Lectures on black-hole perturbation theory

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Dublin School on Gravitational Wave Source Modelling
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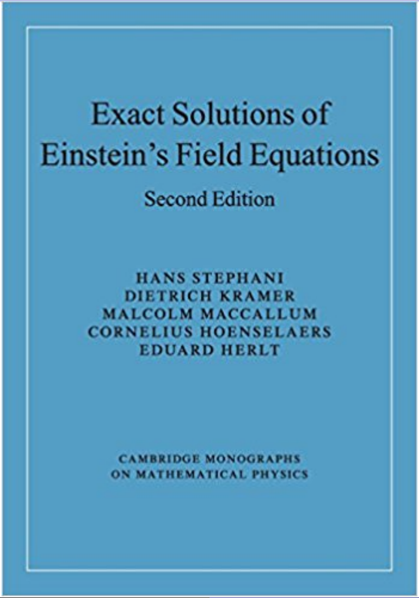
Outline

- Introduction and motivation
- Perturbation theory in general relativity
- Perturbations of a Schwarzschild black hole
- Perturbations of a Kerr black hole
- Practicum

Why perturbation theory?



Why perturbation theory?

The big book lists thousands of exact solutions to the Einstein field equations. Mathematically, this is an impressive feat.

Physically, only a few solutions are useful: Schwarzschild, Kerr, FLRW, Vaidya, Majumdar-Papapetrou, Weyl, C-metric, Robinson-Trautman, ...

There is no exact solution that describes the inspiral of two compact bodies and the emitted gravitational waves.

For this we need sophisticated computational methods, or perturbation theory.

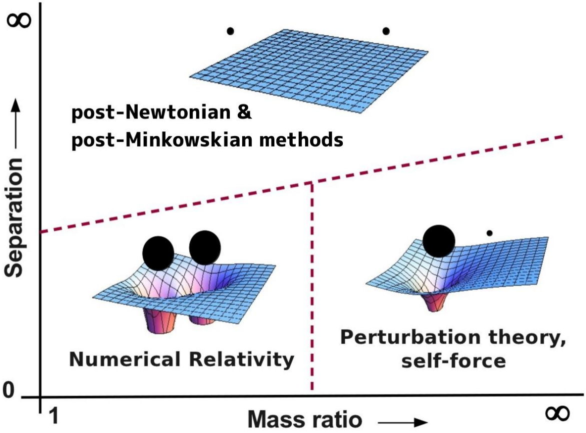
Perturbation theory is also extremely relevant to early-universe cosmology.

Perturbation about what? Expansion parameter?

- post-Minkowskian theory: About Minkowski spacetime, in powers of G
- post-Newtonian theory: About Minkowski spacetime, in powers of c^{-2}
- Schwarzschild perturbation theory: About Schwarzschild spacetime, in powers of a perturbing mass m
- Kerr perturbation theory: About Kerr spacetime, in powers of a perturbing mass m
- FLRW perturbation theory: About FLRW spacetime, in powers of density fluctuation $\delta\rho$

Overlapping domains

Different perturbation methods can overlap in a common domain of validity. [Leor Barack: GR21]



Perturbation theory: Algebra

[C. Bender & S. Orszag, *Advanced Mathematical Methods for Scientists and Engineers*]

Consider the algebraic problem

$$(x-1)(x+2)(x+3)(x+4)+\epsilon=0, \qquad x>0, \qquad \epsilon \ll 1$$

An exact solution is not available.

Postulate the existence of a one-parameter family of variables $x(\lambda)$, with $\lambda \in [0,\epsilon]$, given by

$$x(\lambda)=1+x_1\lambda+\frac{1}{2}x_2\lambda^2+\frac{1}{6}x_3\lambda^3+\frac{1}{24}x_4\lambda^4+\cdots$$

$$x_n=\frac{d^n x(\lambda)}{d\lambda^n}\bigg|_{\lambda=0}$$

Perturbation theory: Algebra

Substitute $x(\lambda)$, set $\lambda = \epsilon$, and solve order by order in ϵ.

The result is

$$x_1 = -\frac{1}{60}, \qquad x_2 = -\frac{47}{108000}$$
$$x_3 = -\frac{1849}{64800000}, \qquad x_4 = -\frac{70703}{23328000000}$$

For $\epsilon = 0.1$, $x[\text{numerical}] - x(\epsilon) \simeq -3.71 \times 10^{-14}$.

Typically one cannot know whether the series in powers of λ converges. Typically the series is only asymptotic.

Perturbation theory: Differential equation

Consider the ordinary, second-order differential equation

$$y'' + (1 - \epsilon x)y = 0, \qquad y(0) = 1, \qquad y(\pi/2) = 0$$

The exact solution involves Airy functions. Pursue instead a perturbative approach.

Postulate the existence of a one-parameter family of functions $y(\lambda, x)$, with $\lambda \in [0, \epsilon]$, given by

$$y(\lambda, x) = \cos x + y_1(x)\lambda + \frac{1}{2}y_2(x)\lambda^2 + \frac{1}{6}y_3(x)\lambda^3 + \cdots$$

$$y_n(x) = \frac{\partial^n y(\lambda, x)}{\partial \lambda^n}\bigg|_{\lambda=0}$$

$$y_n(0) = 0, \qquad y_n(\pi/2) = 0$$

Perturbation theory: Differential equation

Substitute $y(\lambda, x)$ within the differential equation, set $\lambda = \epsilon$, and solve order by order in ϵ.

The result is

$$
\begin{aligned}
y_1 &= \frac{1}{4} x \cos x + \frac{1}{16}(4x^2 - \pi^2)\sin x \\
y_2 &= -\frac{1}{32} x^2 (2x^2 - 10 - \pi^2)\cos x \\
&\quad + \frac{1}{96}(2x - \pi)(10x^2 + 5\pi x - 15 + \pi^2)\sin x \\
y_3 &= \cdots
\end{aligned}
$$

In this case, $|y_1| < 0.18829$, $|y_2| < 0.10029$, and $|y_3| < 0.08101$. Successive terms become progressively smaller.

Singular perturbation theory

A straightforward expansion in powers of λ is not always adequate in perturbation problems.

This is the case, for example, in problems like

$$\epsilon x^4 + 8x^3 + 17x^2 - 2x - 24 = 0$$

$$\epsilon y'' + y' - 1 = 0$$

In such problems, the "small term" changes the character of the equation.

Other perturbative techniques are required: scale transformation, boundary-layer theory, multi-scale analysis, ...

This is the realm of **singular** perturbation theory.

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Linearized general relativity

We wish to solve the Einstein field equations

$$G^{\alpha\beta}[g] = 8\pi T^{\alpha\beta}[g]$$

when the energy-momentum tensor $T^{\alpha\beta} \propto \epsilon$ is small and deviations from flat spacetime are also small.

We postulate the existence of a one-parameter family of metrics

$$g_{\alpha\beta}(\lambda, x) = \eta_{\alpha\beta} + p^1_{\alpha\beta}(x)\,\lambda + \frac{1}{2} p^2_{\alpha\beta}(x)\,\lambda^2 + \cdots$$

$$p^n_{\alpha\beta}(x) = \frac{\partial^n g_{\alpha\beta}(\lambda, x)}{\partial \lambda^n}\bigg|_{\lambda=0}$$

We substitute $g_{\alpha\beta}(\lambda, x)$ within the field equations, set $\lambda = \epsilon$, and solve order by order in ϵ.

Linearized general relativity

Complication: $p^n_{\alpha\beta}$ can be changed arbitrarily by a coordinate transformation

$$x^\alpha \to x^\alpha(\lambda) = x^\alpha + \Xi^\alpha_1(x)\,\lambda + \frac{1}{2}\Xi^\alpha_2(x)\,\lambda^2 + \cdots$$

Solution: Impose a coordinate condition (also known as a gauge condition) to eliminate this redundancy.

Linearized general relativity

$$h_{\alpha\beta} := p^1_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}\big(\eta^{\mu\nu}p^1_{\mu\nu}\big) \qquad \text{(trace reversal)}$$

$$\partial_\beta h^{\alpha\beta} = 0 \qquad \text{(coordinate condition)}$$

$$\Box h^{\alpha\beta} = -16\pi T^{\alpha\beta}[\eta] \qquad \text{(field equations)}$$

Generalization to any background spacetime

A perturbation can be carried out about **any** background spacetime with metric $g(0)$:

$$g_{\alpha\beta}(\lambda, x) = g_{\alpha\beta}(0, x) + p^1_{\alpha\beta}(x)\,\lambda + \frac{1}{2}p^2_{\alpha\beta}(x)\,\lambda^2 + \cdots$$

We assume, for simplicity, that $G^{\alpha\beta}[g(0)] = 0$, and that the source of the perturbation is some energy-momentum tensor $\epsilon\, T^{\alpha\beta}[g]$.

Then

$$G^{\alpha\beta}[g(\lambda)] = G^{\alpha\beta}[g(0) + p^1\,\lambda + \cdots] = \lambda\, \dot{G}^{\alpha\beta}[g(0), p^1] + \cdots$$
$$\epsilon\, T^{\alpha\beta}[g(\lambda)] = \epsilon\, T^{\alpha\beta}[g(0) + p^1\,\lambda + \cdots] = \epsilon\, T^{\alpha\beta}[g(0)] + \cdots$$

where $\dot{G}^{\alpha\beta} = (\partial/\partial\lambda) G^{\alpha\beta}|_{\lambda=0}$.

Generalization to any background spacetime

We then set $\lambda = \epsilon$ and obtain the perturbation equation

$$\dot{G}^{\alpha\beta}[g(0), p^1] = 8\pi T^{\alpha\beta}[g(0)]$$

The left-hand side takes the form of a differential operator acting on $p^1_{\alpha\beta}$, which can be thought of as a tensor field in the background spacetime.

To this we must adjoin a coordinate condition

$$\mathscr{L}^\alpha[g(0), p^1] = 0$$

Geometry of a perturbation

A perturbation takes a background spacetime $(\mathcal{M}(0), g(0))$ to a perturbed spacetime $(\mathcal{M}(\epsilon), g(\epsilon))$.

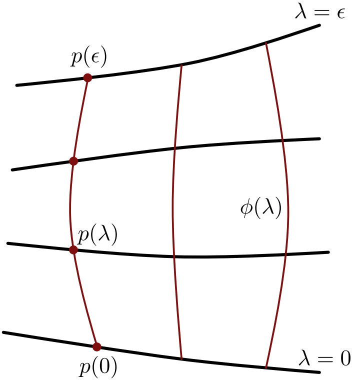
We describe this in terms of a one-parameter family of spacetimes $(\mathcal{M}(\lambda), g(\lambda))$ with $\lambda \in [0, \epsilon]$.

The entire family defines a 5D manifold (with boundaries), and each $\mathcal{M}(\lambda)$ is an embedded submanifold.

We wish to define the perturbation of a tensorial quantity $Q(0, x)$ (for example, the Riemann tensor) on $\mathcal{M}(0)$.

For this we need $Q(\lambda, x)$ and an **identification map** between points on $\mathcal{M}(\lambda)$ and points on $\mathcal{M}(0)$.

Identification map



We introduce a vector field v on the 5D manifold, which is transverse (nowhere tangent) to each $\mathcal{M}(\lambda)$. We let $\phi(\lambda)$ be the integral curves of $v = \partial/\partial\lambda$.

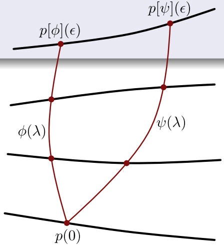
A point $p(\lambda)$ on $\mathcal{M}(\lambda)$ is identified with a point $p(0)$ on $\mathcal{M}(0)$ if they lie on the same integral curve.

Perturbation defined

The perturbation of $Q(0, x)$ is then defined by

$$\dot{Q}(x) := \frac{\partial Q(\lambda, x)}{\partial \lambda}\bigg|_{\lambda=0} = \pounds_v Q\bigg|_{\lambda=0}$$

The perturbation refers to a choice (v, ϕ) of identification map. If we introduce another map (w, ψ), we get a different perturbation.



Difference between equivalent perturbations

$$\dot{Q}[\phi] - \dot{Q}[\psi] = \pounds_v Q - \pounds_w Q = \pounds_{v-w} Q = \pounds_\Xi Q$$

where $\Xi := (v - w)_{\lambda=0}$ is tangent to $\mathcal{M}(0)$.

This is the geometrical view on "gauge transformations".

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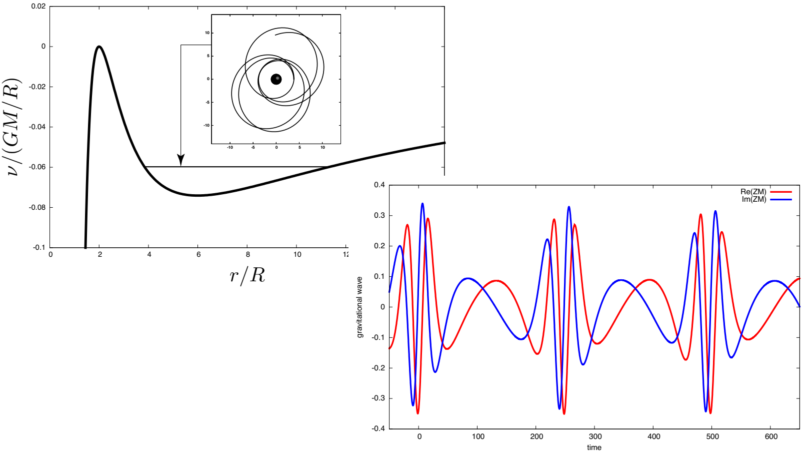
Perturbation of a Schwarzschild black hole

Suppose that we are interested in calculating the gravitational waves emitted by a small particle of mass m on a bound orbit around a Schwarzschild black hole of mass $M \gg m$.



Perturbation of a Schwarzschild black hole

We begin with the background Schwarzschild metric

$$g_{\alpha\beta}(0)\, dx^\alpha dx^\beta = -f\, dt^2 + f^{-1}\, dr^2 + r^2\, d\Omega^2, \qquad f = 1 - 2M/r$$

We place the mass m on a geodesic of this spacetime, and introduce its energy-momentum tensor $T^{\alpha\beta}$, a delta function with support on the geodesic.

Next we introduce the perturbation $p^1_{\alpha\beta} = \dot{g}_{\alpha\beta}$ and impose a coordinate condition $\mathscr{L}^\alpha[p^1] = 0$.

Finally we integrate the perturbation equations

$$\dot{G}^{\alpha\beta}[g(0), p^1] = 8\pi T^{\alpha\beta}[g(0)]$$

and extract the gravitational waves at infinity.

Decomposition in spherical harmonics (3D space)

The Schwarzschild spacetime is **spherically symmetric**. This motivates a decomposition of $p_{\alpha\beta}$ in spherical harmonics.

Scalar harmonics

Any function $h(\theta, \phi)$ can be decomposed in spherical harmonics,

$$h(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} h_{\ell m} Y^{\ell m}(\theta, \phi)$$

Vector harmonics (Cartesian components)

To decompose a vector $\boldsymbol{v}(\theta, \phi)$ we need a vectorial basis,

$$\boldsymbol{Y}_1^{\ell m} = \boldsymbol{r}\, Y^{\ell m} \qquad \text{(vector)}$$

$$\boldsymbol{Y}_2^{\ell m} = \boldsymbol{\nabla} Y^{\ell m} = \boldsymbol{\theta}\, \partial_\theta Y^{\ell m} + \boldsymbol{\phi}\, \partial_\phi Y^{\ell m} \qquad \text{(vector)}$$

$$\boldsymbol{Y}_3^{\ell m} = \boldsymbol{r} \times \boldsymbol{\nabla} Y^{\ell m} = -\boldsymbol{\theta}\, \partial_\phi Y^{\ell m} + \boldsymbol{\phi}\, \partial_\theta Y^{\ell m} \qquad \text{(axial vector)}$$

Vector harmonics (spherical components)

It is useful to distinguish v_r from $v_A = (v_\theta, v_\phi)$.

Decompositions

$$v_r = \sum_{\ell m} a_{\ell m} Y^{\ell m} \qquad \text{(scalar on } S^2\text{)}$$

$$v_A = \sum_{\ell m} \Big(b_{\ell m} Y_A^{\ell m} + c_{\ell m} X_A^{\ell m} \Big)$$

Vector harmonics

$$Y_A^{\ell m} = \partial_A Y^{\ell m} \qquad \text{(vector on } S^2\text{)}$$

$$X_A^{\ell m} = \varepsilon_A^{\ \ B} \partial_B Y^{\ell m} \qquad \text{(axial vector on } S^2\text{)}$$

$$\varepsilon_A^{\ \ B} = \text{Levi-Civita tensor on } S^2$$

Tensor harmonics

For a symmetric tensor we have the components v_{rr} (scalar on S^2), v_{rA} (vector on S^2), and v_{AB} (tensor on S^2).

It is useful to decompose v_{AB} into trace and tracefree pieces,

$$v_{AB} = \frac{1}{2} v\, \Omega_{AB} + \hat{v}_{AB}$$

$$v = \Omega^{AB} v_{AB} \quad \text{(scalar)}, \qquad \hat{v}_{AB} = v_{AB} - \frac{1}{2} v\, \Omega_{AB}$$

where Ω_{AB} is the metric on the unit 2-sphere,

$$d\Omega^2 = \Omega_{AB}\, d\theta^A d\theta^B = d\theta^2 + \sin^2\theta\, d\phi^2$$

To define the tensorial harmonics we shall also need D_A, the covariant derivative on the unit 2-sphere.

Tensor harmonics

Decompositions

$$v_{rr} = \sum_{\ell m} b_{\ell m} Y^{\ell m}$$

$$v_{rA} = \sum_{\ell m} \Bigl(b_{\ell m} Y_A^{\ell m} + c_{\ell m} X_A^{\ell m} \Bigr)$$

$$v_{AB} = \sum_{\ell m} \Bigl(d_{\ell m} \Omega_{AB} Y^{\ell m} + e_{\ell m} Y_{AB}^{\ell m} + f_{\ell m} X_{AB}^{\ell m} \Bigr)$$

Tensor harmonics

$$Y_{AB}^{\ell m} = D_A D_B Y^{\ell m} + \frac{1}{2}\ell(\ell+1)\Omega_{AB} Y^{\ell m} \qquad \text{(tensor on } S^2\text{)}$$

$$X_{AB}^{\ell m} = D_A X_B^{\ell m} + D_B X_A^{\ell m} \qquad \text{(axial tensor on } S^2\text{)}$$

Decomposition of the metric perturbation

[K. Martel and E. Poisson, Phys. Rev. D **71**, 104003 (2005)]

$p_{ab} = (p_{tt}, p_{tr}, p_{rr})$ behaves as a scalar on S^2
$p_{aB} = (p_{t\theta}, p_{t\phi}, p_{r\theta}, p_{r\phi})$ behaves as a vector
$p_{AB} = (p_{\theta\theta}, p_{\theta\phi}, p_{\phi\phi})$ behaves as a tensor

Decomposition

$$p_{ab} = \sum_{\ell m} h_{ab}^{\ell m}(t,r) Y^{\ell m}(\theta,\phi)$$

$$p_{aB} = \sum_{\ell m} j_a^{\ell m}(t,r) Y_A^{\ell m}(\theta,\phi) + \sum_{\ell m} h_a^{\ell m}(t,r) X_A^{\ell m}(\theta,\phi)$$

$$p_{AB} = r^2 \sum_{\ell m} \Big[K^{\ell m}(t,r) \Omega_{AB} Y^{\ell m}(\theta,\phi) + G^{\ell m}(t,r) Y_{AB}^{\ell m}(\theta,\phi) \Big] + \sum_{\ell m} h_2^{\ell m}(t,r) X_{AB}^{\ell m}(\theta,\phi)$$

Parity

Even-parity sector (polar perturbations)

On S^2, $Y^{\ell m}$ behaves as a scalar, $Y_A^{\ell m}$ behaves as a vector, and $Y_{AB}^{\ell m}$ behaves as a tensor. They are said to have even parity under a transformation $\boldsymbol{r} \to -\boldsymbol{r}$.

The associated variables $h_{ab}^{\ell m}$, $j_a^{\ell m}$, $K^{\ell m}$, and $G^{\ell m}$ make up the even-parity sector of the perturbation, also known as polar perturbations.

Odd-parity sector (axial perturbations)

On S^2, $X_A^{\ell m}$ behaves as an axial vector, and $X_{AB}^{\ell m}$ behaves as an axial tensor. They are said to have odd parity.

The associated variables $h_a^{\ell m}$ and $h_2^{\ell m}$ make up the odd-parity sector of the perturbation, also known as axial perturbations.

Coordinate conditions

Two perturbations, $p_{\alpha\beta}$ and $p'_{\alpha\beta}$, are equivalent when they are related by **gauge transformation**

$$p'_{\alpha\beta} - p_{\alpha\beta} = \pounds_{\Xi}\, g_{\alpha\beta} = \nabla_\alpha \Xi_\beta + \nabla_\beta \Xi_\alpha$$

This freedom can be exploited to impose simplifying conditions on the perturbation.

A very popular choice of coordinate condition is the

Regge-Wheeler gauge

$j_a^{\ell m} = 0 = G^{\ell m}$ (even parity), $h_2^{\ell m} = 0$ (odd parity)

Other choices of gauge can be formulated.

Perturbation equations

Decoupling: The perturbation equations for a given ℓm mode decouple from all other modes, and the even-parity sector also decouples from the odd-parity sector.

Perturbed Einstein tensor

$$\dot{G}_{ab} = \sum_{\ell m} Q^{\ell m}_{ab}\, Y^{\ell m}$$

$$\dot{G}_{aB} = \sum_{\ell m} Q^{\ell m}_{a}\, Y^{\ell m}_{A} + \sum_{\ell m} P^{\ell m}_{a}\, X^{\ell m}_{A}$$

$$\dot{G}_{AB} = \sum_{\ell m} \Big[Q^{\ell m}_{\flat}\, \Omega_{AB} Y^{\ell m} + Q^{\ell m}_{\sharp}\, Y^{\ell m}_{AB} \Big] + \sum_{\ell m} P^{\ell m}\, X^{\ell m}_{AB}$$

Explicit expressions for $Q^{\ell m}_{ab}, Q^{\ell m}_{a}, Q^{\ell}_{\flat}, Q^{\ell m}_{\sharp}$ and $P^{\ell m}_{a}, P^{\ell m}$ are provided in the Practicum document.

Master equation (even-parity)

The perturbation equations can be manipulated in the form of two master equations for two master variables.

Zerilli equation

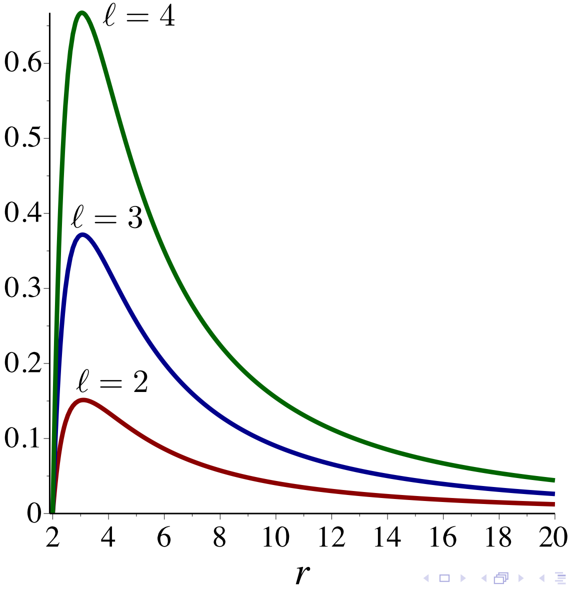
$$\left[-\frac{\partial^2}{\partial t^2} + f\frac{\partial}{\partial r} f\frac{\partial}{\partial r} - V_{\rm even}(r)\right]\Psi_{\rm even} = S_{\rm even}\left[T^{\alpha\beta}\right]$$

$$\Psi_{\rm even} = \frac{2r}{\ell(\ell+1)}\left[K + \frac{2f}{k}\left(f h_{rr} - r\partial_r K\right)\right]$$

$$V_{\rm even} = \frac{f}{k^2}\left[\left[(\ell-1)(\ell+2)\right]^2\left(\frac{\ell(\ell+1)}{r^2} + \frac{6M}{r^3}\right) + \frac{36M^2}{r^4}\left((\ell-1)(\ell+2) + \frac{2M}{r}\right)\right]$$

$$f = 1 - 2M/r, \qquad k = (\ell-1)(\ell+2) + 6M/r$$

Zerilli potential



Master equation (odd parity)

Regge-Wheeler equation

$$\left[-\frac{\partial^2}{\partial t^2} + f\frac{\partial}{\partial r}f\frac{\partial}{\partial r} - V_{\rm odd}(r)\right]\Psi_{\rm odd} = S_{\rm odd}\left[T^{\alpha\beta}\right]$$

$$\Psi_{\rm odd} = \frac{2r}{(\ell-1)(\ell+2)}\left(\partial_r h_t - \partial_t h_r - \frac{2}{r}h_t\right)$$

$$V_{\rm odd} = f\left[\frac{\ell(\ell+1)}{r^2} + \frac{6M}{r^3}\right]$$

In principle, all perturbation variables can be reconstructed from the master functions, $\Psi_{\rm even}$ and $\Psi_{\rm odd}$.

Gravitational waves

Gravitational-wave polarizations

$$h_+(u) = \frac{1}{r}\sum_{\ell m}\left\{\Psi^{\ell m}_{\rm even}(u, r=\infty)\left[\frac{\partial^2}{\partial\theta^2} + \frac{1}{2}\ell(\ell+1)\right]Y^{\ell m} - \Psi^{\ell m}_{\rm odd}(u, r=\infty)\,\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta} - \frac{\cos\theta}{\sin\theta}\right]\frac{\partial}{\partial\phi}Y^{\ell m}\right\}$$

$$h_\times(u) = \frac{1}{r}\sum_{\ell m}\left\{\Psi^{\ell m}_{\rm odd}(u, r=\infty)\left[\frac{\partial^2}{\partial\theta^2} + \frac{1}{2}\ell(\ell+1)\right]Y^{\ell m} + \Psi^{\ell m}_{\rm even}(u, r=\infty)\,\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta} - \frac{\cos\theta}{\sin\theta}\right]\frac{\partial}{\partial\phi}Y^{\ell m}\right\}$$

$$u = t - r^* = t - r - 2M\ln(r/2M - 1) = \text{retarded time}$$

The master functions are therefore especially convenient in the context of calculating gravitational waves.

Numerical integration of the master equations

[C.O. Lousto & R.H. Price, Phys. Rev. D **55**, 2124 (1997); K. Martel & E. Poisson, Phys. Rev. D **66**, 084001 (2002); R. Haas, Phys. Rev. D **75**, 124011 (2007)]

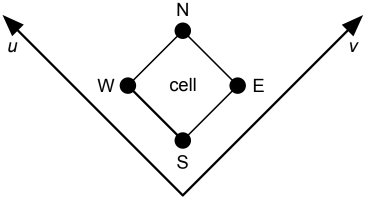
Introduce null coordinates $u = t - r^*$, $v = t + r^*$ with $r^* = \int f^{-1}\, dr = r + 2M \ln(r/2M - 1)$.

Introduce a uniform discretized grid in u and v.

Then integrate the master equation

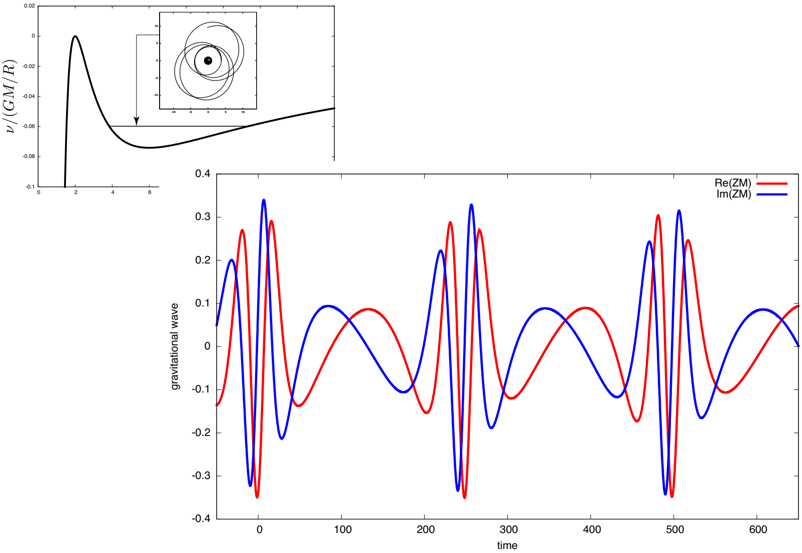
$$\left[-4\frac{\partial^2}{\partial u \partial v} - V(r)\right]\Psi = S$$

over each grid cell, to obtain



$$-4\big[\Psi(N) - \Psi(E) - \Psi(W) + \Psi(S)\big] - \int_{\text{cell}} V\Psi\, du dv = \int_{\text{cell}} S\, du dv$$

Gravitational waves from a particle around a black hole



Analytical integration of the master equations

Fourier transform: $\Psi(t,r) = \int \tilde{\Psi}(\omega,r)e^{-i\omega t}\,d\omega$. Then the master equation becomes an ordinary differential equation.

This can be integrated in terms of a Green's function constructed from solutions to the homogeneous equation.

The homogeneous solutions can be obtained with the **MST method**, in which $\tilde{\Psi}$ is exanded in a series of hypergeometric functions for small r, and in Coulomb wave functions for large r.

[S. Mano, H. Susuki, E. Takasugi, Prog. Theor. Phys. **96**, 549 (1996); **95**, 1079 (1996); **97**, 213 (1997)]

The method works best when $M\omega$ is small and can be used as an (additional) expansion parameter.

R. Fujita calculated the gravitational waves from a particle in circular orbit around a black hole to order $(v/c)^{44}$ beyond leading order.

[R. Fujita, Prog. Theor. Phys. **128**, 971 (2012)]

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Perturbations of the Kerr spacetime

[S.A. Teukolsky, Astrophys. J. **185**, 635 (1973); W.H. Press & S.A. Teukolsky **185**, 649 (1975); **193**, 443 (1974)]

In principle one could linearize the Einstein field equations about the Kerr metric, and write down a set of perturbation equations.

This approach produces equations that don't decouple and don't allow a separation of variables.

Teukolsky found a way to overcome these problems by working with curvature variables instead of the metric (Newman-Penrose formalism).

Kerr spacetime

Kerr metric

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}\, dt d\phi + \frac{\rho^2}{\Delta}\, dr^2 + \rho^2\, d\theta^2 + \frac{\Sigma}{\rho^2}\sin^2\theta\, d\phi^2$$

$$\rho^2 = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - 2Mr + a^2$$

$$\Sigma = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta$$

Null tetrad

$$l^\alpha = \frac{1}{\Delta}\big[r^2 + a^2, \Delta, 0, a\big], \qquad n^\alpha = \frac{1}{2\rho^2}\big[r^2 + a^2, -\Delta, 0, a\big]$$

$$m^\alpha = \frac{1}{\sqrt{2}(r + ia\cos\theta)}\big[ia\sin\theta, 0, 1, i/\sin\theta\big]$$

Curvature variables

Newman-Penrose curvature scalars

$$\psi_0 = -C_{\alpha\mu\beta\nu}\, l^\alpha m^\mu l^\beta m^\nu, \qquad \psi_1 = -C_{\alpha\mu\beta\nu}\, l^\alpha n^\mu l^\beta m^\nu$$

$$\psi_2 = -C_{\alpha\mu\beta\nu}\, l^\alpha m^\mu \bar{m}^\beta n^\nu, \qquad \psi_3 = -C_{\alpha\mu\beta\nu}\, l^\alpha n^\mu \bar{m}^\beta n^\nu$$

$$\psi_4 = -C_{\alpha\mu\beta\nu}\, n^\alpha \bar{m}^\mu n^\beta \bar{m}^\nu$$

For the unperturbed Kerr spacetime, only $\psi_2 \neq 0$.

For a perturbed Kerr spacetime, Teukolsky obtained **decoupled and separable equations** for ψ_0 and ψ_4.

These variables can be related to each other, so only ψ_4 is required in applications.

ψ_4 is analogous to the master functions of a perturbed Schwarzschild spacetime, and its value at $r = \infty$ is simply related to the gravitational-wave polarizations.

Integration of the Teukolsky equation

The Teukolksy equation can be integrated numerically.

[S. Drasco & S.A. Hughes, Phys. Rev. D **73**, 024027 (2006);

A. Zenginoglu & G. Khanna, Phys. Rev. X **1**, 021017 (2011)]

Or it can be integrated analytically using the MST method.

[R. Fujita, Prog. Theor. Exp. Phys. 033E1 (2015)]

There exists a method to reconstruct a metric perturbation from the Teukolsky variable.

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Outline

- Introduction and motivation
- Perturbation theory in general relativity
- Perturbations of a Schwarzschild black hole
- Perturbations of a Kerr black hole
- **Practicum**