquid in the trough V_L. We have already encountered them in the previous subsection, when analysing the cake formation. It turns out that these variables determine all other slow variables and, moreover, we can model their evolution using a system of two nonlinear ordinary differential equations, of first order in time derivatives.

2.2.1 Relation between the total volume in the trough $V = V_S + V_L$ and the submersion angle Θ

Assuming both the trough and the drum are cylindrical and concentric with respective radii R_+ and R, we use the notation from figure 2 in order to compute the area of the blue region (corresponding to a section of the trough). This area is the subtraction of the chord areas of the trough and the drum, giving

$$a = R_+^2 \cos^{-1}(H/R_+) - H\sqrt{R_+^2 - H^2} - \left(R^2 \cos^{-1}(H/R) - H\sqrt{R^2 - H^2}\right),$$

where $H = R\cos(\Theta/2)$. Further, assuming both trough and drum have the same length D, the volume of slurry in the trough is simply $V = a\,D$. For a given filter, the quantities D, R, R_+ are all fixed, so the only quantity that is variable in the formula for the volume is the submerged angle Θ. The explicit formula for V as a function of Θ involves trigonometric functions and their inverses. Defining the ratio

$$q = \frac{R}{R_+} \tag{6}$$

the relation between the total slurry volume and the submersion angle is obtained as follows:

$$\frac{V_S + V_L}{D\,R^2} = \frac{\cos^{-1}(q\cos[\Theta/2])}{q^2} - \frac{\cos(\Theta/2)\sqrt{1 - q^2\cos^2(\Theta/2)}}{q} - \frac{\Theta - \sin[\Theta]}{2}\,. \tag{7}$$

In practice, the quantity that is measured is the height of the surface of slurry (called "level"). Personal communications from Aughinish have provided the relation between the height and the submerged angle. More importantly, they have provided the operational bounds for this submerged angle, corresponding to 0% and 80% of a prescribed operational level of height that was set up by the company. These operational bounds imply that the submerged angle (measured in rads) must be bounded as follows:

$$1.47 \leq \Theta \leq 2.31. \tag{8}$$

In this range, the total slurry volume $V_S + V_L$ can be approximated by a linear function of Θ, leading to the following operational fit, which is a very good approximation even outside the operational range of submerged angles:

$$\frac{V_S + V_L}{D\,R^2} \approx 0.0639379 + 0.170561\,\Theta, \tag{9}$$

where we have used a fixed ratio between the inner and outer radii: $q = R/R_+ = 0.849593$ (taken from actual drum specifications). This fit formula will prove very useful later on, in the analysis of the steady state of the solution to the evolution equations for the slow variables V_S and V_L. For the initial setup of these equations, we will work with the general formula $\Theta = \Theta(V)$, representing that the submerged angle is actually a prescribed function of the total slurry volume. Then, in the steady-state analysis, we will use the linear relation between Θ and $V_S + V_L$.

2.2.2 Modelling the controllable volume flow rate parameters

The volumetric flow rate of slurry mixture that goes into the trough is given in units of volume per unit time and is denoted by $F^{\rm src}$. This is a controllable parameter that can range from about 20 to 60 m^3/h in operational regimes. The proportions of solids and liquid that constitute this source volume flow rate have been determined experimentally. According to data from Aughinish, 44% of the incoming slurry mass is formed by solids. This leads to the following relations for the volumetric flow rates of solids $F_S^{\rm src}$ and liquid $F_L^{\rm src}$, which will be the source terms of our evolution equations:

$$F_S^{\rm src} = \frac{1}{1+\beta} F^{\rm src}, \qquad F_L^{\rm src} = \frac{\beta}{1+\beta} F^{\rm src}, \tag{10}$$

where

$$\beta = \frac{66}{44}\frac{\rho_S}{\rho_L}, \qquad \rho_S = 3200\,\mathrm{kg/m^3}, \qquad \rho_L = 1068\,\mathrm{kg/m^3}.$$

The second controllable volumetric flow rate parameter is the flow rate of wash liquid that is put on the top of the drum. This is denoted by $F_W^{\rm src}$ and is a controllable parameter that can range from about 8 to 24 m^3/h in operational regimes. In our model, some of this wash is filtered through the upper part of the cake (with a volumetric flow rate denoted by $F_W^{\rm filter}$) and some of it continues towards the trough (with volumetric flow rate denoted by $F_W^{\rm trough}$). The defining relation between these three flow rates is

$$F_W^{\rm src} = F_W^{\rm trough} + F_W^{\rm filter}\,. \tag{11}$$

We will leave the modelling of the two flow rates in the RHS for a subsequent section.

2.2.3 Modelling the evolution of the volume of solids in the trough, V_S

The simplest quantity to model is V_S, the volume of solids in the trough. For this quantity, the model is very robust due to the fact that the solids cannot go through the filter so the conservation is easy to establish. The rate of change of V_S is dominated by just two terms: the source term $F_S^{\rm src}$ [given in equation (10)] and a sink term, $F_S^{\rm cake}$, that accounts for the cake that exits the trough (which is then scrapped out via the roller and scraper). To compute this sink term we note that the volume of cake that leaves the trough per unit time (denoted $F^{\rm cake}$) is determined by the cake thickness at exit, L, the frequency of rotation ω and the drum dimensions R, D. Preliminarily, then, we have

$$F^{\rm cake} = \omega\, R\, D\, L\,, \tag{12}$$

but thus flow rate, due to the fact that the cake is porous, consists of a contribution from the solids and a contribution from the liquid. The contribution from the solids is just the above multiplied by $(1-e)$, where e is the porosity of the cake (assumed fixed). Thus, we obtain

$$F_S^{\rm cake} = \omega\, R\, D\, L\, (1-e). \tag{13}$$

Putting together the source term from equation (10) and this sink term we obtain the differential equation

$$\frac{dV_S}{dt} = F_S^{\rm src} - F_S^{\rm cake} = \frac{1}{1+\beta} F^{\rm src} - \omega\, R\, D\, L\, (1-e). \tag{14}$$

In this equation, the only time-dependent variables are V_S and L. Recall that the variable L is given in terms of V_S and V_L via equation (5). In this equation, Θ is understood as a function of the total volume in the trough $V = V_S + V_L$ [see equation (9) and discussion below it].

2.2.4 Modelling the evolution of the volume of liquid in the trough, V_L

To model the rate of change of V_L, we need to consider two source terms: first, the flow rate of liquid from the incoming slurry $F_L^{\rm src}$ given in equation (10); second, the flow rate of wash liquid that enters the trough from above, $F_W^{\rm trough}$, which is related to the wash source via equation (11). Finding a sensible formula for $F_W^{\rm trough}$ requires the appropriate modelling of the flow rate $F_W^{\rm filter}$. Due to its special difficulty we will postpone this modelling for a subsequent section.

There are two sink terms that account for the exit of liquid from the trough. The first sink term, $F_L^{\rm cake}$, is the flow rate of liquid trapped in the cake that exits the trough and is given by $F_L^{\rm cake} = e\, F^{\rm cake}$, where $F^{\rm cake}$ is given in equation (12). We obtain

$$F_L^{\rm cake} = \omega\, R\, D\, L\, e. \tag{15}$$

The second sink term for the liquid in the trough is given by the liquid that is filtered through the submerged part of the cake. We denote this flow rate by $F_L^{\rm filter}$. To model this term we simply use Darcy's law, equation (2), this time applied differentially to the whole angular distribution of cake thickness $l(\theta)$, given in equation (4). We have, by definition,

$$F_L^{\rm filter} = \int_0^{\Theta} \frac{(-\Delta P)}{r\, \mu\, l(\theta)} R\, D\, d\theta\,,$$

where the area A in Darcy's law has been replaced by the differential $R\, D\, d\theta$. Now, using the differential equation (3), we have

$$\frac{(-\Delta P)}{r\, \mu\, l(\theta)} = \omega\, e \left(\frac{(1-e) V_L}{e\, V_S} - 1 \right) \frac{dl}{d\theta},$$

which leads to the intermediate result

$$F_L^{\text{filter}} = \omega\, e \left(\frac{(1-e)V_L}{e\, V_S} - 1 \right) R\, D \int_0^{\Theta} \frac{dl}{d\theta} d\theta\,,$$

and the integral can be performed explicitly, leading finally to

$$F_L^{\text{filter}} = \omega\, e \left(\frac{(1-e)V_L}{e\, V_S} - 1 \right) R\, D\, L. \tag{16}$$

In summary, thus, we obtain the following differential equation for the evolution of the volume of liquid in the trough, V_L:

$$\begin{aligned} \frac{dV_L}{dt} &= F_L^{\text{src}} + F_W^{\text{trough}} - F_L^{\text{cake}} - F_L^{\text{filter}} \\ &= \frac{\beta}{1+\beta} F^{\text{src}} + F_W^{\text{trough}} - \left(\frac{V_L}{V_S} \right) \omega (1-e) R\, D\, L\,. \end{aligned} \tag{17}$$

So far the modelling of the term F_W^{trough} has not been discussed. We will treat this in the next subsection.

2.3 Modelling of the wash terms F_W^{filter} and F_W^{trough}

The wash continuity equation (11) implies that once F_W^{filter}, the flow rate of wash liquid that is filtered through the cake in the upper part of the drum, is properly modelled, the flow rate F_W^{trough} will be obtained.

We discuss the modelling of F_W^{filter}. This is the most difficult term to model because the experimental settings vary a lot and we do not have much information yet about the process. Notice that modelling this term properly might be very important, since the filtering of caustic soda through the cake and the filter (towards the centre of the drum) could be improved if the value of F_W^{filter} was experimentally increased.

Let us assume that the cake that exits the trough has the same resistance r as the cake that is submerged. Also, let us assume that the viscosity of the wash liquid is the same as the viscosity μ of the liquid in the trough. This assumption is not essential but is reasonable within the model since as we will see there is an unknown parameter in the model (the angle of coverage Θ_W of the wash liquid on the cake in the upper part of the drum) which appears as an overall factor in the formula for F_W^{filter}.

Applying Darcy's law in differential form we obtain

$$F_W^{\text{filter}} = \int_0^{\Theta_W} \frac{(-\Delta P)}{r\, \mu\, L} R\, D\, d\theta = \frac{(-\Delta P)}{r\, \mu\, L} R\, D\, \Theta_W\,. \tag{18}$$

Correspondingly we have, assuming $F_W^{\text{src}} - F_W^{\text{filter}} > 0$,

$$F_W^{\text{trough}} = F_W^{\text{src}} - F_W^{\text{filter}} = F_W^{\text{src}} - \frac{(-\Delta P)}{r\, \mu\, L} R\, D\, \Theta_W\,. \tag{19}$$

The assumption $F_W^{\text{src}} - F_W^{\text{filter}} > 0$ is valid if the wash source is strong enough. We have confirmed the validity of this assumption for most of the operational regimes, by replacing typical values of the parameters and variables. In practice, if some of the wash goes into the trough then this assumption is valid.

2.4 Summary of the governing equations

We now summarise our model, stemming from the evolution equations (14), (17), with the definitions (19), (5) and the volume formula in the operational range, equation (9):

$$\begin{aligned}
\frac{dV_S}{dt} &= \frac{F^{\rm src}}{1+\beta} - \omega\, R\, D\, L\,(1-e), \\
\frac{dV_L}{dt} &= \frac{\beta F^{\rm src}}{1+\beta} + F_W^{\rm src} - \frac{(-\Delta P)}{r\,\mu\, L} R\, D\, \Theta_W - \left(\frac{V_L}{V_S}\right)\omega(1-e) R\, D\, L\,, \\
L &= \sqrt{\frac{2\,\Theta\,(-\Delta P)}{e\,\omega\, r\,\mu\,\left(\frac{(1-e)V_L}{e\,V_S} - 1\right)}}\,, \\
\Theta &= a\frac{V_S+V_L}{D\,R^2} - b\,, \qquad 1.47 \le \Theta \le 2.31\,,
\end{aligned} \tag{20}$$

where a, b are fit parameters given by

$$a = 5.86299, \qquad b = 0.374867\,, \tag{21}$$

and the remaining parameters take experimental values in the following ranges:

$$\begin{aligned}
\beta &\approx 4.494, \\
\omega &\approx 0.209\,\mathrm{rad/s}, \\
R &\approx 2.09\,\mathrm{m}, \\
D &\approx 7.5\,\mathrm{m}, \\
e &\approx \frac{0.5\rho_S/\rho_L}{1+0.5(\rho_S/\rho_L - 1)} \approx 0.7498, \\
(-\Delta P) &\approx 45\,k\mathrm{Pa}, \\
r &= 10^{14}\,\mathrm{m}^{-2} - 10^{15}\,\mathrm{m}^{-2}, \\
\mu &\approx 0.55\,m\mathrm{Pa\ s}, \\
\Theta_W &= 0\,\mathrm{rad} - \pi\,\mathrm{rad}\,.
\end{aligned} \tag{22}$$

and finally the two controllable parameters $F^{\rm src}$ and $F_W^{\rm src}$ take values in the following ranges:

$$\begin{aligned}
F^{\rm src} &= 20/3600\,\mathrm{m}^3/\mathrm{s} - 60/3600\,\mathrm{m}^3/\mathrm{s}\,, \\
F_W^{\rm src} &= 8/3600\,\mathrm{m}^3/\mathrm{s} - 24/3600\,\mathrm{m}^3/\mathrm{s}\,.
\end{aligned}$$

3 Analysis of the steady state

The most important result from the practical point of view is that the solution of the above evolution equations converges to a steady state, defined by the condition that the variables V_S and V_L are constant in time. We confirmed this result using numerical simulations of the equations, and obtained a fast convergence to the steady state: in typical circumstances, the relaxation time (to the steady state) is between 10 and 20 minutes, i.e. just 20 to 40 revolutions of the drum.

3.1 Analytical derivation of the steady state

We now perform a thorough analytical derivation of the steady state. At the steady state the time derivatives of the variables V_S and V_L are zero, therefore the right-hand-side of the evolution equations must be equal to zero. This leads to a system of 2 equations for the two unknowns V_S, V_L. We obtain the system:

$$\begin{aligned} \omega\, R\, D\, L\, (1-e) &= \frac{F^{\rm src}}{1+\beta}\,, && (23)\\ \frac{(-\Delta P)}{r\,\mu\, L} R\, D\, \Theta_W + \left(\frac{V_L}{V_S}\right) \omega (1-e) R\, D\, L &= \frac{\beta F^{\rm src}}{1+\beta} + F^{\rm src}_W\,. && (24) \end{aligned}$$

where L is given in the third line of equations (20) above. We will denote the solution to this system with the superscript $^{(0)}$, so for example $V_S^{(0)}, V_L^{(0)}$ is the solution for the volumes. Preliminarily, equation (23) gives the steady-state solution for the variable L:

$$L^{(0)} = \frac{F^{\rm src}}{(1+\beta)\,\omega\, R\, D\, (1-e)}\,. \tag{25}$$

Notice that this equation gives the steady-state cake thickness which could have been obtained from first principles without the need for modelling. The true utility of the modelling is in the solution of the system for the steady-state volumes, which tells us whether the steady state belongs to the operational range. Replacing L by $L^{(0)}$ in equation (24), and using equation (23) we obtain an equation for the ratio between the volumes V_L/V_S:

$$\begin{aligned} \frac{V_L^{(0)}}{V_S^{(0)}} &= \beta + \frac{(1+\beta) F^{\rm src}_W}{F^{\rm src}} - \frac{\gamma}{(F^{\rm src})^2}\frac{\Theta_W}{r}\,, && (26)\\ \gamma &\equiv \frac{(-\Delta P)(1+\beta)^2\,\omega\, R^2\, D^2\,(1-e)}{\mu}\,, && (27) \end{aligned}$$

where we have introduced γ, a positive quantity with dimensions of $\mathrm{length}^4/\mathrm{time}^2$.

In addition, equation (25) along with the last two lines in (20) give, after elimination of $L^{(0)}$ in terms of $\Theta^{(0)}$,

$$\Theta^{(0)} = \frac{(F^{\rm src})^2\, e\, r\left(\frac{(1-e)V_L^{(0)}}{e\,V_S^{(0)}} - 1\right)}{2\,\gamma\,(1-e)}\,. \tag{28}$$

3.2 Steady state in the operational range

Equations (26)–(28) have been obtained without approximation. Does the solution to this system of equations make sense physically? Notice that all steady-state volumes must be positive so one should check that the solution of equations (26)–(28) gives a positive value for the ratio

$\frac{V_L^{(0)}}{V_S^{(0)}}$. The positivity of this ratio is ensured if we restrict the submersion angle to the operational range (20), because then the numerator in the right-hand-side of (28) must be positive, and this implies that $\frac{V_L^{(0)}}{V_S^{(0)}}$ is positive.

We will eliminate the term $\frac{V_L^{(0)}}{V_S^{(0)}}$ from the equations by combining them appropriately. Noticing, from equation (26),

$$\frac{\gamma}{r}\,\Theta_W = (F^{\rm src})^2\left(\beta - \frac{V_L^{(0)}}{V_S^{(0)}}\right) + (1+\beta)\,F^{\rm src}\,F_W^{\rm src}$$

and

$$2\,\frac{\gamma}{r}\,\Theta^{(0)} = (F^{\rm src})^2\frac{V_L^{(0)}}{V_S^{(0)}} - \frac{(F^{\rm src})^2\,e}{1-e},$$

we get readily

$$\frac{\gamma}{r}\left(2\,\Theta^{(0)} + \Theta_W\right) \;=\; (F^{\rm src})^2\left(\beta - \frac{e}{1-e}\right) + F^{\rm src}\,F_W^{\rm src}\,(\beta+1)\,. \tag{29}$$

Equation (29) is remarkable because:

- It has the mathematical interpretation that the modelling of the wash part (represented by Θ_W) basically shifts the origin for the operational range.
- The right-hand side is insensitive to the details of the model: the only modelled parameter is the porosity e. Therefore, optimal control of the flow rates so that the submersion angle is in the operational range can be made in a robust way, i.e. independent of the details of the various model parameters.

3.3 Constraints on the wash and slurry volume flow rates

Back to equation (29), let us set parameter values as given in equations (22), except for r and Θ_W, which we leave free for the moment. We obtain

$$\frac{2.44105\times 10^{10}\,\mathrm{m}^4/\mathrm{s}^2}{r}\left(2\,\Theta^{(0)} + \Theta_W\right) = 0.81716\,(F^{\rm src})^2 + 4.8134\,F^{\rm src}\,F_W^{\rm src}\,.$$

Finally, we impose the condition that the steady state is in the operational range (20). This gives the following "sandwich" inequality involving the flow rates:

$$\frac{2.94}{r} \le \frac{0.81716\,(F^{\rm src})^2 + 4.8134\,F^{\rm src}\,F_W^{\rm src}}{2.44105\times 10^{10}\,\mathrm{m}^4/\mathrm{s}^2} - \frac{\Theta_W}{r} \le \frac{4.62}{r}\,. \tag{30}$$

The important thing about this inequality is that its solution can be found analytically in terms of standard mathematical operations so this can be programmed in any commercial package such as *Microsoft Excel*.

What is the utility of this sandwich inequality? First, it gives us a recipe on how to control the flow rates in order to keep the system in the operational regime. The gradient (with respect to flow-rate changes) of the middle expression in the inequality is proportional to

$$(1.63432F^{\rm src} + 4.8134\,F_W^{\rm src}, 4.8134\,F^{\rm src})\,.$$

In extreme situations when it is necessary to get quickly into the operational regime (for example, when the level is extremely low or extremely high) it will be optimal to produce changes in the flow rates which are proportional to the gradient:

$$(\delta F^{\rm src}, \delta F_W^{\rm src}) \propto (1.63432 F^{\rm src} + 4.8134\, F_W^{\rm src}, 4.8134\, F^{\rm src})\,. \tag{31}$$

The direction of this vector is quite uniform if we remember that the flow rates are allowed to vary in a finite range: the ratio between the components of this vector is

$$\frac{\delta F_W^{\rm src}}{\delta F^{\rm src}} = \frac{4.8134\, F^{\rm src}}{1.63432 F^{\rm src} + 4.8134\, F_W^{\rm src}} = \frac{1}{0.3395 + F_W^{\rm src}/F^{\rm src}}\,.$$

In practice, the ratio $F_W^{\rm src}/F^{\rm src}$ is bounded between 0.133 and 1.2, so we have

$$0.6496 \le \frac{\delta F_W^{\rm src}}{\delta F^{\rm src}} \le 2.116\,.$$

In terms of orientation, denoting the usual slope angle by ϕ where $\tan\phi = \frac{\delta F_W^{\rm src}}{\delta F^{\rm src}}$, we get, in degrees,

$$33.69^{\rm o} \le \phi \le 63.43^{\rm o}. \tag{32}$$

On the other hand, in situations where the system is in the operational range (e.g. when the level is safely in the middle) one would like to produce changes on the flow rates that improve the filtering of caustic soda into the vacuum of the drum, while keeping the level of the system more or less constant. Fortunately, as we will see in the next section, this is possible.

4 Modelling the filtering of caustic soda

4.1 General analysis: away from the steady state

One of the most important quantities to control is the amount of caustic soda mass per unit time that gets trapped in the cake and leaves the drum via the scraper (right part in Fig. 1). We want to minimise this mass flow rate, so that significant savings can be made.

In normal conditions, the slurry that goes into the trough carries an amount of caustic soda, at a mass flow rate of about

$$G_{\text{caus}}^{\text{src}} \approx \frac{987}{3600}\text{kg/s}.$$

Assuming perfect mixing/stirring (as enforced by a helical stirrer that operates in the trough) we deduce that this mass flow will be spread evenly in the liquid. Now, during a time dt, the volume of liquid that exits the trough, trapped in the upper cake, is known: from equation (15) this volume is

$$dV_L = dt\, F_L^{\text{cake}} = dt\, \omega\, R\, D\, L\, e.$$

The ratio between this volume and the total liquid volume V_L determines the proportion of mass of caustic soda that goes into the cake during time dt, trapped with the liquid:

$$dM_{\text{cake}} = M dV_L/V_L = M\, dt\, \omega\, R\, D\, L\, e/V_L, \tag{33}$$

where M is the total mass of caustic soda in the trough, determined by the balance equation

$$dM = dm - dM_{\text{cake}} - dM_{out}\,,$$

where dm is the input mass that gets into the trough during time dt:

$$dm = dt\, G_{\text{caus}}^{\text{src}},$$

and dM_{out} is the amount of mass that goes through the lower part of the cake, all the way through the filter into the vacuum at the centre of the drum: from equation (16) this is

$$dM_{out} = M dt F_L^{\text{filter}}/V_L = M dt\, \omega\, e\left(\frac{(1-e)V_L}{e\, V_S} - 1\right) R\, D\, L/V_L\,.$$

With these definitions, the balance equation for the total mass of caustic soda in the trough becomes

$$dM = dm - M dt(F_L^{\text{cake}} + F_L^{\text{filter}})/V_L = dm - M dt(\omega(1-e)R\, D\, L)/V_S,$$

which is a new ODE for M. Explicitly,

$$\frac{dM}{dt} = G_{\text{caus}}^{\text{src}} - M\omega(1-e)R\, D\, L/V_S\,.$$

This ODE can be solved explicitly once we know V_S and V_L as functions of time.

4.2 Analysis of the steady state

The second term in the right-hand side of the above equation leads to a transient, exponential decay, with typical relaxation time

$$\tau = \frac{V_S}{\omega(1-e)R\, D\, L}.$$

In normal circumstances this time scale is of about 10 minutes, so it coincides with the time scale of the steady-state solution discussed in Section 3. We can then neglect this transient and focus on the steady state for the total mass, given by the equation

$$G_{\rm caus}^{\rm src} - M^{(0)}\omega(1-e)R\,D\,L^{(0)}/V_S^{(0)} = 0, \tag{34}$$

where now all quantities are assumed to be in the steady state. The quantity we wish to study is the mass flow rate of caustic that exits the trough, trapped in the cake: from equation (33) we get

$$\frac{dM_{\rm cake}}{dt} = M^{(0)}\,\omega\,R\,D\,L^{(0)}\,e/V_L^{(0)}$$

but equation (34) allows to eliminate M to give finally

$$\frac{dM_{\rm cake}}{dt} = G_{\rm caus}^{\rm src}\frac{e}{1-e}\frac{V_S^{(0)}}{V_L^{(0)}}. \tag{35}$$

4.2.1 Caustic soda lost via the scraper

So far we have obtained the mass flux rate of caustic to the upper part of the cake. The amount of mass $dM_{\rm cake}$ arrives in the upper part of the cake in a time interval dt. This mass will continue trapped and will exit the system via the scraper, unless the wash liquid pushes the mass into the filter and carries it along. For this to happen, the mass must enter in contact with the wash liquid, which is then sucked through the filter at a volume flow rate given by $F_W^{\rm filter}$, see equation (18):

$$F_W^{\rm filter} = \frac{(-\Delta P)}{r\,\mu\,L^{(0)}}R\,D\,\Theta_W\,.$$

Since a quarter of rotation is a fast time scale, we can say that in a time $T=(\pi/2)/\omega$ the following amount of mass is available for being washed out:

$$M_{\rm cake} = G_{\rm caus}^{\rm src}\frac{e}{1-e}\frac{V_S^{(0)}}{V_L^{(0)}}(\pi/2)/\omega \tag{36}$$

Assuming perfect mixing again we get that this mass spreads in the volume of the liquid in the upper part of cake,

$$V_L^{\rm cake} = T\,F_L^{\rm cake} = (\pi/2)\,R\,D\,L^{(0)}\,e. \tag{37}$$

Finally, during this time, the flow rate of wash liquid volume that goes through the upper part of the cake all the way to the filter and to the centre of the drum, is given by Darcy's law:

$$dV/dt = \Theta_W\frac{RD(-\Delta P)}{r\,\mu\,L^{(0)}}\,, \tag{38}$$

and therefore the mass flow rate that goes through the filter in the upper part of the drum is

$$\frac{dM_{\rm filter}}{dt} = \frac{M_{\rm cake}}{V_L^{\rm cake}}\,dV/dt\,,$$

which, after using equations (36)–(38), gives

$$\frac{dM_{\rm filter}}{dt} = G_{\rm caus}^{\rm src}\frac{e}{1-e}\frac{V_S^{(0)}}{V_L^{(0)}}\left[\frac{(-\Delta P)\,\Theta_W}{r\,\mu\,\omega\,(L^{(0)})^2\,e}\right]$$

and using equation (25) we can eliminate $L^{(0)}$ to obtain

$$\frac{dM_{\rm filter}}{dt} = G^{\rm src}_{\rm caus}\frac{e}{1-e}\frac{V_S^{(0)}}{V_L^{(0)}}\left[\frac{(-\Delta P)\,\Theta_W(1+\beta)^2\,\omega\,R^2\,D^2\,(1-e)^2}{r\,\mu\,(F^{\rm src})^2\,e}\right]$$

or, in terms of the parameter γ defined in equation (27),

$$\frac{dM_{\rm filter}}{dt} = G^{\rm src}_{\rm caus}\frac{V_S^{(0)}}{V_L^{(0)}}\,\frac{\gamma}{(F^{\rm src})^2}\,\frac{\Theta_W}{r}\,.$$

For this model we must impose, for consistency, the inequality

$$\frac{dM_{\rm filter}}{dt} \le \frac{dM_{\rm cake}}{dt}$$

[see equation (35)], which means that we must cap the above formula to the following:

$$\frac{dM_{\rm filter}}{dt} = G^{\rm src}_{\rm caus}\frac{V_S^{(0)}}{V_L^{(0)}}\min\left\{\frac{e}{1-e},\frac{\gamma}{(F^{\rm src})^2}\,\frac{\Theta_W}{r}\right\}\,. \tag{39}$$

In this way, the mass flow rate that is eliminated by the system via the scraper is given by $G^{\rm lost}_{\rm caus} = \frac{dM_{\rm cake}}{dt} - \frac{dM_{\rm filter}}{dt}$ or

$$G^{\rm lost}_{\rm caus} = G^{\rm src}_{\rm caus}\frac{V_S^{(0)}}{V_L^{(0)}}\max\left\{0,\frac{e}{1-e}-\frac{\gamma}{(F^{\rm src})^2}\,\frac{\Theta_W}{r}\right\}\,, \tag{40}$$

where the ratio $\frac{V_S^{(0)}}{V_L^{(0)}}$ is given in equation (26).

4.2.2 How to reduce caustic soda losses?

Let us set parameter values from equations (22). Again, we leave r and Θ_W free for the moment. We obtain

$$\frac{V_L^{(0)}}{V_S^{(0)}} = 3.813 + \frac{4.813F_W^{\rm src}}{F^{\rm src}} - \frac{2.441\times 10^{10}\,{\rm m}^4/{\rm s}^2}{r}\,\frac{\Theta_W}{(F^{\rm src})^2}\,,$$

$$G^{\rm lost}_{\rm caus} = 2.996\,G^{\rm src}_{\rm caus}\,\frac{\max\left\{0,(F^{\rm src})^2-\frac{8.147\times10^9\,{\rm m}^4/{\rm s}^2}{r}\Theta_W\right\}}{3.813\,(F^{\rm src})^2+4.813\,F_W^{\rm src}F^{\rm src}-\frac{2.441\times10^{10}\,{\rm m}^4/{\rm s}^2}{r}\,\Theta_W}\,. \tag{41}$$

In the next subsection we present direct numerical studies of the quantity $G^{\rm lost}_{\rm caus}$ (that measures how much caustic soda mass is lost to the scrap per unit time) in terms of the tuneable flow rates $F^{\rm src}, F_W^{\rm src}$.

For the moment let us consider the worst-case scenario, where the numerator in the above equation is replaced by its upper bound $(F^{\rm src})^2$. The question is then: how to change the flow rates so that the quantity $G^{\rm lost}_{\rm caus}$ is diminished optimally? The answer to this question is given in terms of the gradient of the function

$$f(F^{\rm src},F_W^{\rm src}) = \frac{(F^{\rm src})^2}{3.813\,(F^{\rm src})^2+4.813\,F_W^{\rm src}F^{\rm src}-\frac{2.441\times10^{10}\,{\rm m}^4/{\rm s}^2}{r}\,\Theta_W}\,.$$

We obtain that this gradient is proportional to

$$(4.813F_W^{\rm src}\,F^{\rm src} - \frac{4.882\times10^{10}\,{\rm m}^4/{\rm s}^2}{r}\,\Theta_W, -4.813(F^{\rm src})^2)\,.$$

As a result, a change of the flow rates $(\delta F^{\mathrm{src}}, \delta F_W^{\mathrm{src}})$ in the above gradient's direction will produce an increase of the mass flow rate of caustic soda lost to the scrap. Conversely, a change that is in the opposite direction will reduce the loss rate of caustic soda. Therefore, we conclude that *a positive change of the wash flow rate* $\delta F_W^{\mathrm{src}} > 0$ implies immediately a reduction of the loss rate of caustic soda. Moreover, the most optimal change satisfies the proportionality

$$(\delta F^{\mathrm{src}}, \delta F_W^{\mathrm{src}}) \propto (-4.813 F_W^{\mathrm{src}}\, F^{\mathrm{src}} + \frac{4.882 \times 10^{10}\,\mathrm{m}^4/\mathrm{s}^2}{r}\,\Theta_W, 4.813 (F^{\mathrm{src}})^2) \tag{42}$$

with a positive proportionality constant. To get an idea of this optimal change, notice that when the cake resistance r is sufficiently large then we can approximate this relation by

$$(\delta F^{\mathrm{src}}, \delta F_W^{\mathrm{src}}) \propto (-F_W^{\mathrm{src}}, F^{\mathrm{src}}),$$

which is approximately orthogonal to the direction of optimal change in level, see equations (31), (32). Notice that the optimal change regarding the caustic soda recovery not only requires a raise of wash flow rate, but at the same time it requires a reduction of slurry flow rate.

In summary, this analysis shows that once the operational range (in terms of level or submersion angle) is attained, it is possible to optimize the recovery of caustic soda by changing the flow rate parameters while keeping the system well within the operational range.

5 Numerical parameter studies of the steady state

We apply numerically the formulas obtained in the previous sections, and produce several studies.

5.1 Graphical plots

For selected choices of cake resistance r and wash contact angle Θ_W, we produce 3D plots of the mass flow rate of lost caustic soda $G^{\text{lost}}_{\text{caus}}$ as a function of the controllable volume flux parameters F^{src} (slurry source) and F^{src}_W (wash liquid source), within the operational range (figures 3, 4, 5, 6).

The analysis of the figures is as follows. Let us focus on figure 4 as an example. The top-left panel shows the region of flow-rate parameter space where the steady state of the system is compliant with the "operational range" of submersion levels, as indicated in inequalities (8). In this respect, the oblique curves in the figure represent level overflow (top curve) and level underflow (bottom curve). The overall bounding box of the figure represents the factory ranges 8–24 for the wash source flow rate and 20–60 for the slurry source flow rate, both measured in cubic metres per hour.

As for the top-right panel in figure 4, it shows a close-up of the same region in the left panel, with the inclusion of a third dimension representing the value of the mass flow rate of caustic soda lost to the scraper, in units of kilograms per second. The "rainbow" colour code is juts indicative of the relative value of the quantity, so red is maximum and blue is minimum. It is evident that the colour gradient is roughly orthogonal to the overflow-underflow gradients, in agreement with the theoretical discussion in previous sections. Moreover, we confirm the result that the caustic soda loss can be reduced by going to the "blue" regions, i.e. by increasing the wash flow rate while at the same time decreasing the slurry flow rate.

5.2 Tables

We provide, for selected choices of cake resistance r, wash contact angle Θ_W and volume flux parameters F^{src} and F^{src}_W, a table with steady-state values of the main parameters of interest, such as final cake thickness $L^{(0)}$, final mass flow rate of lost caustic soda $G^{\text{lost}}_{\text{caus}}$ and final density of lost caustic soda $\rho^{\text{lost}}_{\text{caus}}$.

The analysis of the table is as follows. There are three main row-blocks in the table, corresponding respectively to the top panels of figures 3, 4 and 5. In each case, selected flow-rate parameter values within the operational range are considered, and for each choice we compute the steady-state values of the main quantities of interest, in particular the mass flow rate of scrapped (i.e. lost) caustic soda, which is of economic importance. We confirm again that this quantity is reduced by simultaneously increasing the wash flow rate and decreasing the slurry flow rate, leading to savings of caustic soda from 20% (case $r = 2 \times 10^{14}\mathrm{m}^{-2}$, first row-block) to 41% (case $r = 3 \times 10^{14}\mathrm{m}^{-2}$, second row-block) to 79% (case $r = 6 \times 10^{14}\mathrm{m}^{-2}$, third row-block).

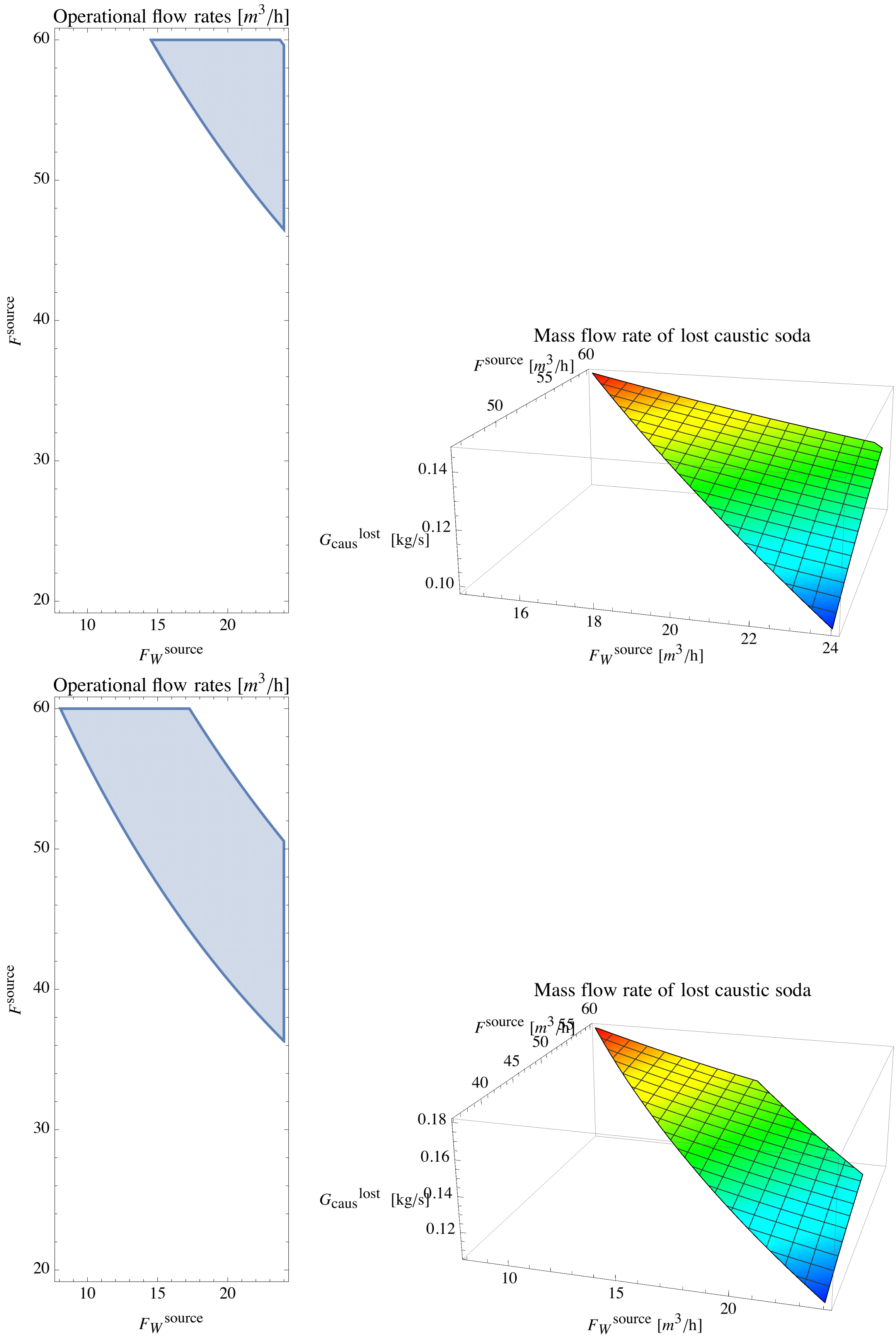

Figure 3: In all panels, cake resistance value is $r = 2 \times 10^{14} \mathrm{m}^{-2}$. In all top panels, contact wash angle is $\Theta_W = \pi/2$. In all bottom panels, contact wash angle is $\Theta_W = \pi/8$. Left panels: 2D shade plots of operational ranges for volume flow rates of source slurry ($F^{\rm src}$) and source wash liquid ($F_W^{\rm src}$), according to operational bounds in equation (8). Right panels: 3D plots of mass flow rate of caustic soda lost to the scraper $G_{\rm caus}^{\rm lost}$ as a function of volume flow rates of source slurry ($F^{\rm src}$) and source wash liquid ($F_W^{\rm src}$).

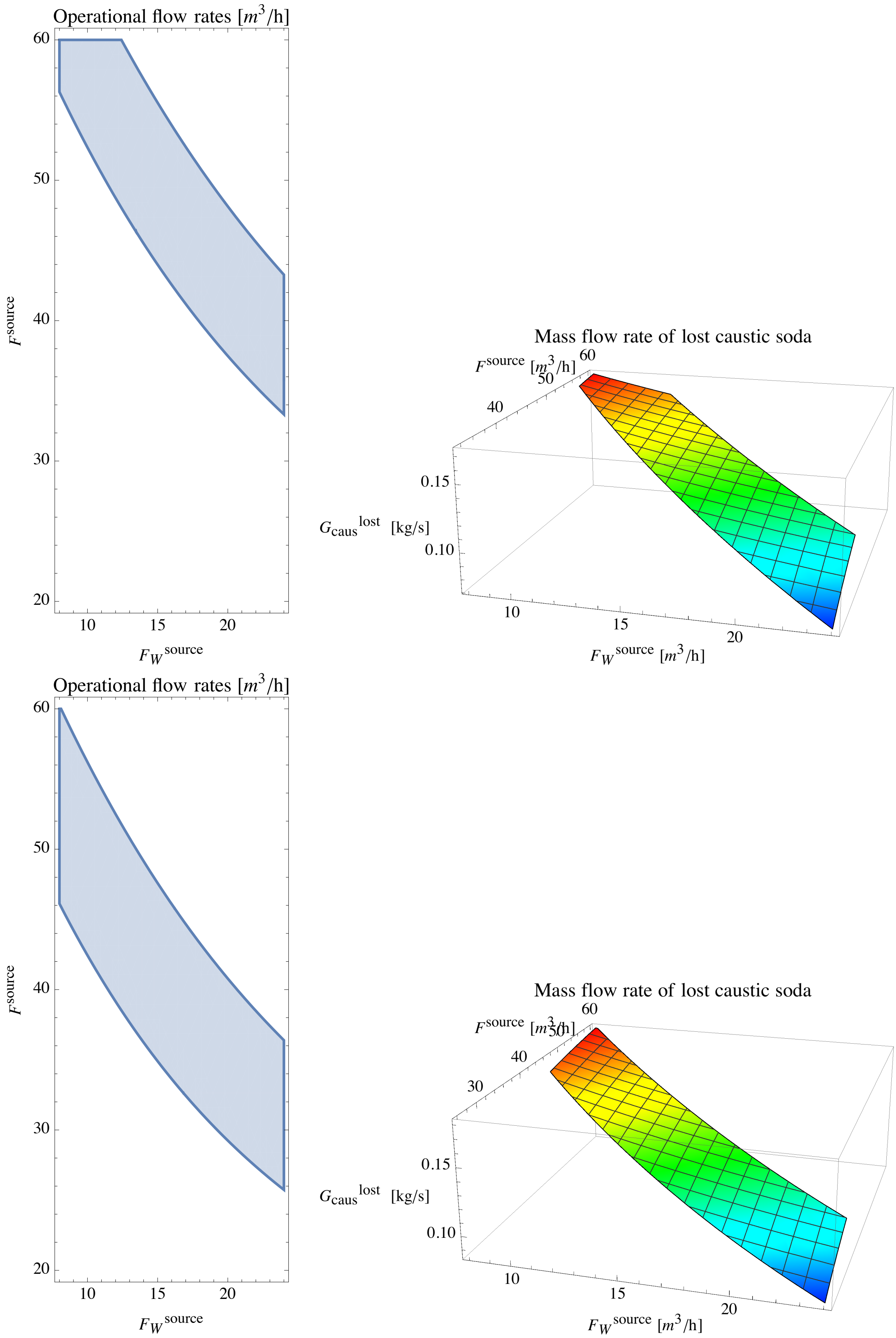

Figure 4: In all panels, cake resistance value is $r = 3 \times 10^{14} \mathrm{m}^{-2}$. In all top panels, contact wash angle is $\Theta_W = \pi/2$. In all bottom panels, contact wash angle is $\Theta_W = \pi/8$. Left panels: 2D shade plots of operational ranges for volume flow rates of source slurry (F^{src}) and source wash liquid (F_W^{src}), according to operational bounds in equation (8). Right panels: 3D plots of mass flow rate of caustic soda lost to the scraper $G_{\mathrm{caus}}^{\mathrm{lost}}$ as a function of volume flow rates of source slurry (F^{src}) and source wash liquid (F_W^{src}).

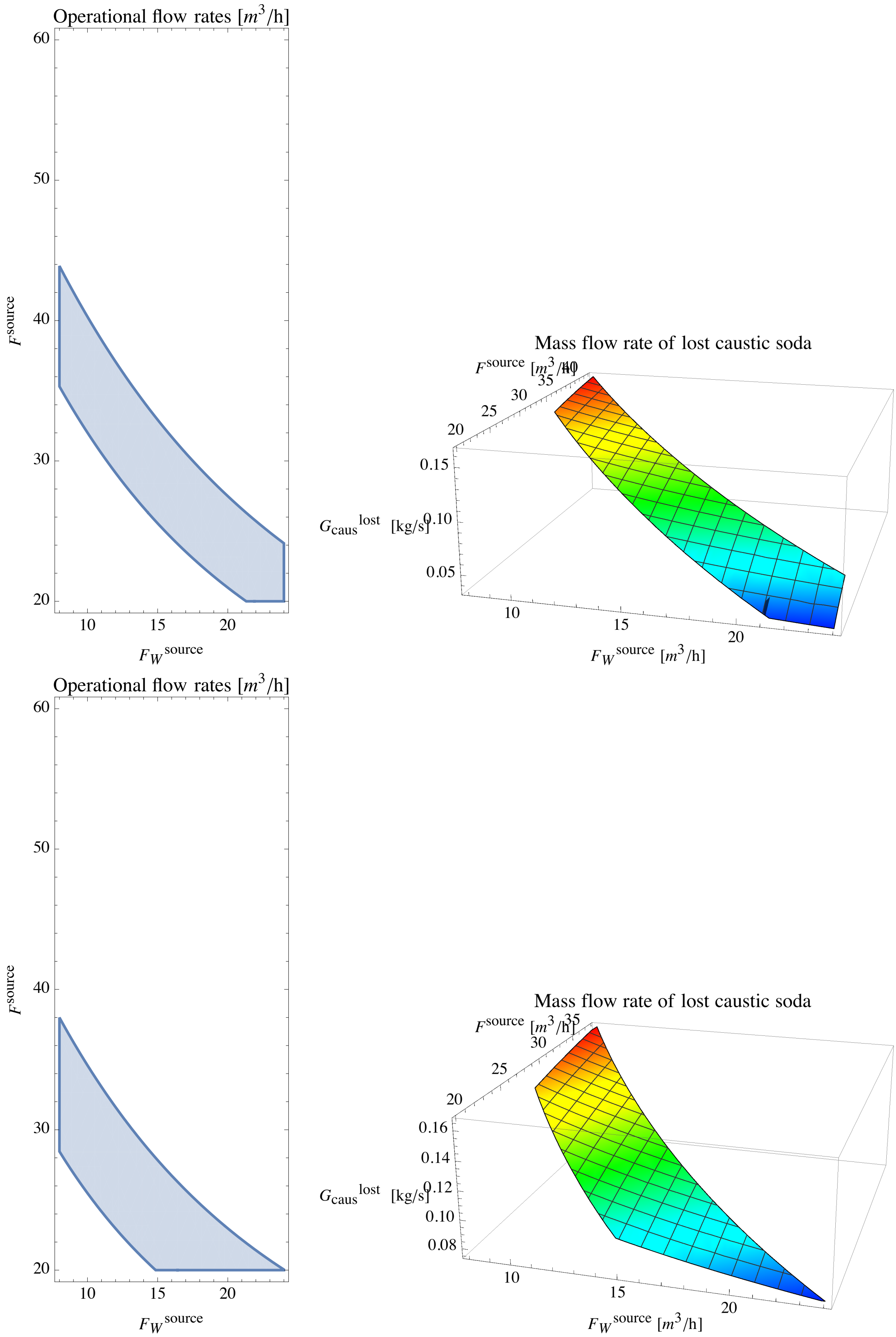

Figure 5: In all panels, cake resistance value is $r = 6 \times 10^{14}\mathrm{m}^{-2}$. In all top panels, contact wash angle is $\Theta_W = \pi/2$. In all bottom panels, contact wash angle is $\Theta_W = \pi/8$. Left panels: 2D shade plots of operational ranges for volume flow rates of source slurry (F^{src}) and source wash liquid (F_W^{src}), according to operational bounds in equation (8). Right panels: 3D plots of mass flow rate of caustic soda lost to the scraper $G_{\mathrm{caus}}^{\mathrm{lost}}$ as a function of volume flow rates of source slurry (F^{src}) and source wash liquid (F_W^{src}).

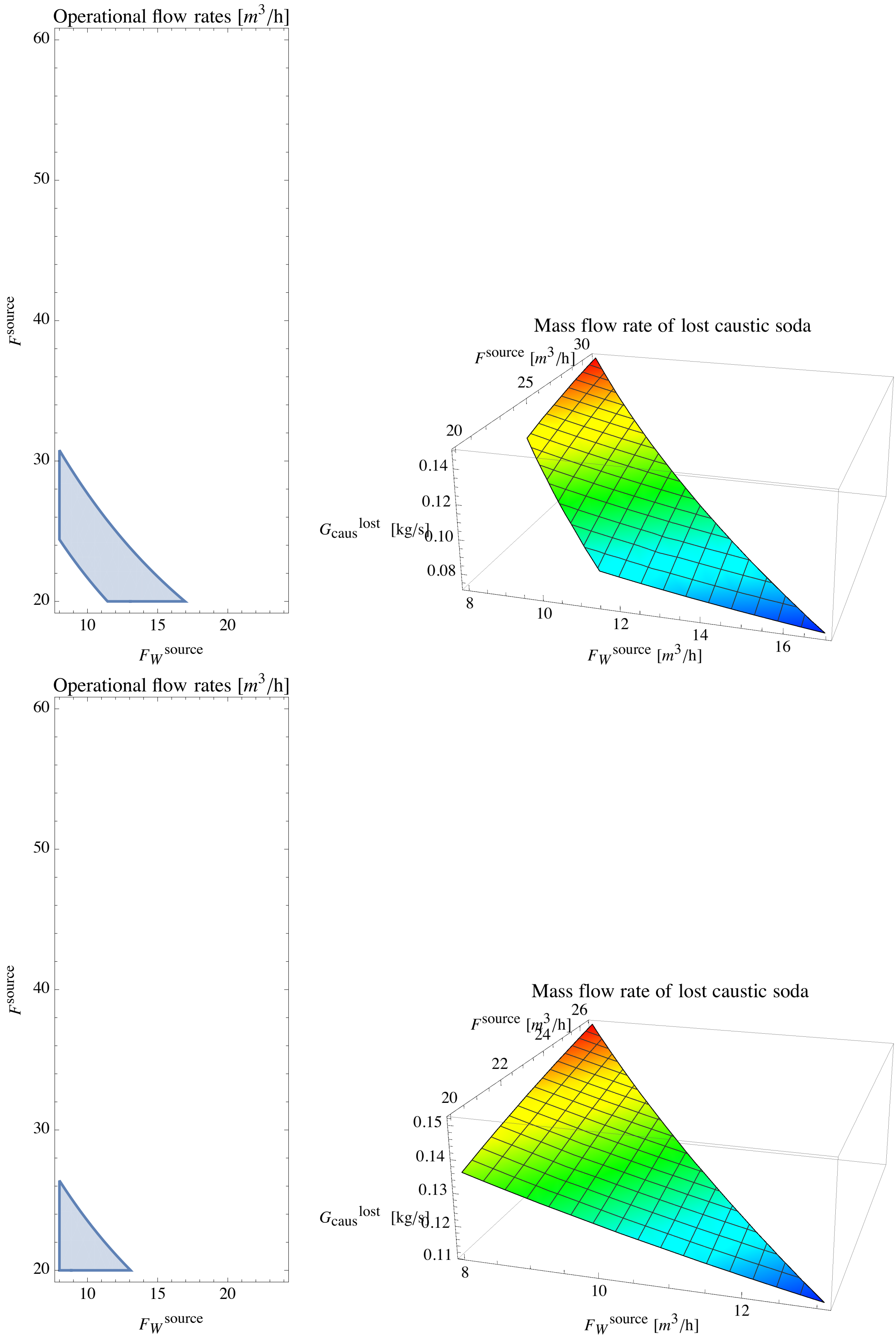

Figure 6: In all panels, cake resistance value is $r = 10 \times 10^{14}\mathrm{m}^{-2}$. In all top panels, contact wash angle is $\Theta_W = \pi/2$. In all bottom panels, contact wash angle is $\Theta_W = \pi/8$. Left panels: 2D shade plots of operational ranges for volume flow rates of source slurry (F^{src}) and source wash liquid (F_W^{src}), according to operational bounds in equation (8). Right panels: 3D plots of mass flow rate of caustic soda lost to the scraper $G_{\mathrm{caus}}^{\mathrm{lost}}$ as a function of volume flow rates of source slurry (F^{src}) and source wash liquid (F_W^{src}).

Cake resistance $r\,[\mathrm{m}^{-2}]$	Wash vol. flow $F_W^{\mathrm{src}}\,[\mathrm{m}^3/\mathrm{h}]$	Slurry vol. flow $F^{\mathrm{src}}\,[\mathrm{m}^3/\mathrm{h}]$	Cake thickness $L^{(0)}\,[\mathrm{mm}]$	Caustic scrapped $G_{\mathrm{caus}}^{\mathrm{lost}}\,[\mathrm{kg/s}]$	Caustic density $\rho_{\mathrm{caus}}^{\mathrm{lost}}\,[\mathrm{g/l}]$
2×10^{14}	20	60	4.214	0.1337	12.89
	24	50	3.512	0.1070	12.38
3×10^{14}	12	60	4.214	0.1611	15.53
	16	50	3.512	0.1364	15.78
	24	40	2.810	0.0949	13.72
6×10^{14}	8	40	2.810	0.1596	23.07
	16	30	2.107	0.1042	20.09
	24	20	1.405	0.0338	9.760

Table 1: Last three columns: steady-state values for the cake thickness (25), scrapped caustic soda mass flux rate (41), and scrapped caustic soda density $G_{\mathrm{caus}}^{\mathrm{lost}}/F_L^{\mathrm{filter}}$ (16), as functions of cake resistance, wash source volume flow rate and slurry source volume flow rate. Notice that increasing the wash flow rate and reducing the slurry flow rate leads to a significant reduction of the scrapped caustic soda mass flux rate, of direct economic relevance. In all cases, the nominal value $\Theta_W = \pi/2$ is chosen for the contact wash angle. All cases correspond to operational regimes, i.e. the level is well within the allowed values (8).

6 Conclusions

We have modelled successfully the operational behaviour of a single drum filter. Despite the significant amount of physical parameters involved in the problem, the main results depend mostly on the cake porosity e, the ratio β between the liquid and solid volume flow rates in the incoming slurry, and a single combination of the remaining parameters, which we call γ. There are two tuneable quantities, the total slurry volume flow rate and the wash liquid flow rate. The main result is a prescription ("recipe") to change these quantities at the same time in order to move optimally either towards better savings of caustic soda, or towards better compliance with the operational regime in terms of level of slurry in the trough. These two options of change are "orthogonal" or in other words, independent, and can be programmed using standard algorithms and softwares. If, in a real experiment, the steady-state cannot be attained because it is out of the operational range (in terms of the level of slurry in the trough), then any automatic control method must ensure that the flow rate parameters change towards the operational range, as fast as possible. This is ensured by going along the direction formulated in equation (31). Once the operational range is attained, it is possible to make significant savings of caustic soda by changing the parameters along the direction formulated in equation (42).

We have presented detailed results for the steady-state for simplicity of presentation, but this is not essential. For the non-steady case the analysis becomes more difficult, and this normally requires more advanced computer software such as *Matlab* or *Mathematica*.

There is room for improvement in several places. For example, one could upgrade the liquid's density and viscosity to the status of dynamical variables, and allow the cake's resistance and porosity to be modelled from first principles, which could then depend on the dynamical variables. However, the results so far have been robust, in the sense that only a few parameters dominate the results, which indicates that new models might not produce huge improvements. It remains to test the results experimentally to see if a new model is needed.

A final comment regarding the original problem statement by Rusal Aughinish. It is straightforward to consider the case of several drums and to try to optimise the changes in flow rates for each of the drums independently. This could be done in a future collaboration.