Newtonian and relativistic Love numbers

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Setting and assumptions

We consider a self-gravitating body of mass M and radius R; the body is spherical when unperturbed.

The body is placed in a tidal potential created by remote bodies; the length scale of variation is assumed to be long compared with R.

The tidal potential is expressed as a Taylor expansion in powers of x^a, the position relative to the body's centre of mass,

$$U_{\rm tidal} = -\sum_{\ell=2}^{\infty} \frac{1}{\ell!} \mathcal{E}_{a_1 a_2 \cdots a_\ell}(t)\, x^{a_1} x^{a_2} \cdots x^{a_\ell}$$

$$\mathcal{E}_{a_1 a_2 \cdots a_\ell}(t) = -\partial_{a_1 a_2 \cdots a_\ell} U_{\rm remote}(t, 0)$$

The tidal potential varies on an external time scale $t_{\rm ext} \sim \sqrt{r'^3/GM}$, where r' is the inter-body separation, and the time scale for hydrodynamical processes inside the body is $t_{\rm int} \sim \sqrt{R^3/GM}$.

We assume that $t_{\rm ext} \gg t_{\rm int}$, and that the tidal field is weak.

We wish to calculate the tidal deformation of the body.

Governing equations

Poisson's equation: $\nabla^2 U = -4\pi G\rho$

Euler's equation: $\rho \dfrac{dv_a}{dt} = -\partial_a p + \rho\,\partial_a U$

Continuity equation: $\partial_t \rho + \partial_a(\rho v^a) = 0$

Equation of state: $p = p(\rho)$

Unperturbed body (hydrostatic equilibrium)

$$\nabla^2 U = -4\pi G\rho, \qquad \partial_a p = \rho\,\partial_a U$$

Perturbed body (hydrostatic equilibrium)

$$U \to U + \delta U + U_{\rm tidal}, \qquad \rho \to \rho + \delta\rho, \qquad p \to p + \delta p$$

$$\nabla^2 \delta U = -4\pi G\delta\rho, \qquad \partial_a \delta p = \delta\rho\,\partial_a U + \rho\,\partial_a(\delta U + U_{\rm tidal})$$

Expansion in spherical harmonics

Because $\mathcal{E}_{a_1 a_2 \cdots a_\ell}$ is a symmetric, tracefree tensor, each term in the tidal potential is a solid harmonic,

$$\mathcal{E}_{a_1 a_2 \cdots a_\ell} \, x^{a_1} x^{a_2} \cdots x^{a_\ell} = \sum_m \mathcal{E}^m_\ell \, r^\ell Y^m_\ell(\theta, \phi)$$

So expand all other variables in spherical harmonics,

$$\delta U = \sum_{\ell m} U^m_\ell(r) \, Y^m_\ell(\theta, \phi)$$

$$\delta \rho = \sum_{\ell m} \rho^m_\ell(r) \, Y^m_\ell(\theta, \phi)$$

$$\delta p = \sum_{\ell m} p^m_\ell(r) \, Y^m_\ell(\theta, \phi)$$

Reduced equations

The perturbed Poisson equation becomes

$$r^2 \frac{d^2 U_\ell^m}{dr^2} + 2r \frac{dU_\ell^m}{dr} - \ell(\ell+1) U_\ell^m = -4\pi G r^2 \rho_\ell^m$$

The equation of perturbed hydrostatic equilibrium yields

$$p_\ell^m = \rho(U_\ell^m + r^\ell \mathcal{E}_\ell^m)$$

The equation of state implies that $\delta p = (dp/d\rho)\delta\rho$, or

$$\rho_\ell^m = \frac{d\rho}{dp} p_\ell^m$$

The end result is a decoupled ODE for U_ℓ^m.

Love numbers

The internal solution for U_ℓ^m is matched to the external solution

$$U_\ell^m = \frac{4\pi G}{2\ell+1}\frac{I_\ell^m}{r^{\ell+1}}, \qquad I_\ell^m = \int (\rho + \delta\rho)\, r^\ell\, Y_\ell^m(\theta,\phi)\, dV$$

This determines the relationship between the multipole moments I_ℓ^m and the tidal moments $\mathcal{E}_\ell^m$,

$$GI_\ell^m = \frac{2\ell+1}{2\pi\ell!}\, k_\ell\, R^{2\ell+1}\, \mathcal{E}_\ell^m$$

where k_ℓ are the gravitational Love numbers.

Tensorial relation between multipole and tidal moments

$$GI_{a_1 a_2 \cdots a_\ell} = -\frac{2}{(2\ell-1)!!}\, k_\ell\, R^{2\ell+1}\, \mathcal{E}_{a_1 a_2 \cdots a_\ell}$$

$$I^{a_1 a_2 \cdots a_\ell} = \int (\rho + \delta\rho)\, x^{\langle a_1} x^{a_2} \cdots x^{a_\ell\rangle}\, dV$$

Quadrupole deformation

For $\ell = 2$,

$$GI_{ab} = -\frac{2}{3} k_2 R^5 \mathcal{E}_{ab}$$

$$I^{ab} = \int (\rho + \delta\rho) \left(x^a x^b - \tfrac{1}{3} r^2 \delta^{ab}\right) dV$$

$$\mathcal{E}_{ab} = -\partial_{ab} U_{\rm remote}$$

External gravitational potential

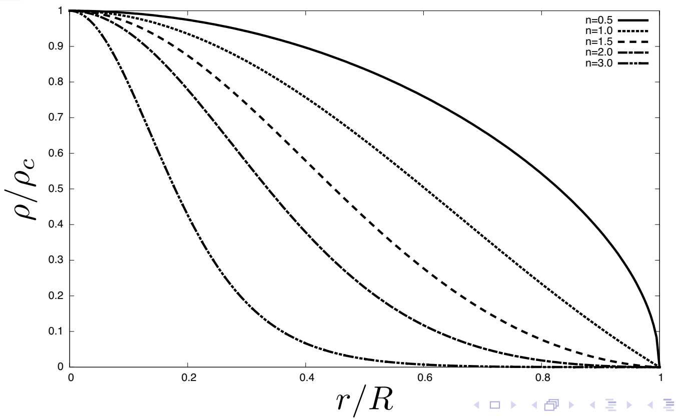
$$U = \frac{GM}{r} - \frac{1}{2}\left[1 + 2k_2 (R/r)^5\right] \mathcal{E}_{ab} x^a x^b + \cdots$$

Polytropes: $p = K\rho^{1+1/n}$

Quadrupole Love number

n	$1/2$	1	$3/2$	2	3
k_2	0.449	0.260	0.143	0.0739	0.0144

The Love number is larger for stiffer equations of state.



Tides in general relativity

The relativistic setting is the same as in the Newtonian discussion.

We continue to consider a self-gravitating body of mass M and radius R immersed in a tidal environment created by remote bodies.

We continue to assume that the tidal field is weak, and varies on long spatial and time scales.

The tidally deformed star continues to be in hydrostatic equilibrium.

But there are now two types of tidal fields: gravitoelectric and gravitomagnetic.

$$\mathcal{E}_{\alpha\beta} = u^\mu u^\nu C_{\alpha\mu\beta\nu} \quad \Longrightarrow \quad \mathcal{E}_{a_1 a_2 \cdots a_\ell} \quad \Longrightarrow \quad \mathcal{E}^m_\ell$$

$$\mathcal{B}_{\alpha\beta} = \frac{1}{2} u^\mu u^\nu \epsilon_{\mu\alpha\gamma\delta} C^{\gamma\delta}{}_{\beta\nu} \quad \Longrightarrow \quad \mathcal{B}_{a_1 a_2 \cdots a_\ell} \quad \Longrightarrow \quad \mathcal{B}^m_\ell$$

Governing equations

Einstein's equation: $G_{\alpha\beta} = 8\pi T_{\alpha\beta}$

Euler's equation: $(\mu + p)u^\beta \nabla_\beta u^\alpha = -\big(g^{\alpha\beta} + u^\alpha u^\beta\big)\nabla_\beta p$

Continuity equation: $\nabla_\alpha(\rho u^\alpha) = 0$

Equations of state: $p = p(\rho), \qquad \mu = \mu(\rho)$

Unperturbed body

The body's structure is determined by the TOV equations.

Perturbed body

$$g_{\alpha\beta} \to g_{\alpha\beta} + \delta g_{\alpha\beta}, \qquad \rho \to \rho + \delta\rho, \qquad u^\alpha \to u^\alpha + \delta u^\alpha$$

A gauge condition is imposed on the metric perturbation.

All variables are expanded in (tensorial) spherical harmonics.

Gravitoelectric tides

The field and fluid equations can be manipulated into a decoupled ODE for $h_\ell^m(r)$, defined by $\delta g_{tt} = \sum_{\ell m} h_\ell^m(r)\, Y_\ell^m(\theta,\phi)$.

The equation is of the same general form as the Newtonian equation,

$$r^2 \frac{d^2 h_\ell^m}{dr^2} + 2rV \frac{dh_\ell^m}{dr} + W h_\ell^m = 0$$

The internal solution is matched to the external solution

$$h_\ell^m = \frac{2}{(\ell-1)} \left[A(r) r^\ell + 2k_\ell^{\rm el}\, R^{2\ell+1} \frac{B(r)}{r^{\ell+1}} \right] \mathcal{E}_\ell^m$$

$A(r) =$ regular at $r = 2M$, tends to 1 at $r = \infty$

$B(r) =$ singular at $r = 2M$, tends to 1 at $r = \infty$

This determines the gravitoelectric Love numbers $k_\ell^{\rm el}$.

Gravitomagnetic tides

The field and fluid equations can be manipulated into a decoupled ODE for $j_\ell^m(r)$, defined by $\delta g_{t\theta} = \sum_{\ell m} j_\ell^m(r)\, \partial_\phi Y_\ell^m(\theta,\phi)/\sin(\theta)$.

The equation is again of the same general form as the Newtonian equation.

The internal solution is matched to the external solution

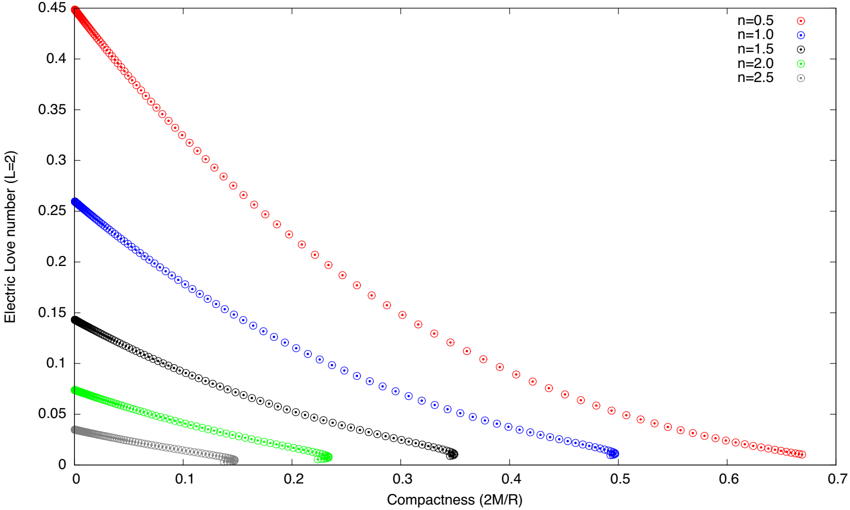
$$j_\ell^m = \frac{2}{3(\ell-1)\ell}\left[C(r)r^\ell - 4\frac{\ell+1}{\ell}k_\ell^{\rm mag}\, MR^{2\ell}\,\frac{D(r)}{r^{\ell+1}}\right]\mathcal{B}_\ell^m$$

$C(r)$ = regular at $r = 2M$, tends to 1 at $r = \infty$

$D(r)$ = singular at $r = 2M$, tends to 1 at $r = \infty$

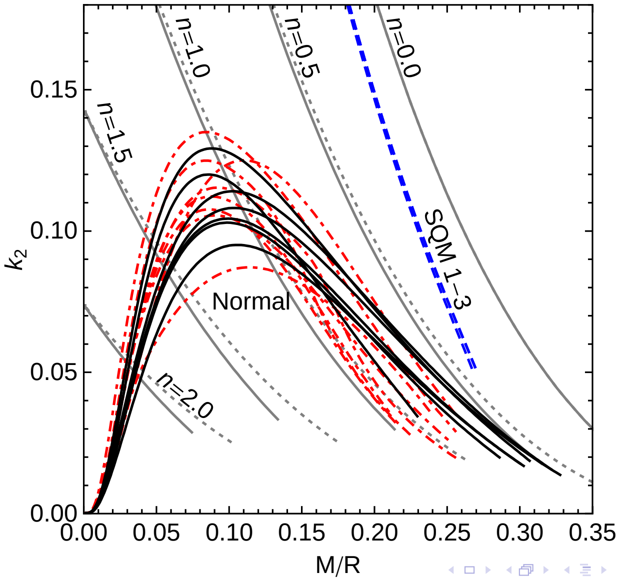
This determines the gravitomagnetic Love numbers $k_\ell^{\rm mag}$.

Gravitoelectric Love number: Polytropes

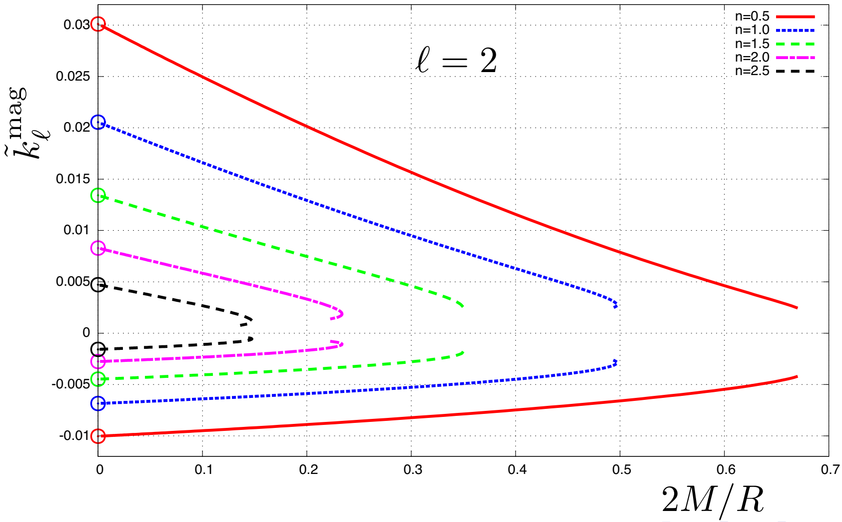


Gravitoelectric Love number: Realistic equations of state

[Hinderer, Lackey, Lang, Read (2010)]



Gravitomagnetic Love number: Polytopes



Black hole

For a black hole, the metric perturbation must be regular at the horizon.

This requires the elimination of the decaying solutions in h_ℓ^m and j_ℓ^m.

Love numbers of a black hole

$$k_\ell^{\rm el} = 0, \qquad k_\ell^{\rm mag} = 0$$

References

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