Exercises

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1. In some relativistic theories of gravity, the "graviton" is not massless as in general relativity, but possesses a mass m_g. In the Newtonian limit, the graviton mass gives rise to a modified Poisson equation of the form

$$\left(\nabla^2 - \frac{1}{\lambda_g^2}\right) U = -4\pi G\rho,$$

in which $\lambda_g = h/(m_g c)$ is the Compton wavelength of the graviton.

   (a) Show that the spherically symmetric potential of a body of mass m is given by $U = (Gm/r)e^{-r/\lambda_g}$.

   (b) Consider a body in orbit around the body of mass m. Assuming that $\lambda_g \gg$ the semimajor axis of the orbit, find the leading non-trivial "disturbing acceleration" in the Keplerian equation of motion in powers of λ_g^{-1}.

   (c) Using the Lagrange planetary equations, calculate the secular variation of the orbit elements a, e, ω, i and Ω induced by this perturbation.

   (d) The table gives orbital elements for various planets, together with the current uncertainties in their measured perihelion advances in arcseconds per century, after accounting for perturbations due to the other planets and general relativity. Using these uncertainties, find the best lower bound on the graviton Compton wavelength λ_g. Compare your result with the lower bound recently reported by the LIGO/Virgo collaboration of $\lambda_g > 1.6 \times 10^{13}$ km. Recall that the orbital period is given by $P = 2\pi(a^3/Gm)^{1/2}$ and that one radian $= 2.06 \times 10^5$ arcseconds.

Table 1: Orbital elements of selected planets. The astronomical unit (AU) is the Earth-Sun distance, equal to 149.60×10^6 km.

Planet	Semi-major axis (AU)	Orbital period (yr)	Eccentricity	Uncertainty in $\dot{\omega}$ (as/ca)
Mercury	0.387099	0.24085	0.205628	1.0×10^{-3}
Venus	0.723332	0.61521	0.006787	1.6×10^{-3}
Earth	1.000000	1.00004	0.016722	1.9×10^{-4}
Mars	1.523691	1.88089	0.093377	3.7×10^{-5}
Jupiter	5.202803	11.86223	0.04845	2.8×10^{-2}
Saturn	9.53884	29.4577	0.05565	4.7×10^{-4}

2. The radiation-reaction contribution to the relative acceleration in a two-body system is given by

$$\boldsymbol{a}[\mathrm{rr}] = \frac{8}{5}\eta\frac{(Gm)^2}{c^5 r^3}\left[\left(3v^2 + \frac{17}{3}\frac{Gm}{r}\right)\dot{r}\,\boldsymbol{n} - \left(v^2 + 3\frac{Gm}{r}\right)\boldsymbol{v}\right].$$

Show that the energy and angular-momentum losses that result from this balance the fluxes emitted to infinity, given by

$$\frac{dE}{dt} = \frac{8}{15}\eta^2\frac{c^3}{G}\left(\frac{Gm}{c^2 r}\right)^4\left(12v^2 - 11\dot{r}^2\right),$$
$$\frac{dJ^j}{dt} = \frac{8}{5}h^j\eta^2\frac{c}{G}\left(\frac{Gm}{c^2 r}\right)^3\left(2v^2 - 3\dot{r}^2 + 2\frac{Gm}{r}\right),$$

where $\dot{r} = dr/dt$ and $\boldsymbol{h} = \boldsymbol{x} \times \boldsymbol{v}$.

3. Consider a Keplerian orbit that is circular apart from the slow decrease in radius a caused by the energy lost to gravitational radiation. The energy loss rate is given by

$$\frac{dE}{dt} = -\frac{32}{5}\eta^2\frac{c^5}{G}\left(\frac{Gm}{c^2 a}\right)^5, \tag{1}$$

where $\eta = m_1 m_2/m$, and $E = -G\eta m^2/2a$. As a function of η, m, and the initial orbital period P_0, calculate the lifetime of the binary system before radiation reaction brings the semimajor axis to zero. Using this result, carry out the following estimates:

(a) The remaining lifetime of the Hulse-Taylor binary pulsar PSR 1913+16, with $M_1 \approx M_2 \approx 1.4 M_\odot$ and $P_0 = 7.75$ hours (assume that the orbit is circular). Compare this with the 300 Myr mentioned in the lecture. Explain any difference.

(b) The total time in the gravitational-wave signal from an inspiralling binary system of two $1.4 M_\odot$ compact objects, from the time it enters the LIGO-Virgo frequency band with a gravitational-wave frequency of 10 Hz to the end of the inspiral (when $a = 0$).

(c) The time between when an insipiralling black-hole binary system of 25 $M_\odot$ each leaves the LISA band at 0.1 Hz and enters the LIGO band at 10 Hz. Assume a circular orbit.

(d) the remaining lifetime of the Earth-Sun system.