Practicum: Quadrupole deformation of the Schwarzschild spacetime

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I. BACKGROUND AND MOTIVATION FROM NEWTONIAN THEORY

A. Newtonian potential

The potential U of Newtonian gravity is a solution to Poisson's equation $\nabla^2 U = -4\pi G\rho$, where ρ is the mass density. The solution is given by

$$U(\boldsymbol{r}) = G\int \frac{\rho(\boldsymbol{r'})}{|\boldsymbol{r}-\boldsymbol{r'}|}\,dV', \tag{1.1}$$

where $\boldsymbol{r}$ is the position at which the potential is evaluated, $\boldsymbol{r'}$ is a position within the matter distribution, and dV' is the element of volume at position $\boldsymbol{r'}$.

A spherical body of mass M and radius R has a potential given simply by $U = GM/r$ outside the body ($r > R$).

B. Quadrupole deformation

We take the body to be slightly nonspherical, and describe its deformation from spherical symmetric by a perturbation of the potential $U = GM/r$. We take $r > r'$ in Eq. (1.1), and insert the identity

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r'}|} = \sum_{\ell=0}^{\infty} \frac{r'^\ell}{r^{\ell+1}} P_\ell(\cos\gamma), \tag{1.2}$$

where γ is the angle between the vectors $\boldsymbol{r}$ and $\boldsymbol{r'}$, and P_ℓ are Legendre polynomials. The term $\ell = 0$ reproduces GM/r, the term $\ell = 1$ vanishes by virtue of the definition of the centre of mass, and the leading term describing the deformation is $\ell = 2$. For an axisymmetric body, we have that the potential becomes

$$U = \frac{GM}{r} + \frac{GQ}{r^3} P_2(\cos\theta), \tag{1.3}$$

where θ is the angle between $\boldsymbol{r}$ and the body's symmetry axis, and

$$Q = \int \rho(r',\theta') r'^2 P_2(\cos\theta')\,dV' \tag{1.4}$$

is the body's quadrupole moment. The second term in the potential describes the body's quadrupole deformation.

One of the themes explored below (Sec. II C) is whether a black hole is capable of supporting a permanent quadrupole moment.

C. Tidal field

Next we take the body to be spherical, and put it in the presence of a remote companion of mass M' at position $\boldsymbol{b}$ relative to the body's centre of mass. The total potential is now

$$U = \frac{GM}{r} + U_{\rm ext}, \qquad U_{\rm ext} = \frac{GM'}{|\boldsymbol{r}-\boldsymbol{b}|}. \tag{1.5}$$

We assume that $b \gg r$, and exploit the identity

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r'}|} = \sum_{\ell=0}^{\infty} \frac{r^\ell}{b^{\ell+1}} P_\ell(\cos\theta), \tag{1.6}$$

where θ is now the angle between $\boldsymbol{r}$ and $\boldsymbol{b}$. The term $\ell = 0$ in $U_{\rm ext}$ gives rise to an irrelevant constant, the term $\ell = 1$ gives rise to a constant force responsible for the body's centre-of-mass motion around its companion, and the $\ell = 2$ term gives a leading-order description of the tidal forces exerted on M by M'. The total potential is then

$$U = \frac{GM}{r} + \frac{GM'}{b^3} r^2 P_2(\cos\theta), \tag{1.7}$$

and the second term describes the small perturbation associated with the tidal forces exerted by M'.

One of the themes explored below (Sec. II D) is whether a black hole undergoes a deformation when it is subjected to a tidal field.

D. Love number

Even when it is originally spherical, a body will respond to tidal forces and undergo a slight deformation. We can expect that the deformation, measured by the quadrupole moment Q, will be proportional to the tidal field, measured by GM'/b^3. We express this as

$$GQ = 2kR^5 \frac{GM'}{b^3}, \tag{1.8}$$

where $2k$ is a dimensionless factor, and the factor R^5 was inserted to respect the dimensionality of Q, which has units of [mass] × [length]^2. The number k is known as the quadrupolar Love number, and its value depends on the body's internal structure. The total potential is

$$U = \frac{GM}{r} + \frac{GM'}{b^3}\Big[1 + 2k(R/r)^5\Big] r^2 P_2(\cos\theta). \tag{1.9}$$

In the square brackets, the first term represents the applied tidal field, and the Love-number term represents the body's tidal deformation.

One of the themes explored below (Sec. II E) is the calculation of the quadrupolar Love number for a black hole.

II. QUADRUPOLE DEFORMATION OF A SCHWARZSCHILD SPACETIME

The main goal of this Practicum is to construct a time-independent, quadrupole perturbation of a Schwarzschild spacetime, and to extract its physical significance for black holes. A complete listing of the perturbation equations is given in the Appendix. To connect the tasks below to the Newtonian discussion of Sec. I, it is good to define an effective Newtonian potential $U_{\rm eff}$ by

$$g_{tt} = -1 + 2U_{\rm eff}. \tag{2.1}$$

For the Schwarzschild solution, $g_{tt} = -(1-2M/r)$, and $U_{\rm eff} = M/r$, the same expression as in Newtonian gravity (in geometrized units with $G = c = 1$).

A. Task 1: Reduction of the equations

We insert $\ell = 2$ in the equations listed in the Appendix, and since the perturbation is assumed to be t-independent, we eliminate all time derivatives. To keep the task reasonable and connected to the Newtonian discussion of Sec. I, we take the perturbation to be of even parity, and ignore all odd-parity terms. In the Regge-Wheeler gauge, the perturbation is therefore described entirely by $h_{tt}(r)$, $h_{tr}(r)$, $h_{rr}(r)$, and $K(r)$. We construct a *vacuum* perturbation of the Schwarzschild spacetime, for which $G^{\alpha\beta} = 0$.

1. Show that the $Q^{tr} = 0$ equation implies that $h_{tr} = 0$.

2. Show that the $Q^\sharp = 0$ equation implies that $h_{tt} = f^2 h_{rr}$, where $f = 1 - 2M/r$. The primary variables become h_{rr} and K.

3. Combine the $Q^{rr} = 0$ and $Q^r = 0$ equations to eliminate the K' term, in which a prime indicates differentiation with respect to r, and obtain

$$K = \frac{1}{2} M f h'_{rr} + (1 - M/r) h_{rr}. \tag{2.2}$$

   This equation determines K once h_{rr} is known.

4. Differentiate the previous equation with respect to r, and insert it within the $Q^r = 0$ equation. Show that the equation becomes

$$r(r-2M)h''_{rr} + 2(r+M)h'_{rr} - 6h_{rr} = 0. \tag{2.3}$$

   This is the equation that determines h_{rr}.

B. Task 2: Solutions to the equations

Equation (2.3) takes the form of the hypergeometric equation, but the resulting series doesn't converge. We must proceed differently to find its solution. Before we do we recall how one can find a second solution to a second-order differential equation once a first solution is known. For a differential equation of the form $y'' + p(r)y' + q(r)y = 0$, if $y^{(1)}$ is a solution, then

$$y^{(2)} = y^{(1)}(r) \int^r e^{-w(r')} \left[y^{(1)}(r') \right]^{-2} dr' \tag{2.4}$$

is another solution, where $w(r) = \int^r p(r')\, dr'$.

1. Show that

$$h^{(1)}_{rr} = r^2 \tag{2.5}$$

   is a solution to Eq. (2.3).

2. Find a second solution using the strategy described previously. Show that it can be put in the form

$$h^{(2)}_{rr} = -10 \biggl[3r^2 \ln f + \frac{2M(r-M)(3r^2 - 6Mr - 2M^2)}{(r-2M)^2} \biggr]. \tag{2.6}$$

3. Show that when $M/r \ll 1$, $h^{(2)}_{rr} \sim (2M)^5/r^3$. This requires taking a Taylor expansion in powers of M/r to at least five orders.

4. Find the corresponding expressions for K.

C. Task 3: Black hole with a quadrupole moment

In this section we ask whether it is possible for a black hole to support a permanent quadrupole moment, like a material body can (in both Newtonian theory and general relativity).

1. Identify the solution to the perturbation equations that would describe a quadrupole deformation of a black hole.

2. Is this solution acceptable physically? If so, why? If not, why not?

3. If you concluded that the solution is unacceptable, and that a black hole cannot support a quadrupole deformation, explain why the objection doesn't hold for a material body.

D. Task 4: Black hole in a tidal field

In this section we ask whether a black hole deforms when it is subjected to tidal forces exerted by a remote companion.

1. Identify the solution to the perturbation equations that would describe a black hole immersed in a tidal field.

2. Is this solution acceptable physically? If so, why? If not, why not?

3. If you concluded that the solution is acceptable, describe what changes, if any, would be required if the black hole were replaced by a material body.

E. Task 5: Love number of a black hole

In this section we calculate the quadrupolar Love number of a black hole.

1. Identify the solution to the perturbation equations that would describe a black hole deforming under the action of an applied tidal field.

2. Is this solution acceptable physically? If so, why? If not, why not?

3. If you concluded that the solution is unacceptable, what is k for a black hole?

4. If you concluded that the solution is unacceptable for a black hole, would it be acceptable for a material body? Describe what would be required to calculate k in this case.

F. Task 6: Regularity at $r = 2M$

The answer to many of the preceding questions relied on the property that the second solution to the perturbation equations becomes infinite at $r = 2M$. The argument is that such a perturbation cannot describe a black hole with a nonsingular event horizon.

The argument is made subtle by the fact that the coordinates (t, r, θ, ϕ) are singular at $r = 2M$. Regularity of the event horizon is therefore difficult to verify in this coordinate system. A way to cope is to work with coordinates that are regular across $r = 2M$. The simplest system is (v, r, θ, ϕ), where $v = t + r^*$ with

$$r^* = \int f^{-1}\, dr = r + 2M \ln(r/2M - 1). \tag{2.7}$$

The new time coordinate, known as *advanced time*, is constant on radial null geodesics that converge toward the black hole.

1. Verify that the Schwarzschild metric is regular at $r = 2M$ when expressed in the (v, r, θ, ϕ) coordinates.

2. Verify that the first solution to the perturbation equations is regular at $r = 2M$ when transformed to the (v, r, θ, ϕ) coordinates.

3. Verify that the second solution to the perturbation equations diverges logarithmically at $r = 2M$ when transformed to the (v, r, θ, ϕ) coordinates.

Appendix A: Perturbation equations

We consider a perturbation of the Schwarzschild spacetime, with unperturbed metric

$$ds^2 = -f\, dt^2 + f^{-1}\, dr^2 + r^2\, d\Omega^2, \tag{A1}$$

where $f = 1 - 2M/r$. In the Regge-Wheeler gauge, the metric perturbation is decomposed as

$$p_{ab} = \sum_{\ell m} h^{\ell m}_{ab}(t, r) Y^{\ell m}(\theta, \phi), \tag{A2a}$$

$$p_{aB} = \sum_{\ell m} h^{\ell m}_{a}(t, r) X^{\ell m}_{A}(\theta, \phi), \tag{A2b}$$

$$p_{AB} = r^2 \sum_{\ell m} K^{\ell m}(t, r) \Omega_{AB} Y^{\ell m}(\theta, \phi), \tag{A2c}$$

where indices $a, b, \cdots$ refer to the coordinates (t, r), while indices $A, B, \cdots$ refer to (θ, ϕ). The perturbed Einstein tensor is decomposed as

$$\dot{G}_{ab} = \sum_{\ell m} Q^{\ell m}_{ab}\, Y^{\ell m}, \tag{A3a}$$

$$\dot{G}_{aB} = \sum_{\ell m} Q^{\ell m}_{a}\, Y^{\ell m}_{A} + \sum_{\ell m} P^{\ell m}_{a}\, X^{\ell m}_{A}, \tag{A3b}$$

$$\dot{G}_{AB} = \sum_{\ell m} \Big[Q^{\ell m}_{\flat}\, \Omega_{AB} Y^{\ell m} + Q^{\ell m}_{\sharp}\, Y^{\ell m}_{AB} \Big] + \sum_{\ell m} P^{\ell m}\, X^{\ell m}_{AB}. \tag{A3c}$$

Apart from unimportant numerical factors, the explicit expressions are

$$Q^{tt} = -\frac{\partial^2}{\partial r^2}K - \frac{3r-5M}{r^2 f}\frac{\partial}{\partial r}K + \frac{f}{r}\frac{\partial}{\partial r}h_{rr} + \frac{(\lambda+2)r+4M}{2r^3}h_{rr} + \frac{\mu}{2r^2 f}K, \tag{A4a}$$

$$Q^{tr} = \frac{\partial^2}{\partial t\partial r}K + \frac{r-3M}{r^2 f}\frac{\partial}{\partial t}K - \frac{f}{r}\frac{\partial}{\partial t}h_{rr} - \frac{\lambda}{2r^2}h_{tr}, \tag{A4b}$$

$$Q^{rr} = -\frac{\partial^2}{\partial t^2}K + \frac{(r-M)f}{r^2}\frac{\partial}{\partial r}K + \frac{2f}{r}\frac{\partial}{\partial t}h_{tr} - \frac{f}{r}\frac{\partial}{\partial r}h_{tt} + \frac{\lambda r+4M}{2r^3}h_{tt} - \frac{f^2}{r^2}h_{rr} - \frac{\mu f}{2r^2}K \tag{A4c}$$

$$Q^{t} = \frac{\partial}{\partial t}h_{rr} - \frac{\partial}{\partial r}h_{tr} + \frac{1}{f}\frac{\partial}{\partial t}K - \frac{2M}{r^2 f}h_{tr}, \tag{A4d}$$

$$Q^{r} = -\frac{\partial}{\partial t}h_{tr} + \frac{\partial}{\partial r}h_{tt} - f\frac{\partial}{\partial r}K - \frac{r-M}{r^2 f}h_{tt} + \frac{(r-M)f}{r^2}h_{rr}, \tag{A4e}$$

$$\begin{aligned} Q^{\flat} = &-\frac{\partial^2}{\partial t^2}h_{rr} + 2\frac{\partial^2}{\partial t\partial r}h_{tr} - \frac{\partial^2}{\partial r^2}h_{tt} - \frac{1}{f}\frac{\partial^2}{\partial t^2}K + f\frac{\partial^2}{\partial r^2}K + \frac{2(r-M)}{r^2 f}\frac{\partial}{\partial t}h_{tr} - \frac{r-3M}{r^2 f}\frac{\partial}{\partial r}h_{tt} \\ &- \frac{(r-M)f}{r^2}\frac{\partial}{\partial r}h_{rr} + \frac{2(r-M)}{r^2}\frac{\partial}{\partial r}K + \frac{\lambda r^2 - 2(2+\lambda)Mr + 4M^2}{2r^4 f^2}h_{tt} - \frac{\lambda r^2 - 2\mu M r - 4M^2}{2r^4}h_{rr}, \end{aligned} \tag{A4f}$$

$$Q^{\sharp} = \frac{1}{f}h_{tt} - f h_{rr} \tag{A4g}$$

in the even-parity sector, and

$$P^{t} = -\frac{\partial^2}{\partial t\partial r}h_r + \frac{\partial^2}{\partial r^2}h_t - \frac{2}{r}\frac{\partial}{\partial t}h_r - \frac{\lambda r - 4M}{r^3 f}h_t, \tag{A5a}$$

$$P^{r} = \frac{\partial^2}{\partial t^2}h_r - \frac{\partial^2}{\partial t\partial r}h_t + \frac{2}{r}\frac{\partial}{\partial t}h_t + \frac{\mu f}{r^2}h_r, \tag{A5b}$$

$$P = -\frac{1}{f}\frac{\partial}{\partial t}h_t + f\frac{\partial}{\partial r}h_r + \frac{2M}{r^2}h_r \tag{A5c}$$

in the odd-parity sector. We write $\lambda = \ell(\ell+1)$ and $\mu = (\ell-1)(\ell+2)$. By virtue of the Bianchi identities, the components of $\dot{G}^{\alpha\beta}$ are not all independent from each other. There is redundancy in the system of perturbation equations.