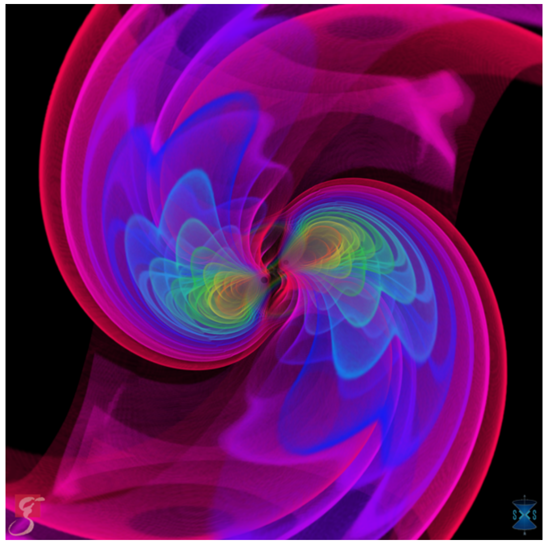

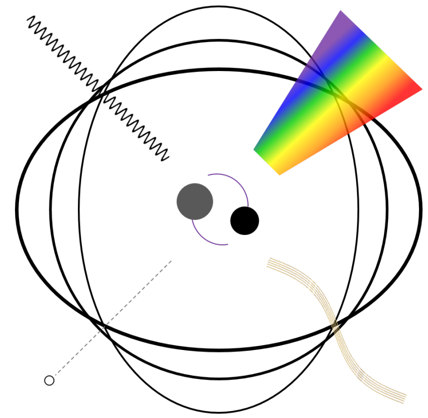

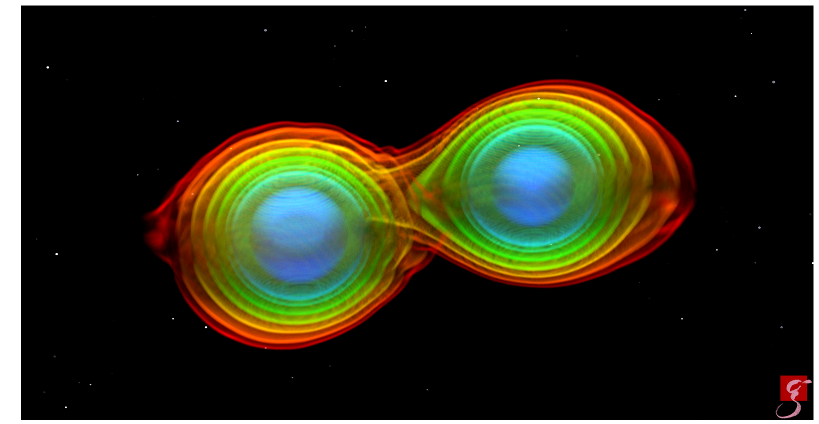

Effective-One-Body Theory & Applications to Gravitational-Wave Observations

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Max Planck Institute for Gravitational Physics

(Albert Einstein Institute)

Department of Physics, University of Maryland

"Dublin School on Gravitational-Wave School Modeling", Dublin **June 11-12, 2018**

Outline

- **Lectures I-II:** Basics of gravitational waves

- **Lectures III-IV:** Motivations and development of effective-one-body (EOB) theory (two-body dynamics and waveforms)

- **Lecture V:** Using waveform models to infer astrophysical and cosmological information of gravitational-wave observations

- **Lecture VI:** Using waveform models to probe dynamical gravity and extreme matter with gravitational-wave observations

(NR simulation: Ossokine, AB & SXS @AEI)

(visualization credit: Benger @ Airborne Hydro Mapping Software & Haas @AEI)

References:

- **M. Maggiore's books**: "Gravitational Waves Volume I: Theory and Experiments" (2007) & "Gravitational Waves Volume II: Astrophysics and Cosmology" (2018).

- **E. Poisson & C. Will's book**: "Gravity" (2015).

- **E.E. Flanagan & S.A. Hughes' review**: arXiv:0501041.

- **AB's Les Houches School Proceedings**: arXiv:0709.4682.

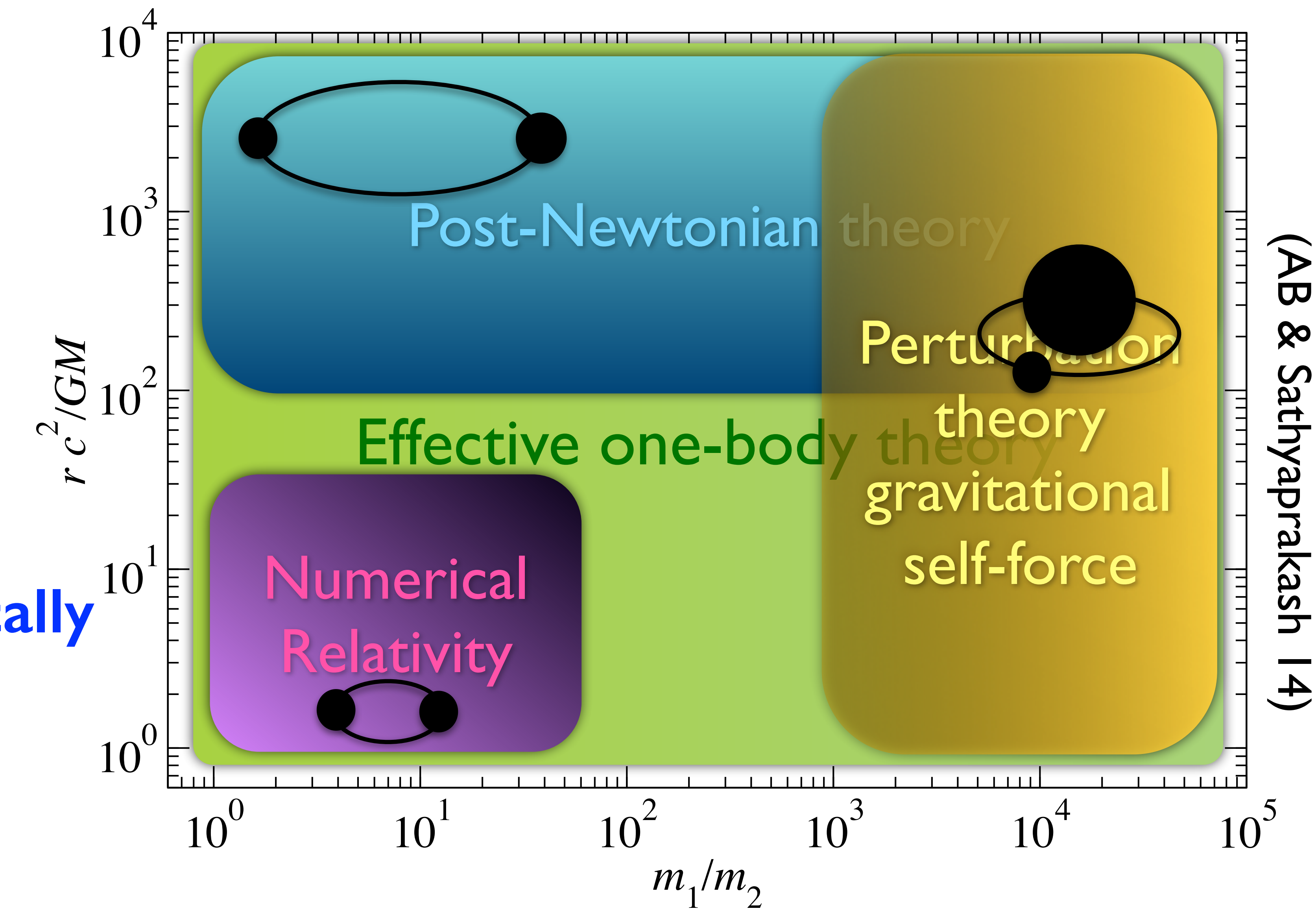
- **AB & B. Sathyaprakash's review**: arXiv:1410.7832.

- **UMD/AEI graduate course on GW Physics & Astrophysics** taught in Winter-Spring 2017: http://www.aei.mpg.de/2000472.

Solving two-body problem in General Relativity (including radiation)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

$v^2/c^2 \sim GM/rc^2$



- **GR** is **non-linear theory.**
- Einstein's field equations can be solved:
  - **approximately,** but **analytically** (**fast** way)
  - **exactly,** but **numerically** on supercomputers (**slow** way)
- **Synergy** between **analytical** and **numerical relativity** is **crucial.**
- Physical (EOBNR) and phenomenological (Phenom) **inspiral-merger-ringdown waveforms.**

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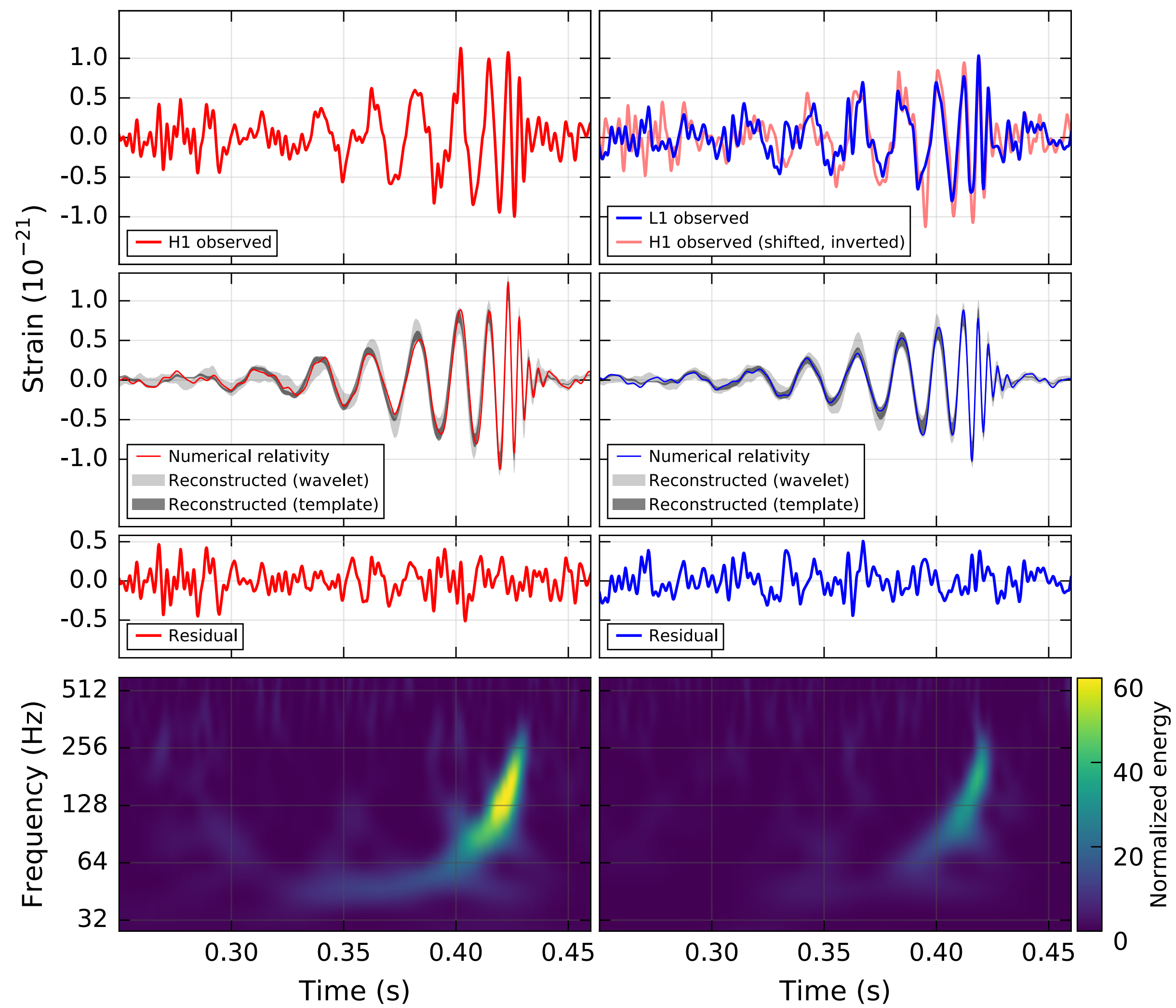
(Abbott et al. PRL 116 (2016) 061102)

- **GW151226**: SNR=23, 10 cycles (from 30 Hz), **0.2 sec.**



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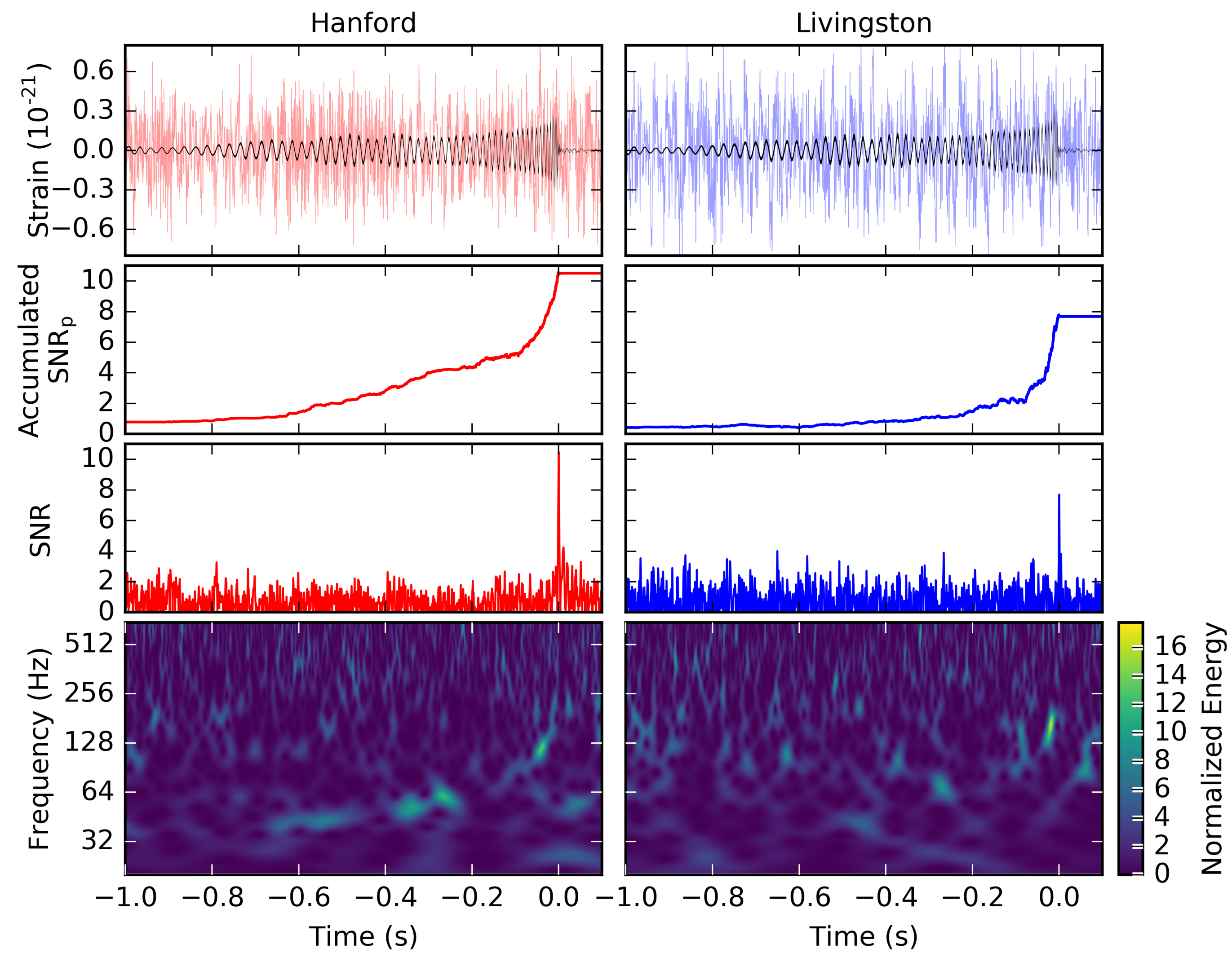
(Abbott et al. PRL 116 (2016) 241103)

- **GW151226**: SNR=13, 55 cycles (from 35 Hz), **1 sec.**



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Solving two-body problem in General Relativity (including radiation)

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(Abbott et al. PRL 119 (2017) 161101)

- **GW170817**: SNR=32, 3000 cycles (from 30 Hz), one **minute.**

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- Einstein's field equations can be solved:
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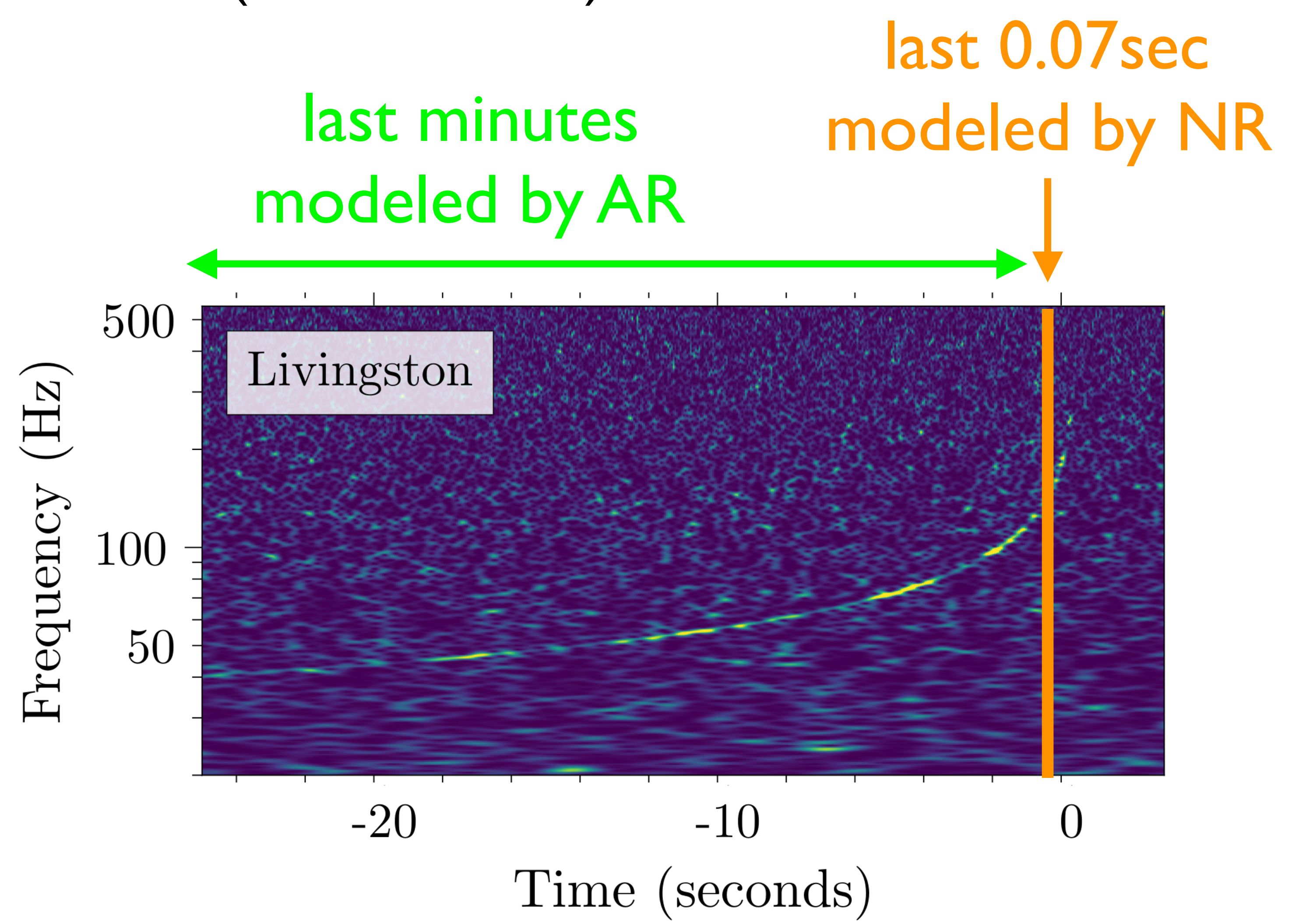

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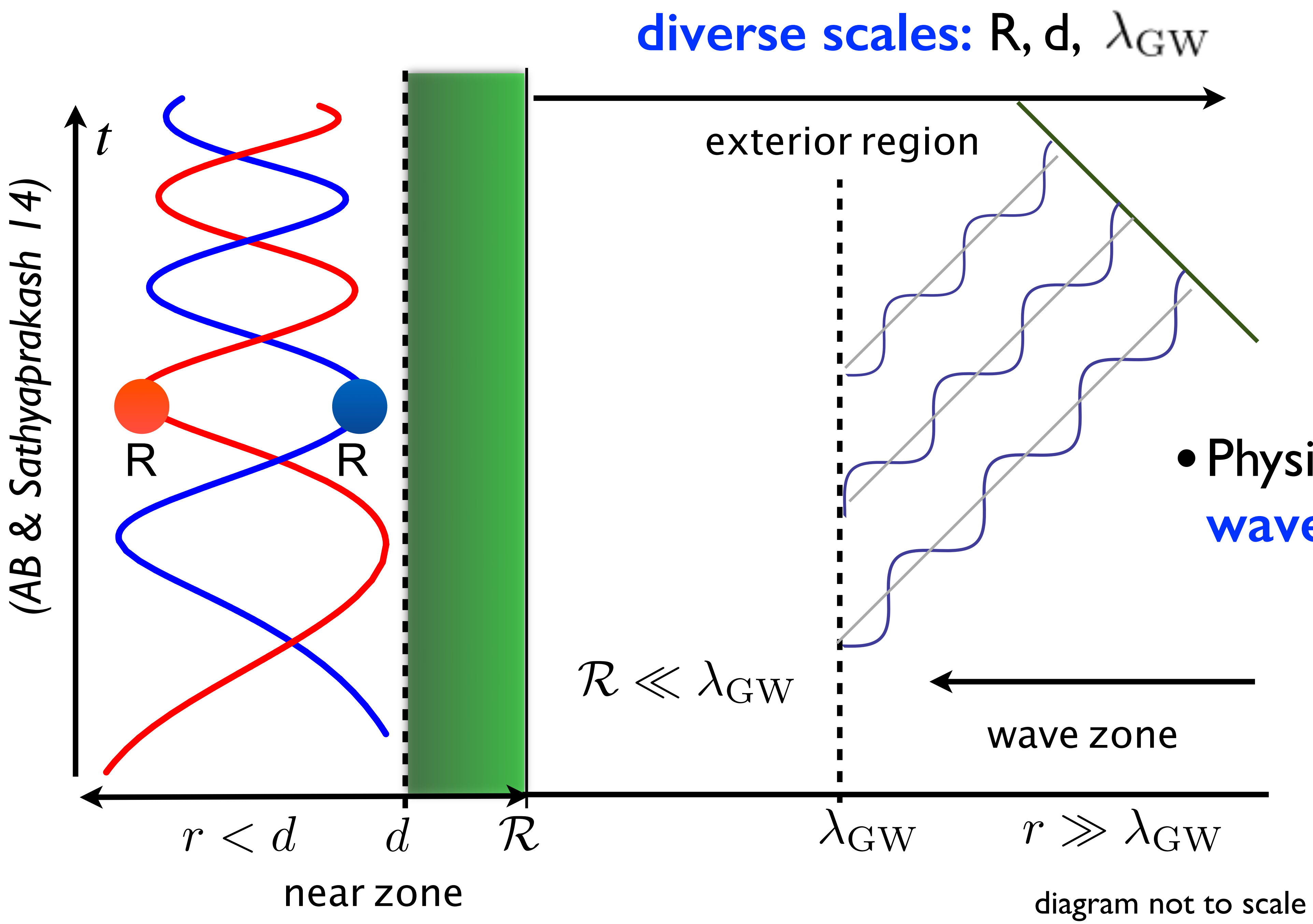
Post-Newtonian/post-Minkowskian formalism/effective field theory



- Physical **observables are waveforms** at null infinity

- **Equations of motion** of compact objects.
- **Generation** problem and **radiation-reaction** problem.

Post-Newtonian/post-Minkowskian formalism

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{8\pi G}{c^4} T_{\alpha\beta}$$

Einstein equations can be **recast in convenient form** introducing:

$$h^{\alpha\beta} = \sqrt{-g}\, g^{\alpha\beta} - \eta^{\alpha\beta}$$

in weak-field limit it coincides with reverse-trace tensor

depends on non-linear terms in $g_{\mu\nu}$ and $h_{\mu\nu}$, & derivatives

imposing $\partial_\beta h^{\alpha\beta} = 0 \;\Rightarrow\; \Box h^{\alpha\beta} = \frac{16\pi G}{c^4} |g| T^{\alpha\beta} + \Lambda^{\alpha\beta} \equiv \frac{16\pi G}{c^4} \tau^{\alpha\beta}$

no-incoming radiation boundary conditions

$$h^{\alpha\beta}(t, \boldsymbol{r}) = \frac{16\pi G}{c^4} \Box^{-1}_{\rm ret} \tau^{\alpha\beta} = -\frac{4G}{c^4} \int \frac{\tau^{\alpha\beta}(t - |\boldsymbol{r} - \boldsymbol{r}'|/c, \boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} \mathrm{d}^3 r'.$$

Post-Newtonian/post-Minkowskian formalism

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}\, R = \frac{8\pi G}{c^4}\, T_{\alpha\beta}$$

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at leading order in G

$$h^{\alpha\beta}(t, \boldsymbol{r}) = -\frac{4G}{c^4\, r} \int T^{\alpha\beta}(t - r/c + \boldsymbol{n} \cdot \boldsymbol{r}'/c, \boldsymbol{r}')\mathrm{d}^3 r'$$

$$r \gg d$$

Far-field quadrupole formula

$$h^{\alpha\beta}(t,\boldsymbol{r}) = -\frac{4G}{c^4\,r}\int T^{\alpha\beta}(t - r/c + \boldsymbol{n}\cdot\boldsymbol{r}'/c, \boldsymbol{r}')\mathrm{d}^3 r'$$

Assuming slow motion, and using conservation law of matter energy-momentum tensor

$$\partial_\alpha T^{\alpha\beta} = 0 \qquad\qquad r \gg d, \quad r \gg \lambda_{\mathrm{GW}}, \quad \lambda_{\mathrm{GW}} \gg d$$

In suitable radiative coordinates, in TT gauge:

$$\boxed{h_{ij}^{\mathrm{TT}} = \frac{2G}{c^4 R}\sum_{k,l}\mathcal{P}_{ijkl}(\boldsymbol{N})\left[\frac{d^2}{dT^2}Q_{kl}\left(T - \frac{R}{c}\right) + \mathcal{O}\left(\frac{1}{c}\right)\right] + \mathcal{O}\left(\frac{1}{R^2}\right)}$$

Mass-quadrupole moment: $Q_{ij}(t) = \int_{\mathrm{source}} d^3\boldsymbol{x}'\,\rho(t,\boldsymbol{x}')\left(x_i'\,x_j' - \frac{1}{3}\delta_{ij}\,x'^2\right)$

$$\mathcal{P}_{ijkl} = \mathcal{P}_{ik}\,\mathcal{P}_{jl} - \mathcal{P}_{ij}\,\mathcal{P}_{kl}/2 \qquad \mathcal{P}_{ij} = \delta_{ij} - N_i\,N_j \qquad \boldsymbol{N} = \boldsymbol{X}/R$$

Einstein quadrupole formula

$$h^{\alpha\beta}(t,\boldsymbol{r}) = -\frac{4G}{c^4\, r}\int T^{\alpha\beta}(t - r/c + \boldsymbol{n}\cdot\boldsymbol{r'}/c, \boldsymbol{r'})\mathrm{d}^3 r'$$

Assuming slow motion, and using conservation law of matter energy-momentum tensor

$$\partial_\alpha T^{\alpha\beta} = 0 \qquad\qquad r \gg d, \quad r \gg \lambda_{\mathrm{GW}}, \quad \lambda_{\mathrm{GW}} \gg d$$

In suitable radiative coordinates, in TT gauge:

$$\boxed{\mathcal{L} = \frac{G}{5c^5}\left[\frac{d^3 Q_{ij}}{dT^3}\frac{d^3 Q_{ij}}{dT^3} + \mathcal{O}\left(\frac{1}{c^2}\right)\right]}$$

Mass-quadrupole moment: $Q_{ij}(t) = \int_{\mathrm{source}} d^3\boldsymbol{x'}\,\rho(t,\boldsymbol{x'})\left(x'_i\, x'_j - \frac{1}{3}\delta_{ij}\, x'^2\right)$

$$\mathcal{P}_{ijkl} = \mathcal{P}_{ik}\mathcal{P}_{jl} - \mathcal{P}_{ij}\mathcal{P}_{kl}/2 \qquad \mathcal{P}_{ij} = \delta_{ij} - N_i N_j \qquad \boldsymbol{N} = \boldsymbol{X}/R$$

Equations of motion/Hamiltonian in post-Newtonian theory

- **Work started in 1917** *(Droste & Lorentz 1917, and Einstein, Infeld & Hoffmann 1938)*

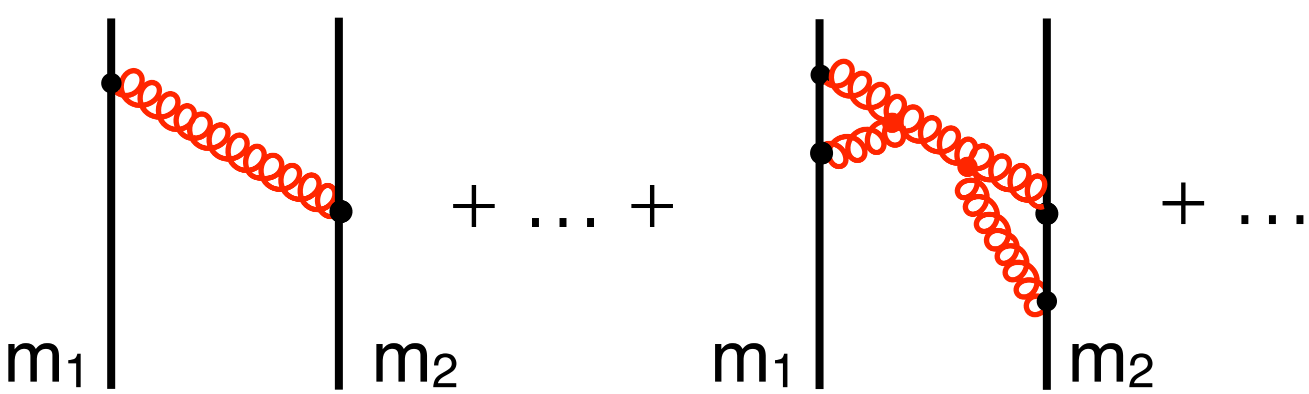
(Blanchet, Damour, Iyer, Faye, Bernard, Bohe', AB, Marsat; Jaranowski, Schaefer, Steinhoff; Will, Wiseman; Flanagan, Hinderer, Vines; Goldberger, Porto, Rothstein; Kol, Levi, Smolkin; Foffa, Sturani; …)

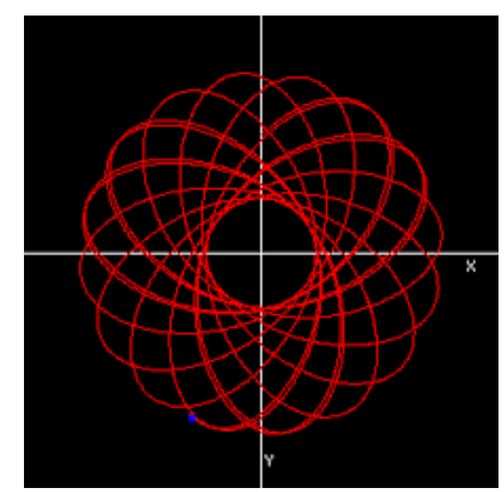
$$\widehat{H}_{\mathrm{N}}\left(\mathbf{r},\mathbf{p}\right) = \frac{\mathbf{p}^2}{2} - \frac{1}{r}$$

Small parameter is **$v/c << 1$, $v^2/c^2 \sim GM/rc^2$**

$$\widehat{H}_{1\mathrm{PN}}\left(\mathbf{r},\mathbf{p}\right) = \frac{1}{8}(3\nu-1)(\mathbf{p}^2)^2 - \frac{1}{2}\left\{(3+\nu)\mathbf{p}^2+\nu(\mathbf{n}\cdot\mathbf{p})^2\right\}\frac{1}{r} + \frac{1}{2r^2}$$

$$\widehat{H}_{2\mathrm{PN}}\left(\mathbf{r},\mathbf{p}\right) = \frac{1}{16}\left(1-5\nu+5\nu^2\right)\left(\mathbf{p}^2\right)^3 + \frac{1}{8}\left\{\left(5-20\nu-3\nu^2\right)\left(\mathbf{p}^2\right)^2 \quad + \ldots\right.$$





- Compact object is **point-like body endowed** with time-dependent **multipole moments**.

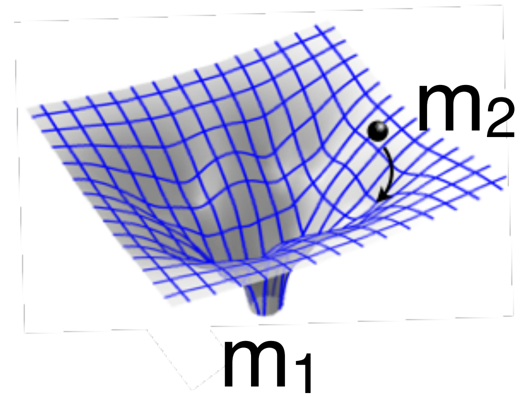
Small mass-ratio expansion/gravitational self-force formalism

- **First works in 50-70s** *(Regge & Wheeler 56, Zerilli 70, Teukolsky 72)*

  Small parameter is **$m_2/m_1 \ll 1$, $v^2/c^2 \sim GM/rc^2 \sim 1$, $M = m_1 + m_2$**

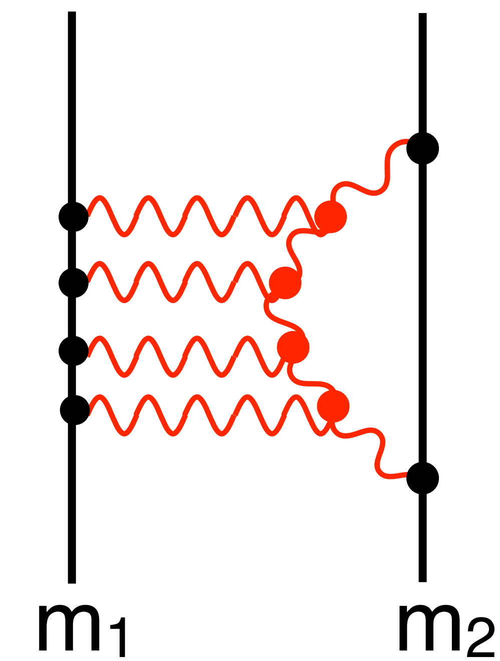
  Equation of gravitational perturbations in black-hole spacetime:

$$\frac{\partial^2\Psi}{\partial t^2} - \frac{\partial^2\Psi}{\partial r_\star^2} + V_{\ell m}\Psi = S_{\ell m}$$



  Green functions in Schwarzschild/Kerr spacetimes.

(Fujita, Poisson, Sasaki, Shibata, Khanna, Hughes, Bernuzzi, Harms, Nagar…)



- Accurate modeling of **relativistic dynamics of large mass-ratio** inspirals **requires** to include **back-reaction effects** due to interaction of small object with its own gravitational perturbation field.

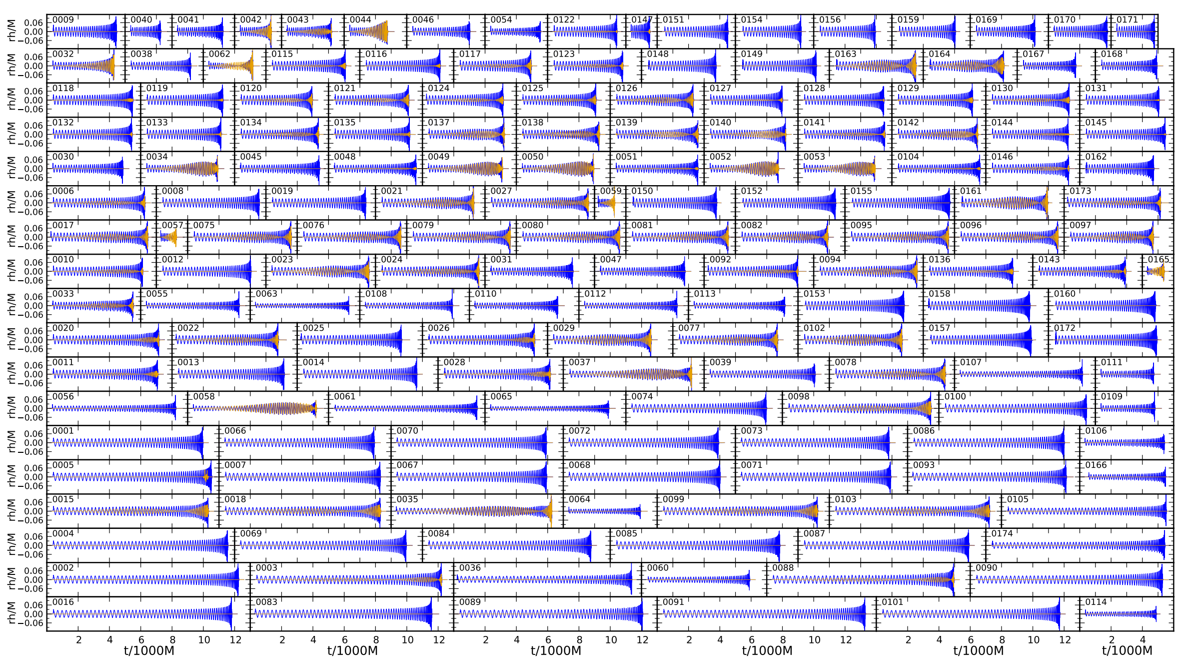
(Deitweiler, Whiting, Mino, Poisson, Quinn, Wald, Sasaki, Tanaka, Barack, Ori, Pound, van de Meent, …)

Numerical Relativity: binary black holes

- **Breakthrough** in 2005 *(Pretorius 05, Campanelli et al. 06, Baker et al. 06)*

(Kidder, Pfeiffer, Scheel, Lindblom, Szilagyi; Bruegmann; Hannam, Husa, Tichy; Laguna, Shoemaker; …)

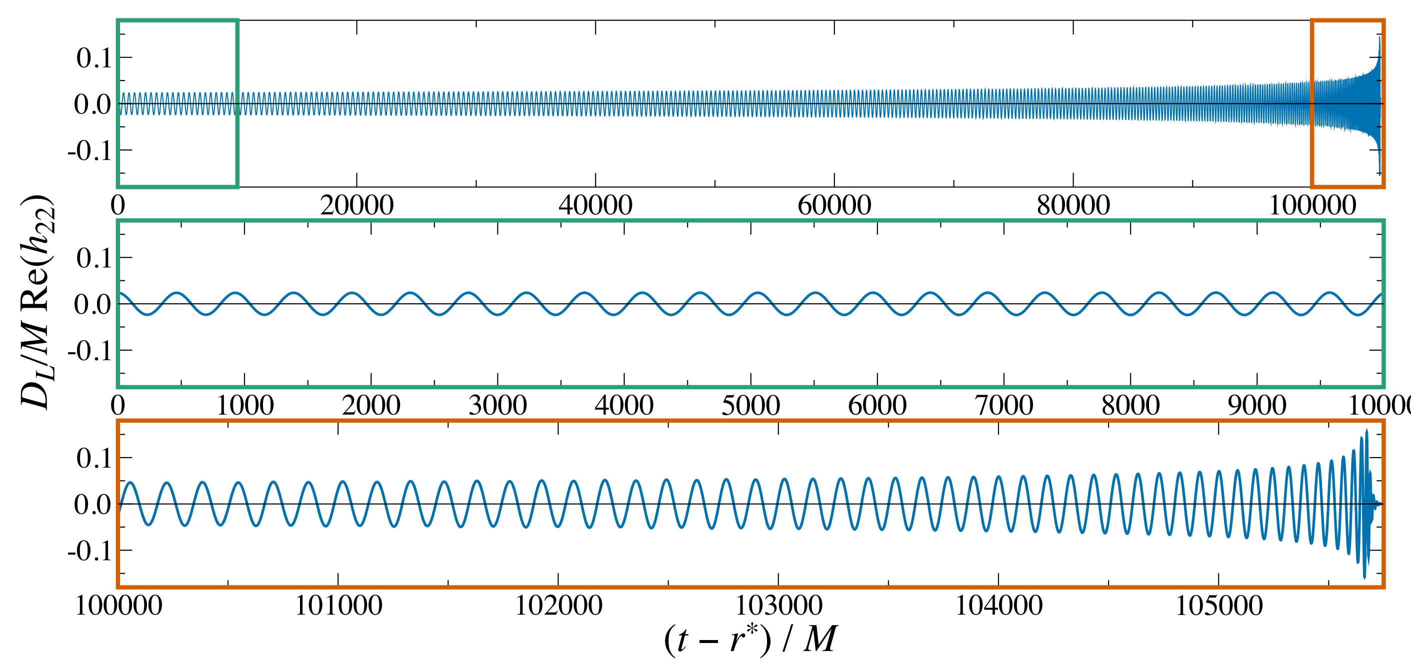
$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$



- **Simulating eXtreme Spacetimes (SXS)** collaboration *(Mroue et al. 13)*

- **376 GW cycles**, zero spins & mass-ratio 7 (8 months, few millions CPU-h)

(Szilagyi, Blackman, AB, Taracchini et al. 15)



- **Numerical-Relativity & Analytical-Relativity** collaboration *(Hinder et al. 13)*

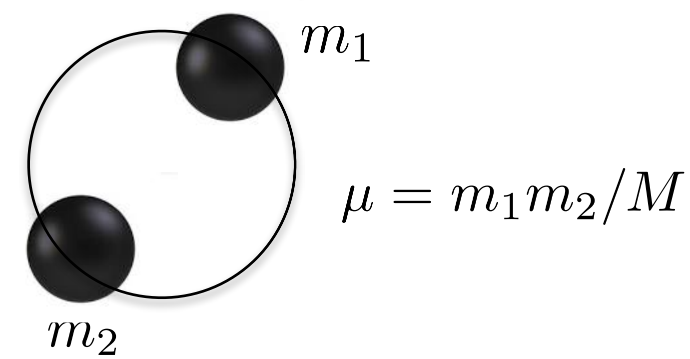
Gravitational waveforms from inspiraling binaries

- GW from time-dependent **quadrupole moment**: $$h_{ij} \sim \frac{G}{c^4}\frac{\ddot{Q}_{ij}}{\mathrm{R}}$$

$$h = \nu \left(\frac{GM}{c^2\,\mathrm{R}}\right)\frac{v^2}{c^2}\cos 2\Phi$$

$$\frac{v}{c} = \left(\frac{GM\omega}{c^3}\right)^{1/3}$$

$$\nu = \mu/M$$



- Center-of-mass energy: $E(\omega)$
- GW luminosity: $\mathcal{L}_{\mathrm{GW}}(\omega) \equiv F(\omega)$

- Balance equation: $$\frac{dE(\omega)}{dt} = -F(\omega) \quad \longrightarrow \quad \dot{\omega}(t) = -\frac{F(\omega)}{dE(\omega)/d\omega}$$

- Gravitational-wave **phase**: $$\Phi_{\mathrm{GW}}(t) = 2\Phi(t) = \frac{1}{\pi}\int^t \omega(t')dt'$$

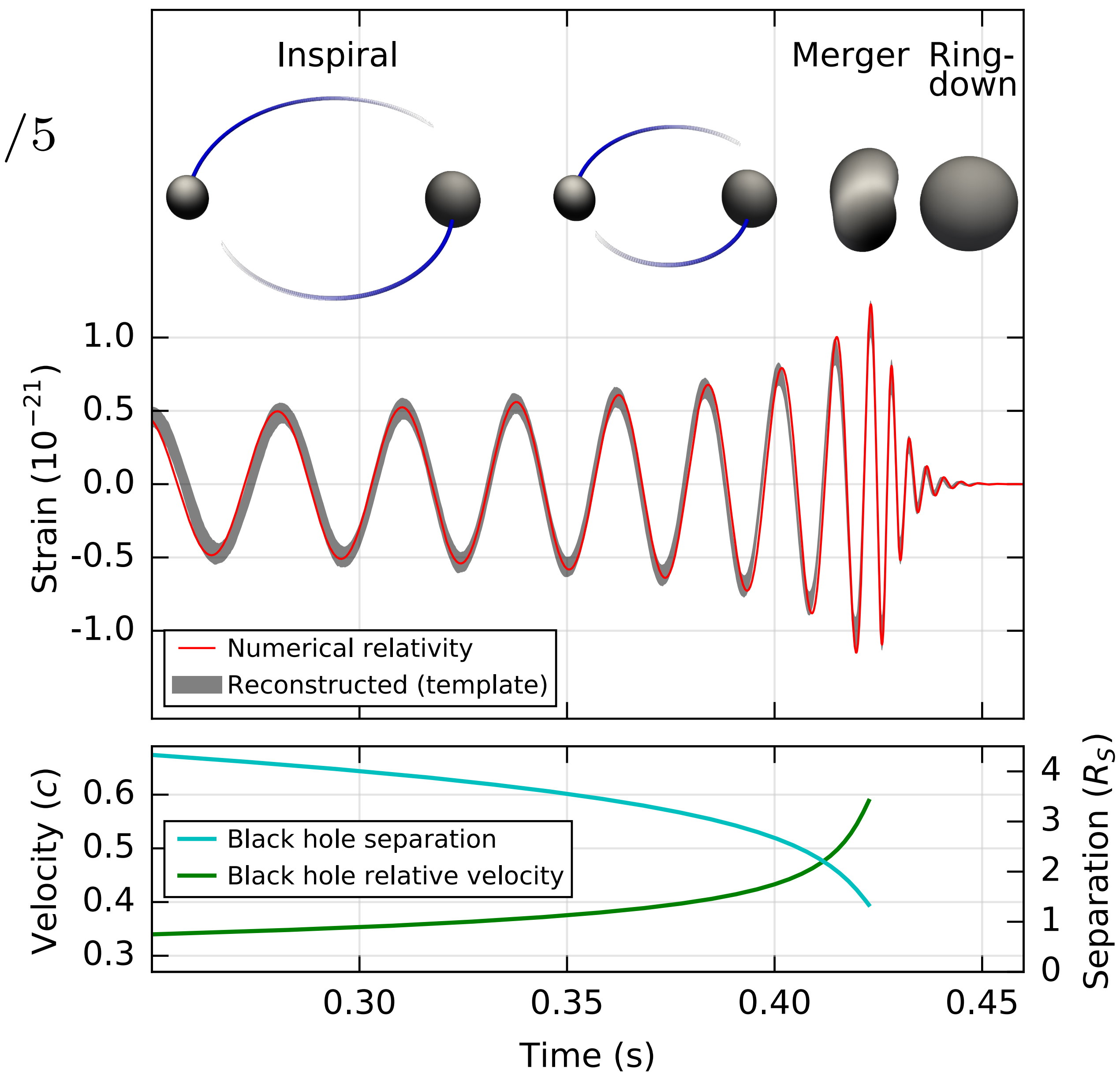
GW150914: binary composed of two compact objects, no neutron star

$$\nu = \frac{\mu}{M} \qquad 0 \leq \nu \leq 1/4$$

(Abbott et al. PRL 116 (2016) 061102)

$$\mu = \frac{m_1 m_2}{M} \qquad M = m_1 + m_2$$

$$\mathcal{M} = \nu^{3/5} M = \left(\frac{5}{96} \pi^{-8/3} f_{\rm GW}^{-11/3} \dot{f}_{\rm GW} \right)^{3/5}$$



- We **measured**:

$$\mathcal{M} \simeq 30 M_\odot \Rightarrow M \geq 70 M_\odot$$

$$f_{\rm GW} \sim 150 \mathrm{Hz}\,, \omega^2 r^2 = \frac{M}{r}$$

$$\Rightarrow r \simeq 350 \mathrm{km}, 2M \sim 210 \mathrm{km}$$

- If **neutron star** were **present**:

$$m_{\rm NS} \sim 2 M_\odot, m_{\rm BH} \sim 1700 M_\odot$$

binary would **merge** at **lower frequencies**!

(see also Abbott et al. Annalen Phys. 529 (2017) 0209)

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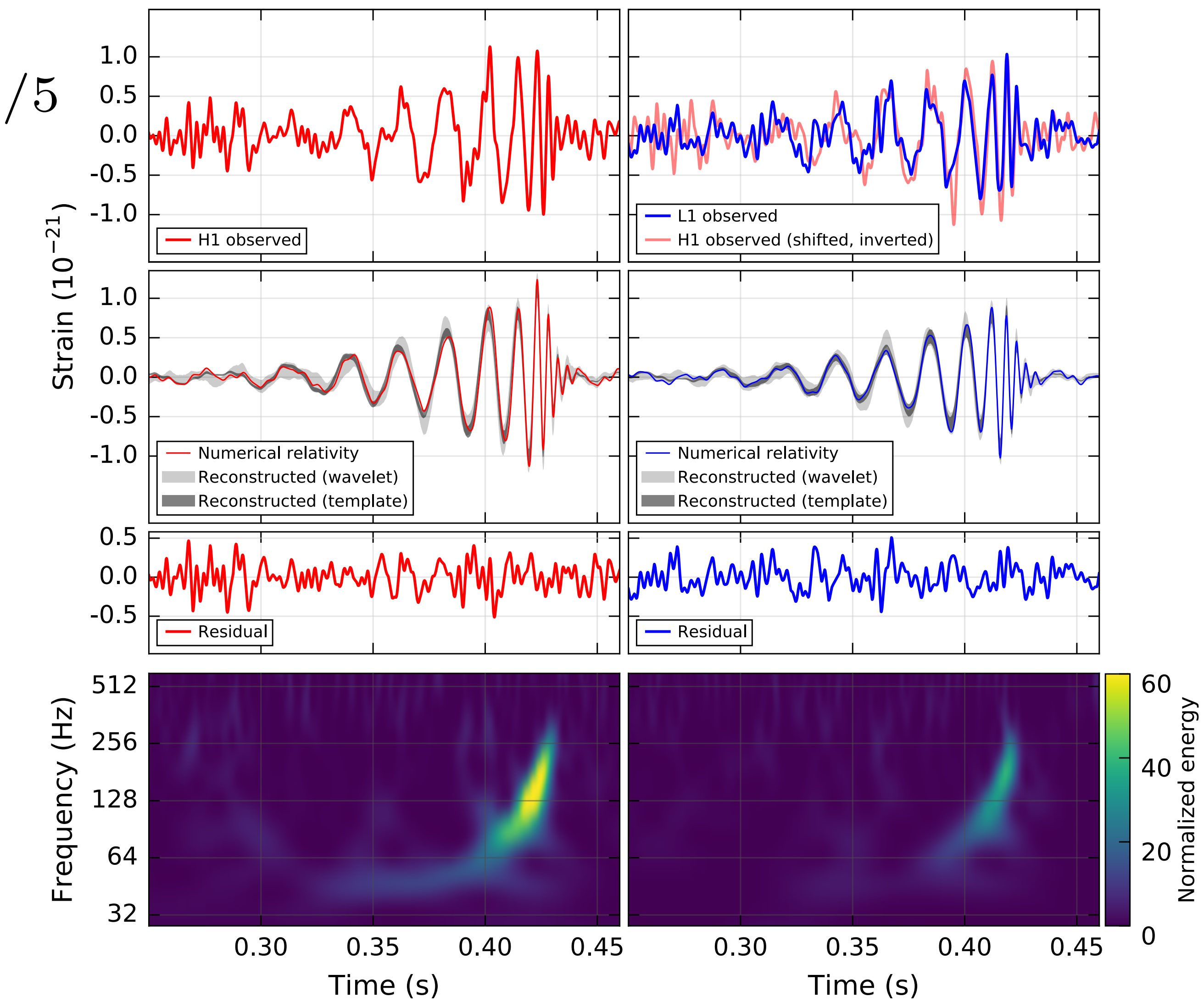

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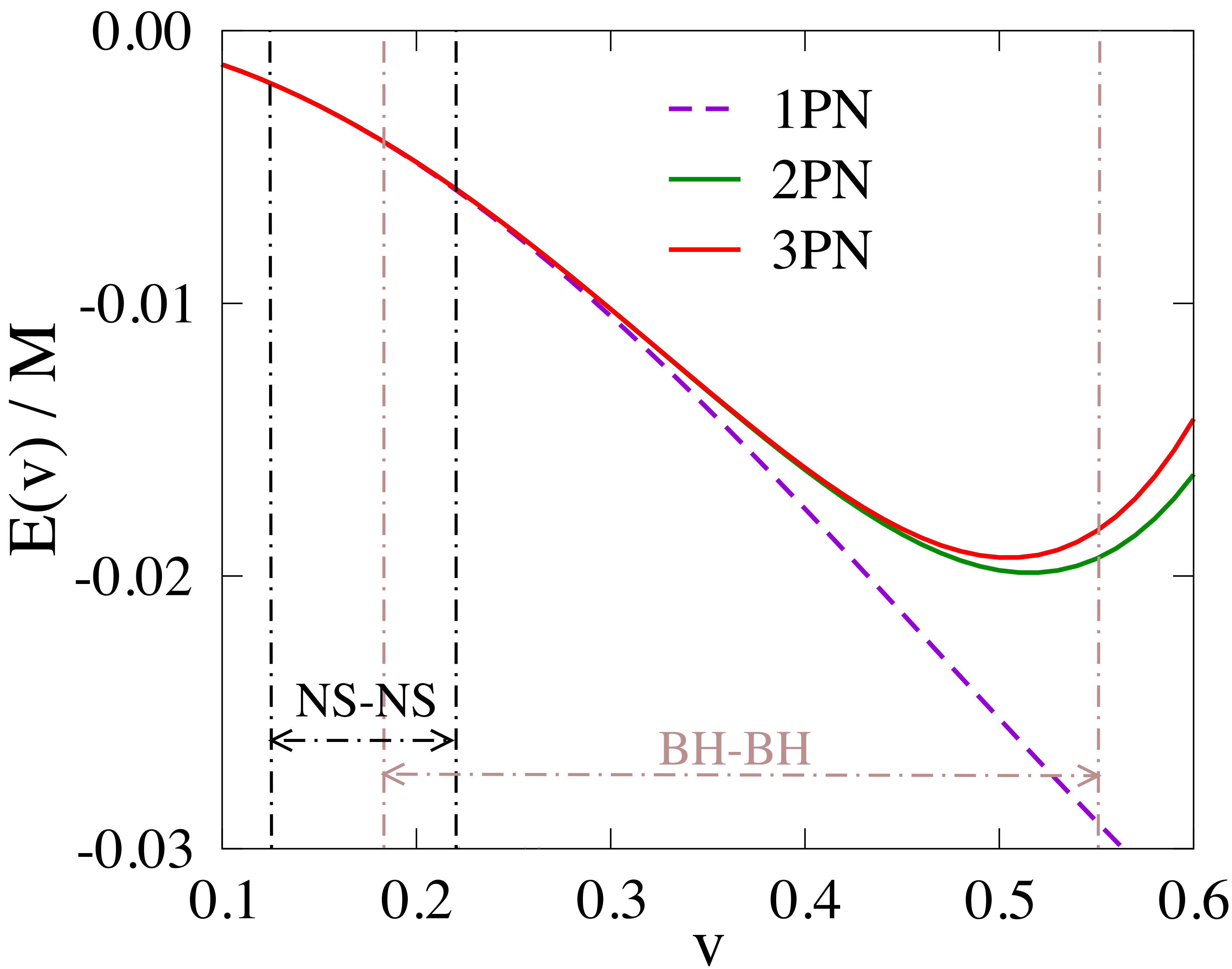
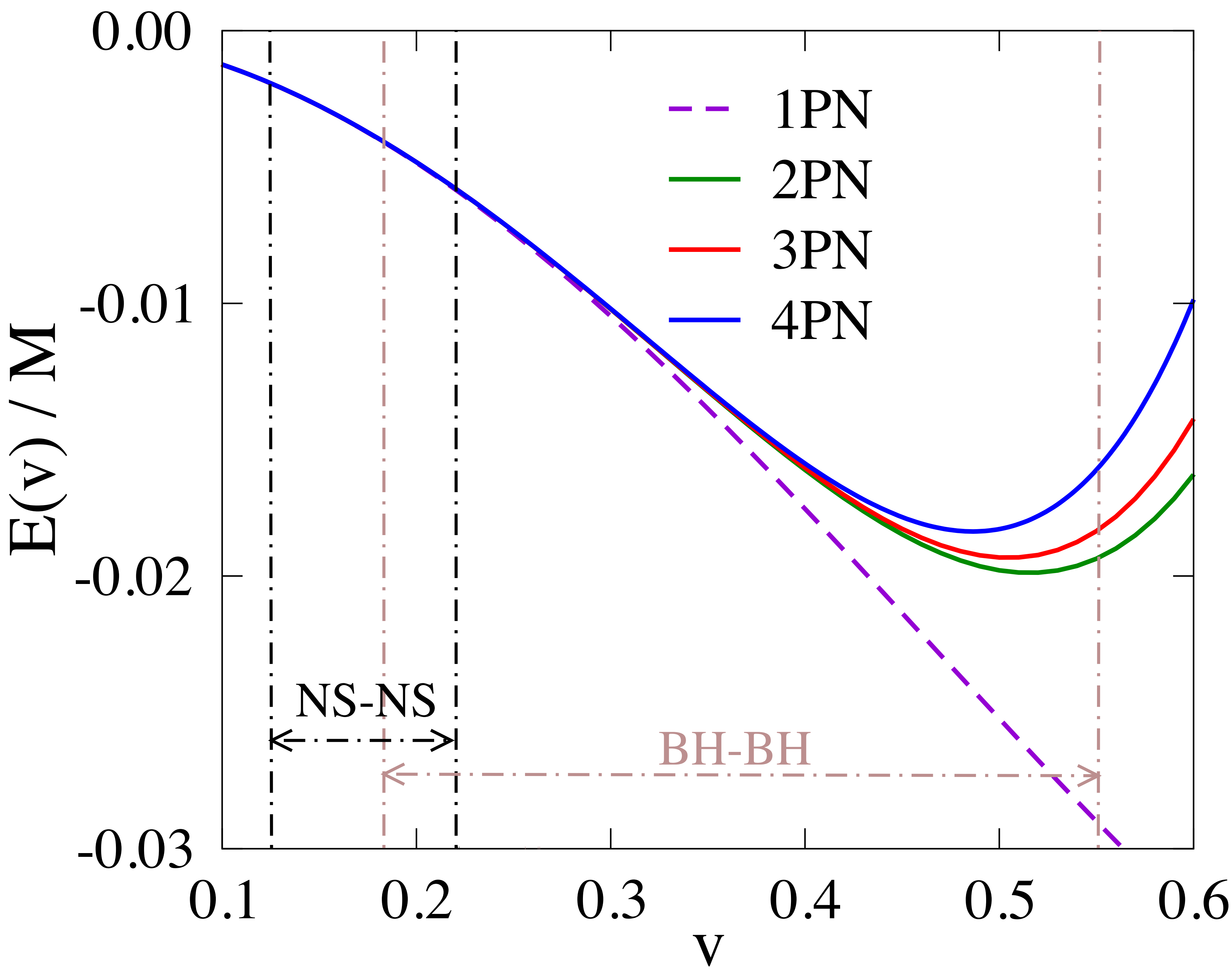
PN (binding) energy versus velocity

- Equal-mass, non-spinning binary



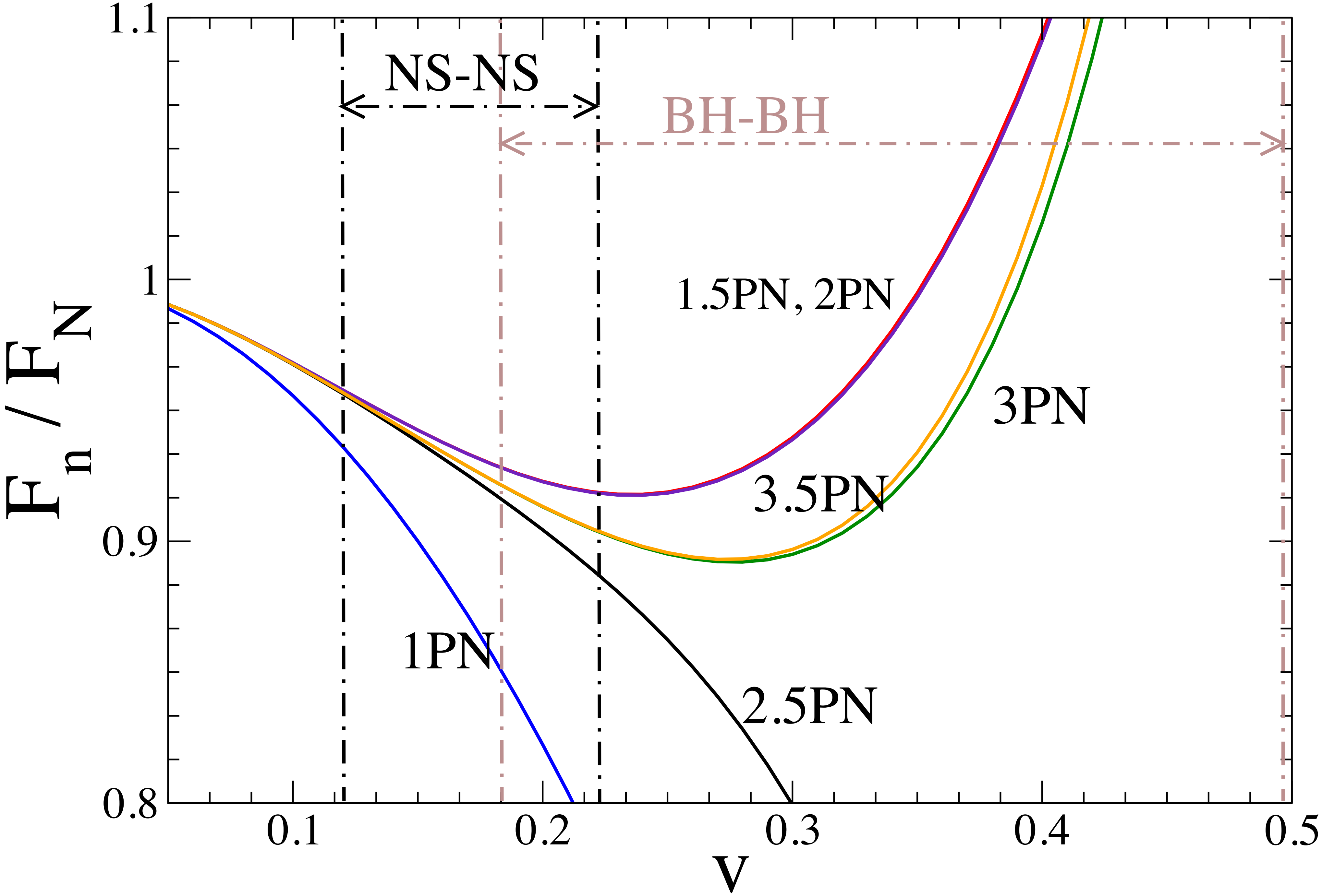
PN (binding) energy versus velocity

- Equal-mass, non-spinning binary (known today through 4PN order)



PN gravitational-wave flux versus velocity

- Equal-mass, non-spinning binary



Number of GW cycles predicted by PN theory

$$M = (1.4 + 1.4) M_\odot$$
$$f_{\rm in} = 40 \text{ Hz};\ f_{\rm fin} = 1570 \text{ Hz}$$
$$\chi = |\mathbf{S}|/m^2$$

$$N_{\rm tot} = \frac{1}{\pi}(\Phi_{\rm max} - \Phi_{\rm min}) = \frac{1}{\pi}\int_{f_{\rm min}}^{f_{\rm max}} df\, \frac{d\Phi(f)}{df} = \int_{f_{\rm min}}^{f_{\rm max}} \frac{df}{f}\, N_{\rm inst}(f)$$

$$N_{\rm inst}(f) = \frac{f^2}{df/dt}, \qquad f = \frac{1}{\pi}\frac{d\Phi}{dt}$$

	Number of cycles	Number of *useful* cycles:
Newtonian:	16034	247.8
1PN:	$+441$	$+24.0$
1.5PN	-211	-20.0
Spin-orbit:	$+65.7\chi_1 + 65.7\chi_2$	$6.2\chi_1 + 6.2\chi_2$
2PN	$+9.9$	$+1.5$
2.5PN	$-11.7 + 9.2\chi_1 + 9.2\chi_2$	$-2.3\ +0.8\chi_1 + 0.8\chi_2$
3PN:	$+2.6$	$+0.6$
3.5PN:	-0.9	-0.2

Number of GW cycles predicted by PN theory

$$N_{\rm useful} = \frac{\int_{f_{\rm min}}^{f_{\rm max}} w(f)\, N_{\rm inst}(f)\, df/f}{\int_{f_{\rm min}}^{f_{\rm max}} w(f)\, df/f}$$

$M = (1.4 + 1.4) M_\odot$

$f_{\rm in} = 40$ Hz; $f_{\rm fin} = 1570$ Hz

$\chi = |\mathbf{S}|/m^2$

$$w(f) = a^2(f)/[f S_n(f)]\,, \quad h(t) = 2a(t)\,\cos 2\Phi(t)$$

(Damour, Iyer & Sathyaprakash 03)

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Number of GW cycles predicted by PN theory

$$N_{\rm useful} = \frac{\int_{f_{\rm min}}^{f_{\rm max}} w(f)\, N_{\rm inst}(f)\, df/f}{\int_{f_{\rm min}}^{f_{\rm max}} w(f)\, df/f}$$

$M = (15 + 15) M_\odot$

$f_{\rm in} = 40$ Hz; $f_{\rm fin} = 147$ Hz

$\chi = |\mathbf{S}|/m^2$

$$w(f) = a^2(f)/[f S_n(f)]\,, \quad h(t) = 2a(t)\, \cos 2\Phi(t)$$

	Number of cycles	**Number of *useful* cycles:**
Newtonian:	302	10.7
1PN:	$+39$	$+4.0$
1.5PN	-37	-6.2
Spin-orbit:	$+11.7\chi_1 + 11.7\chi_2$	$1.9\chi_1 + 1.9\chi_2$
2PN	$+3.3$	$+0.8$
Spin-spin:	$-1.7\chi_1\,\chi_2$	$-0.4\chi_1\,\chi_2$
2.5PN	$-6.2 + 3.6\chi_1 + 3.6\chi_2$	$-2.3\ +0.8\chi_1 + 0.8\chi_2$
3PN:	$+2$	$+1.2$
3.5PN:	-0.8	-0.5

PN approximants for inspiraling waveforms

$$E(\omega) = E_0 v^2 \left[1 + E_{1\mathrm{PN}} v^2 + E_{2\mathrm{PN}} v^4 + \cdots\right]$$

$$F(\omega) = F_0 v^{10} \left[1 + F_{1\mathrm{PN}} v^2 + F_{1.5\mathrm{PN}} v^3 + F_{2\mathrm{PN}} v^4 + \cdots\right]$$

$$\Rightarrow \dot{\omega}(t) = -\frac{F(\omega)}{dE(\omega)/d\omega}$$

- **T4-PN-approximant at 2PN order:**

$$[\dot{\omega}(t)]_{2\mathrm{PN}} = -\mathcal{T}_{2\mathrm{PN}} \left\{ \frac{F(\omega)}{dE(\omega)/d\omega} \right\} \qquad \mathcal{T}_{\mathrm{nPN}} \Rightarrow \text{Taylor exp. up to nPN}$$

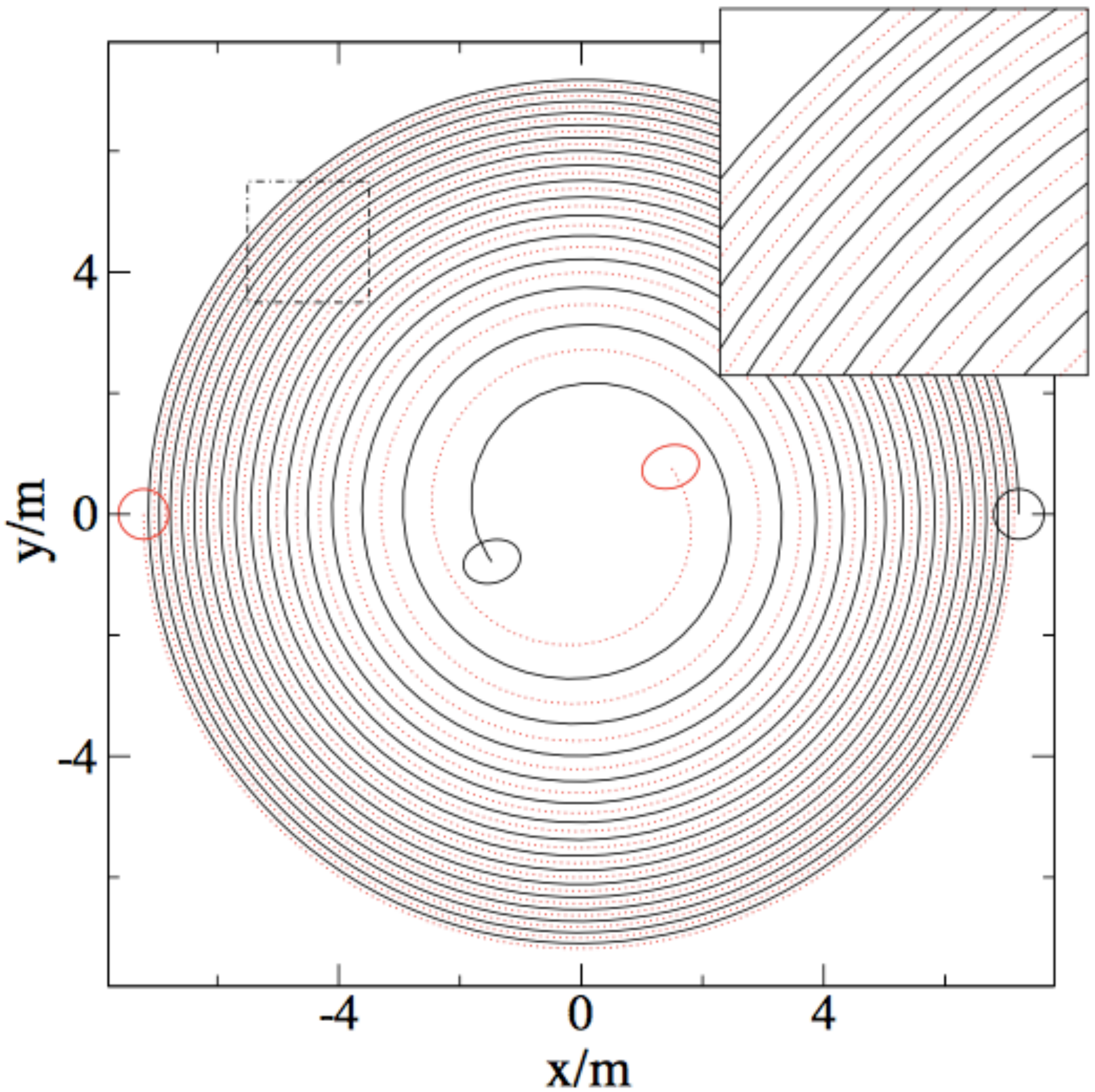
- **T1–PN-approximant at 2PN order:**

$$[\dot{\omega}(t)]_{2\mathrm{PN}} = -\frac{\mathcal{T}_{2\mathrm{PN}}\{F(\omega)\}}{\mathcal{T}_{2\mathrm{PN}}\{dE(\omega)/d\omega\}}$$

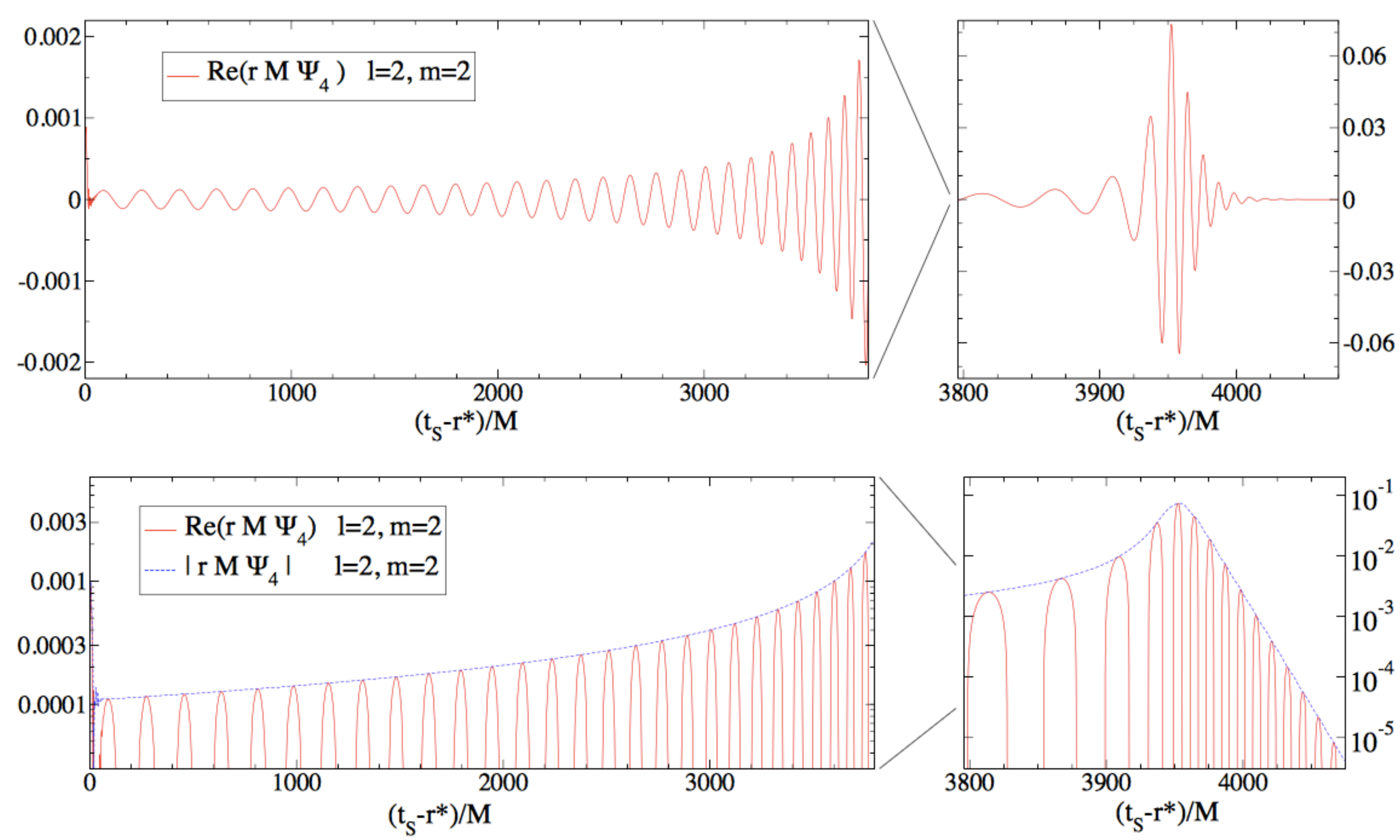
(see, e.g., AB, Iyer, Ochsner, Pan & Sathyaprakash 2011)

Numerical-relativity waveform

(Boyle et al. 2007)



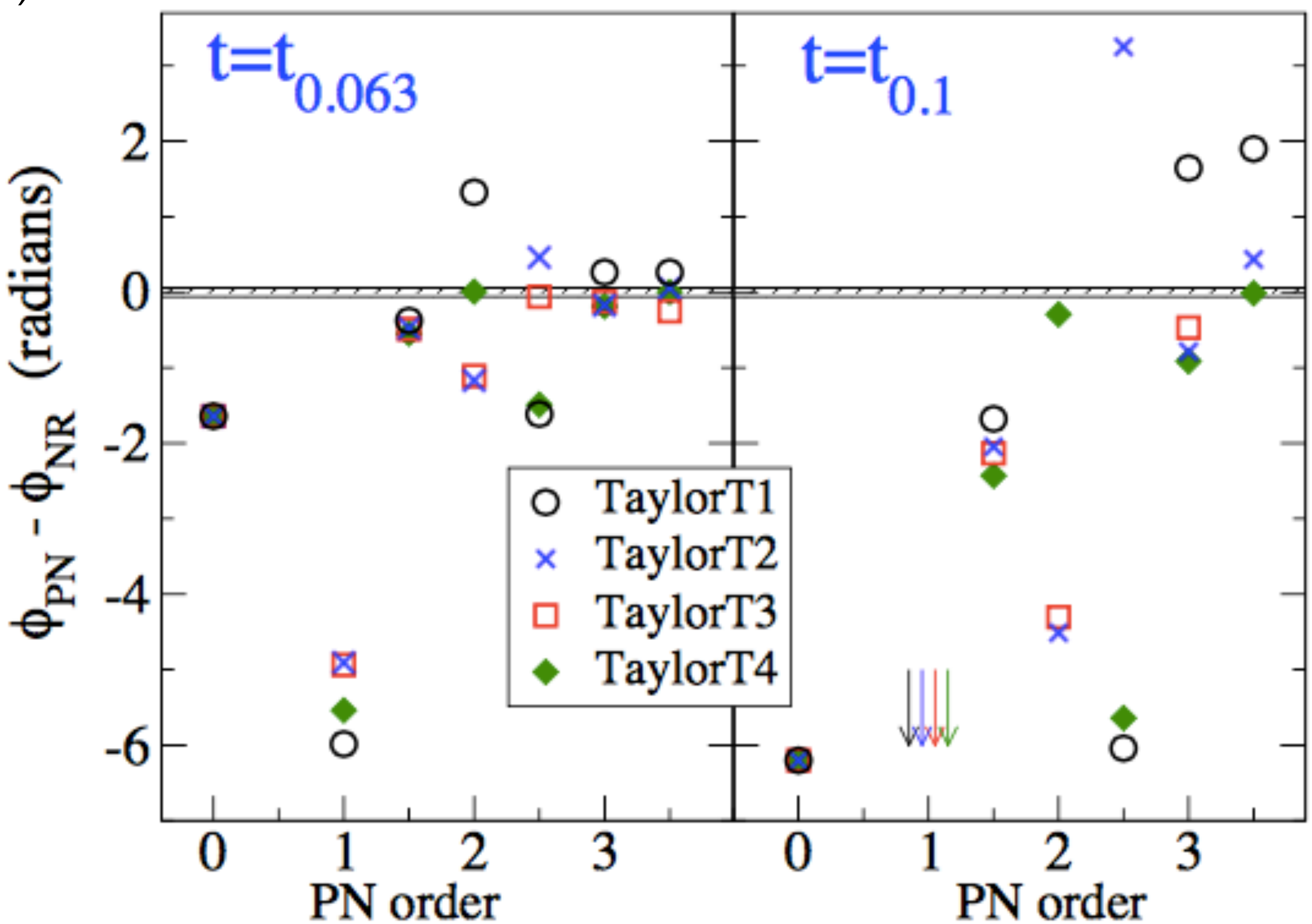
- **Highly-accurate equal-mass, non-spinning** binary waveform



Comparing GW phase in NR and PN

(Boyle et al. 2007)

- Equal-mass, non-spinning binary

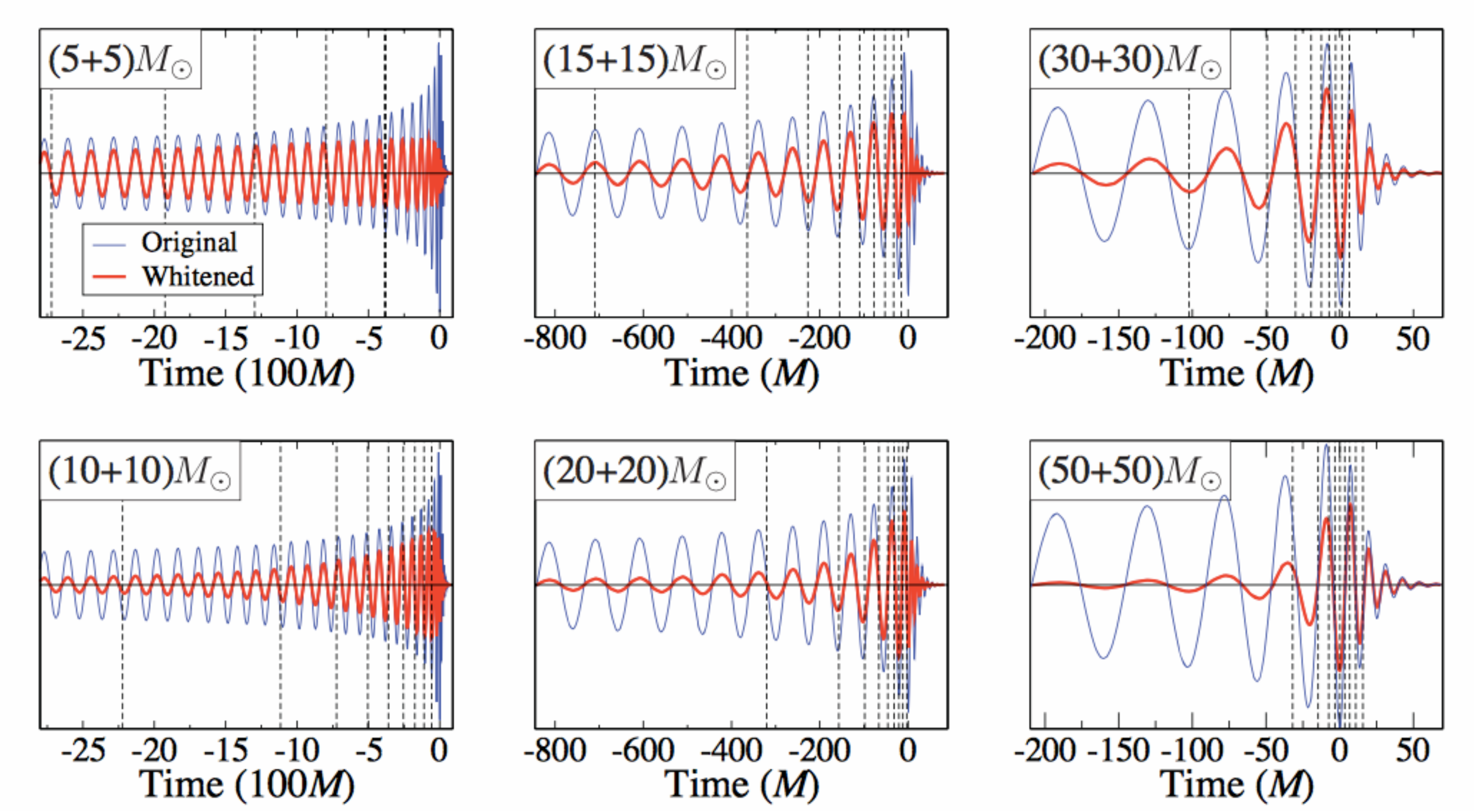


$$M = 20 M_\odot \Rightarrow f^{GW}_{0.063} = 98\,\mathrm{Hz},\, f^{GW}_{0.1} = 161\,\mathrm{Hz},\, f^{GW}_{\mathrm{ISCO}} = 220\,\mathrm{Hz}$$

$$M = 2.8 M_\odot \Rightarrow f^{GW}_{0.063} = 807\,\mathrm{Hz},\, f^{GW}_{0.1} = 1350\,\mathrm{Hz},\, f^{GW}_{\mathrm{ISCO}} = 1570\,\mathrm{Hz}$$

The significance of inspiral, merger & ringdown

(Pan, AB, Baker, Centrella, Kelly, McWilliams & Pretorius 2007)



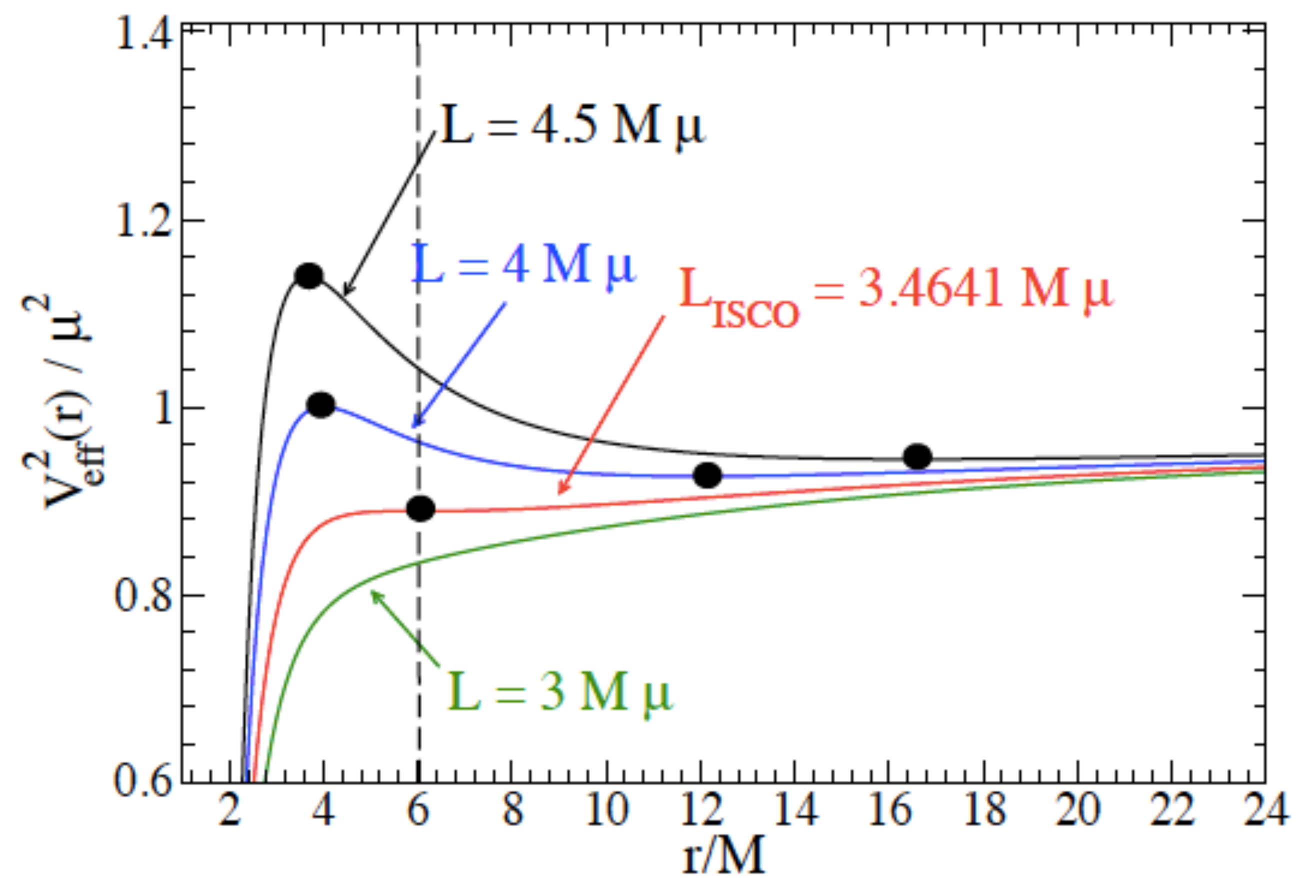
How far in strong-field regime can we push PN approximation?

- **Circular-orbit energy for a test-particle in Schwarzschild:**

$$E_{\rm circ} = \mu \frac{1-2M/r}{\sqrt{1-3M/r}} \qquad E_{\rm circ}^{\rm PN} = \mu - \frac{\mu M}{2r}\left[1 - \frac{3}{4}\frac{M}{r} - \frac{27}{8}\frac{M^2}{r^2} - \frac{675}{64}\frac{M^3}{r^3}\cdots\right]$$

miminum of $E_{\rm circ}$ **gives ISCO** $\Rightarrow r_{\rm ISCO} = 6M$

$$E_{\rm circ}^{\rm 1PN} \Rightarrow r_{\rm ISCO}^{\rm 1PN} = 1.5M$$
$$E_{\rm circ}^{\rm 2PN} \Rightarrow r_{\rm ISCO}^{\rm 2PN} = 4.019M$$
$$E_{\rm circ}^{\rm 3PN} \Rightarrow r_{\rm ISCO}^{\rm 3PN} = 5.104M$$
$$E_{\rm circ}^{\rm 4PN} \Rightarrow r_{\rm ISCO}^{\rm 4PN} = 5.572M$$
$$E_{\rm circ}^{\rm 5PN} \Rightarrow r_{\rm ISCO}^{\rm 5PN} = 5.788M$$
$$E_{\rm circ}^{\rm 6PN} \Rightarrow r_{\rm ISCO}^{\rm 6PN} = 5.892M$$
$$\cdots$$
$$E_{\rm circ}^{\rm 10PN} \Rightarrow r_{\rm ISCO}^{\rm 10PN} = 5.992M$$



Can we get insights on accuracy from PN equations of motion?

(Blanchet et al. 04)

$$
\begin{aligned}
\boldsymbol{a}_1 = & -\frac{G m_2}{r_{12}^2}\boldsymbol{n}_{12} \\
& + \frac{1}{c^2}\left\{\left[\frac{5G^2 m_1 m_2}{r_{12}^3} + \frac{4G^2 m_2^2}{r_{12}^3} + \frac{G m_2}{r_{12}^2}\left(\frac{3}{2}(n_{12}v_2)^2 - v_1^2 + 4(v_1 v_2) - 2v_2^2\right)\right]\boldsymbol{n}_{12}\right. \\
& \qquad \left. + \frac{G m_2}{r_{12}^2}\left(4(n_{12}v_1) - 3(n_{12}v_2)\right)\boldsymbol{v}_{12}\right\} \\
& + \frac{1}{c^4}\left\{\left[-\frac{57G^3 m_1^2 m_2}{4r_{12}^4} - \frac{69G^3 m_1 m_2^2}{2r_{12}^4} - \frac{9G^3 m_2^3}{r_{12}^4}\right.\right. \\
& \qquad + \frac{G m_2}{r_{12}^2}\left(-\frac{15}{8}(n_{12}v_2)^4 + \frac{3}{2}(n_{12}v_2)^2 v_1^2 - 6(n_{12}v_2)^2(v_1v_2) - 2(v_1v_2)^2 + \frac{9}{2}(n_{12}v_2)^2 v_2^2\right. \\
& \qquad\qquad \left. + 4(v_1v_2)v_2^2 - 2v_2^4\right) \\
& \qquad + \frac{G^2 m_1 m_2}{r_{12}^3}\left(\frac{39}{2}(n_{12}v_1)^2 - 39(n_{12}v_1)(n_{12}v_2) + \frac{17}{2}(n_{12}v_2)^2 - \frac{15}{4}v_1^2 - \frac{5}{2}(v_1v_2) + \frac{5}{4}v_2^2\right) \\
& \qquad \left. + \frac{G^2 m_2^2}{r_{12}^3}\left(2(n_{12}v_1)^2 - 4(n_{12}v_1)(n_{12}v_2) - 6(n_{12}v_2)^2 - 8(v_1v_2) + 4v_2^2\right)\right]\boldsymbol{n}_{12} \\
& \qquad + \left[\frac{G^2 m_2^2}{r_{12}^3}\left(-2(n_{12}v_1) - 2(n_{12}v_2)\right) + \frac{G^2 m_1 m_2}{r_{12}^3}\left(-\frac{63}{4}(n_{12}v_1) + \frac{55}{4}(n_{12}v_2)\right)\right. \\
& \qquad\quad + \frac{G m_2}{r_{12}^2}\left(-6(n_{12}v_1)(n_{12}v_2)^2 + \frac{9}{2}(n_{12}v_2)^3 + (n_{12}v_2)v_1^2 - 4(n_{12}v_1)(v_1v_2)\right. \\
& \qquad\qquad \left.\left.\left. + 4(n_{12}v_2)(v_1v_2) + 4(n_{12}v_1)v_2^2 - 5(n_{12}v_2)v_2^2\right)\right]\boldsymbol{v}_{12}\right\}
\end{aligned}
$$

Can we get insights on accuracy from PN equations of motion?

$$
\begin{aligned}
&+\frac{1}{c^5}\Bigg\{\left[\frac{208G^3m_1m_2^2}{15r_{12}^4}(n_{12}v_{12})-\frac{24G^3m_1^2m_2}{5r_{12}^4}(n_{12}v_{12})+\frac{12G^2m_1m_2}{5r_{12}^3}(n_{12}v_{12})v_{12}^2\right]\boldsymbol{n}_{12}\\
&\qquad+\left[\frac{8G^3m_1^2m_2}{5r_{12}^4}-\frac{32\,G^3m_1m_2^2}{5r_{12}^4}-\frac{4G^2m_1m_2}{5r_{12}^3}v_{12}^2\right]\boldsymbol{v}_{12}\Bigg\}\\
&+\frac{1}{c^6}\Bigg\{\Bigg[\frac{Gm_2}{r_{12}^2}\bigg(\frac{35}{16}(n_{12}v_2)^6-\frac{15}{8}(n_{12}v_2)^4v_1^2+\frac{15}{2}(n_{12}v_2)^4(v_1v_2)+3(n_{12}v_2)^2(v_1v_2)^2\\
&\qquad\qquad-\frac{15}{2}(n_{12}v_2)^4v_2^2+\frac{3}{2}(n_{12}v_2)^2v_1^2v_2^2-12(n_{12}v_2)^2(v_1v_2)v_2^2-2(v_1v_2)^2v_2^2\\
&\qquad\qquad+\frac{15}{2}(n_{12}v_2)^2v_2^4+4(v_1v_2)v_2^4-2v_2^6\bigg)\\
&\qquad+\frac{G^2m_1m_2}{r_{12}^3}\bigg(-\frac{171}{8}(n_{12}v_1)^4+\frac{171}{2}(n_{12}v_1)^3(n_{12}v_2)-\frac{723}{4}(n_{12}v_1)^2(n_{12}v_2)^2\\
&\qquad\qquad+\frac{383}{2}(n_{12}v_1)(n_{12}v_2)^3-\frac{455}{8}(n_{12}v_2)^4+\frac{229}{4}(n_{12}v_1)^2v_1^2\\
&\qquad\qquad-\frac{205}{2}(n_{12}v_1)(n_{12}v_2)v_1^2+\frac{191}{4}(n_{12}v_2)^2v_1^2-\frac{91}{8}v_1^4-\frac{229}{2}(n_{12}v_1)^2(v_1v_2)\\
&\qquad\qquad+244(n_{12}v_1)(n_{12}v_2)(v_1v_2)-\frac{225}{2}(n_{12}v_2)^2(v_1v_2)+\frac{91}{2}v_1^2(v_1v_2)\\
&\qquad\qquad-\frac{177}{4}(v_1v_2)^2+\frac{229}{4}(n_{12}v_1)^2v_2^2-\frac{283}{2}(n_{12}v_1)(n_{12}v_2)v_2^2\\
&\qquad\qquad+\frac{259}{4}(n_{12}v_2)^2v_2^2-\frac{91}{4}v_1^2v_2^2+43(v_1v_2)v_2^2-\frac{81}{8}v_2^4\bigg)\\
&\qquad+\frac{G^2m_2^2}{r_{12}^3}\bigg(-6(n_{12}v_1)^2(n_{12}v_2)^2+12(n_{12}v_1)(n_{12}v_2)^3+6(n_{12}v_2)^4\\
&\qquad\qquad+4(n_{12}v_1)(n_{12}v_2)(v_1v_2)+12(n_{12}v_2)^2(v_1v_2)+4(v_1v_2)^2\\
&\qquad\qquad-4(n_{12}v_1)(n_{12}v_2)v_2^2-12(n_{12}v_2)^2v_2^2-8(v_1v_2)v_2^2+4v_2^4\bigg)\\
&\qquad+\frac{G^3m_2^3}{r_{12}^4}\bigg(-(n_{12}v_1)^2+2(n_{12}v_1)(n_{12}v_2)+\frac{43}{2}(n_{12}v_2)^2+18(v_1v_2)-9v_2^2\bigg)\\
&\qquad+\frac{G^3m_1m_2^2}{r_{12}^4}\bigg(\frac{415}{8}(n_{12}v_1)^2-\frac{375}{4}(n_{12}v_1)(n_{12}v_2)+\frac{1113}{8}(n_{12}v_2)^2-\frac{615}{64}(n_{12}v_{12})^2\pi^2\\
&\qquad\qquad+18v_1^2+\frac{123}{64}\pi^2v_{12}^2+33(v_1v_2)-\frac{33}{2}v_2^2\bigg)\\
&\qquad+\frac{G^3m_1^2m_2}{r_{12}^4}\bigg(-\frac{45887}{168}(n_{12}v_1)^2+\frac{24025}{42}(n_{12}v_1)(n_{12}v_2)-\frac{10469}{42}(n_{12}v_2)^2+\frac{48197}{840}v_1^2\\
&\qquad\qquad-\frac{36227}{420}(v_1v_2)+\frac{36227}{840}v_2^2+110(n_{12}v_{12})^2\ln\left(\frac{r_{12}}{r'_1}\right)-22v_{12}^2\ln\left(\frac{r_{12}}{r'_1}\right)\bigg)\\
&\qquad+\frac{16G^4m_2^4}{r_{12}^5}+\frac{G^4m_1^2m_2^2}{r_{12}^5}\left(175-\frac{41}{16}\pi^2\right)+\frac{G^4m_1^3m_2}{r_{12}^5}\left(-\frac{3187}{1260}+\frac{44}{3}\ln\left(\frac{r_{12}}{r'_1}\right)\right)\\
&\qquad+\frac{G^4m_1m_2^3}{r_{12}^5}\left(\frac{110741}{630}-\frac{41}{16}\pi^2-\frac{44}{3}\ln\left(\frac{r_{12}}{r'_2}\right)\right)\Bigg]\boldsymbol{n}_{12}\\
&\qquad+\Bigg[\frac{Gm_2}{r_{12}^2}\bigg(\frac{15}{2}(n_{12}v_1)(n_{12}v_2)^4-\frac{45}{8}(n_{12}v_2)^5-\frac{3}{2}(n_{12}v_2)^3v_1^2+6(n_{12}v_1)(n_{12}v_2)^2(v_1v_2)
\end{aligned}
$$

Can we get insights on accuracy from PN equations of motion?

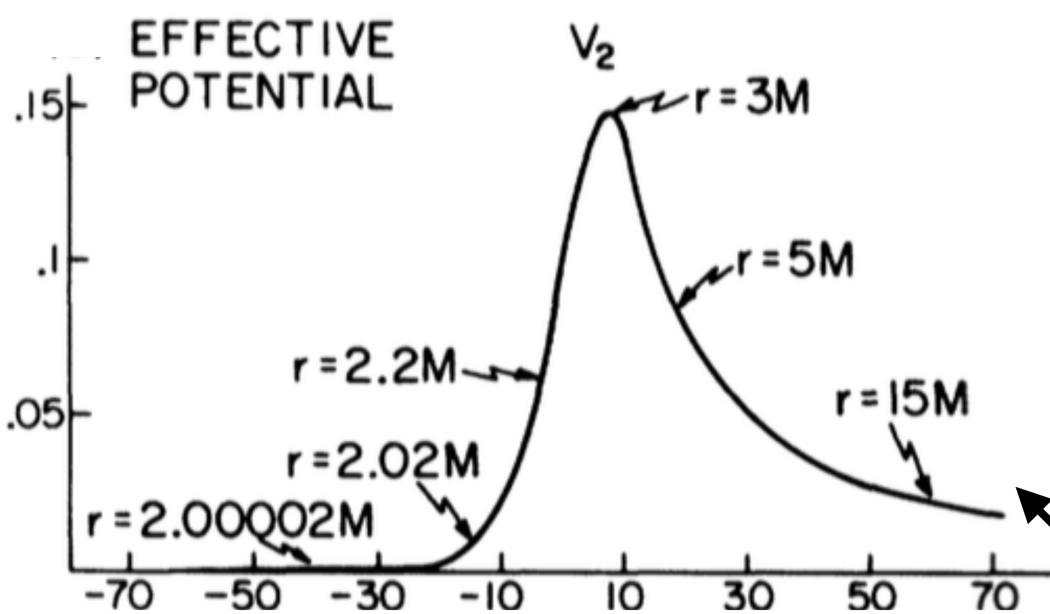
$$
\begin{aligned}
&- 6(n_{12}v_2)^3(v_1v_2) - 2(n_{12}v_2)(v_1v_2)^2 - 12(n_{12}v_1)(n_{12}v_2)^2v_2^2 + 12(n_{12}v_2)^3v_2^2 \\
&+ (n_{12}v_2)v_1^2v_2^2 - 4(n_{12}v_1)(v_1v_2)v_2^2 + 8(n_{12}v_2)(v_1v_2)v_2^2 + 4(n_{12}v_1)v_2^4 \\
&- 7(n_{12}v_2)v_2^4\Big) \\
&+ \frac{G^2m_2^2}{r_{12}^3}\Big(-2(n_{12}v_1)^2(n_{12}v_2) + 8(n_{12}v_1)(n_{12}v_2)^2 + 2(n_{12}v_2)^3 + 2(n_{12}v_1)(v_1v_2) \\
&\quad + 4(n_{12}v_2)(v_1v_2) - 2(n_{12}v_1)v_2^2 - 4(n_{12}v_2)v_2^2\Big) \\
&+ \frac{G^2m_1m_2}{r_{12}^3}\Big(-\frac{243}{4}(n_{12}v_1)^3 + \frac{565}{4}(n_{12}v_1)^2(n_{12}v_2) - \frac{269}{4}(n_{12}v_1)(n_{12}v_2)^2 \\
&\quad - \frac{95}{12}(n_{12}v_2)^3 + \frac{207}{8}(n_{12}v_1)v_1^2 - \frac{137}{8}(n_{12}v_2)v_1^2 - 36(n_{12}v_1)(v_1v_2) \\
&\quad + \frac{27}{4}(n_{12}v_2)(v_1v_2) + \frac{81}{8}(n_{12}v_1)v_2^2 + \frac{83}{8}(n_{12}v_2)v_2^2\Big) \\
&+ \frac{G^3m_2^3}{r_{12}^4}\left(4(n_{12}v_1) + 5(n_{12}v_2)\right) \\
&+ \frac{G^3m_1m_2^2}{r_{12}^4}\left(-\frac{307}{8}(n_{12}v_1) + \frac{479}{8}(n_{12}v_2) + \frac{123}{32}(n_{12}v_{12})\pi^2\right) \\
&+ \frac{G^3m_1^2m_2}{r_{12}^4}\left(\frac{31397}{420}(n_{12}v_1) - \frac{36227}{420}(n_{12}v_2) - 44(n_{12}v_{12})\ln\left(\frac{r_{12}}{r'_1}\right)\right)\Big]\boldsymbol{v}_{12}\Big\} \\
&+ \frac{1}{c^7}\Big\{\Big[\frac{G^4m_1^3m_2}{r_{12}^5}\left(\frac{3992}{105}(n_{12}v_1) - \frac{4328}{105}(n_{12}v_2)\right) \\
&\quad + \frac{G^4m_1^2m_2^2}{r_{12}^6}\left(-\frac{13576}{105}(n_{12}v_1) + \frac{2872}{21}(n_{12}v_2)\right) - \frac{3172}{21}\frac{G^4m_1m_2^3}{r_{12}^6}(n_{12}v_{12}) \\
&\quad + \frac{G^3m_1^2m_2}{r_{12}^4}\Big(48(n_{12}v_1)^3 - \frac{696}{5}(n_{12}v_1)^2(n_{12}v_2) + \frac{744}{5}(n_{12}v_1)(n_{12}v_2)^2 - \frac{288}{5}(n_{12}v_2)^3 \\
&\qquad - \frac{4888}{105}(n_{12}v_1)v_1^2 + \frac{5056}{105}(n_{12}v_2)v_1^2 + \frac{2056}{21}(n_{12}v_1)(v_1v_2) \\
&\qquad - \frac{2224}{21}(n_{12}v_2)(v_1v_2) - \frac{1028}{21}(n_{12}v_1)v_2^2 + \frac{5812}{105}(n_{12}v_2)v_2^2\Big) \\
&\quad + \frac{G^3m_1m_2^2}{r_{12}^4}\Big(-\frac{582}{5}(n_{12}v_1)^3 + \frac{1746}{5}(n_{12}v_1)^2(n_{12}v_2) - \frac{1954}{5}(n_{12}v_1)(n_{12}v_2)^2 \\
&\qquad + 158(n_{12}v_2)^3 + \frac{3568}{105}(n_{12}v_{12})v_1^2 - \frac{2864}{35}(n_{12}v_1)(v_1v_2) \\
&\qquad + \frac{10048}{105}(n_{12}v_2)(v_1v_2) + \frac{1432}{35}(n_{12}v_1)v_2^2 - \frac{5752}{105}(n_{12}v_2)v_2^2\Big) \\
&\quad + \frac{G^2m_1m_2}{r_{12}^3}\Big(-56(n_{12}v_{12})^5 + 60(n_{12}v_1)^3v_{12}^2 - 180(n_{12}v_1)^2(n_{12}v_2)v_{12}^2 \\
&\qquad + 174(n_{12}v_1)(n_{12}v_2)^2v_{12}^2 - 54(n_{12}v_2)^3v_{12}^2 - \frac{246}{35}(n_{12}v_{12})v_1^4 \\
&\qquad + \frac{1068}{35}(n_{12}v_1)v_1^2(v_1v_2) - \frac{984}{35}(n_{12}v_2)v_1^2(v_1v_2) - \frac{1068}{35}(n_{12}v_1)(v_1v_2)^2
\end{aligned}
$$

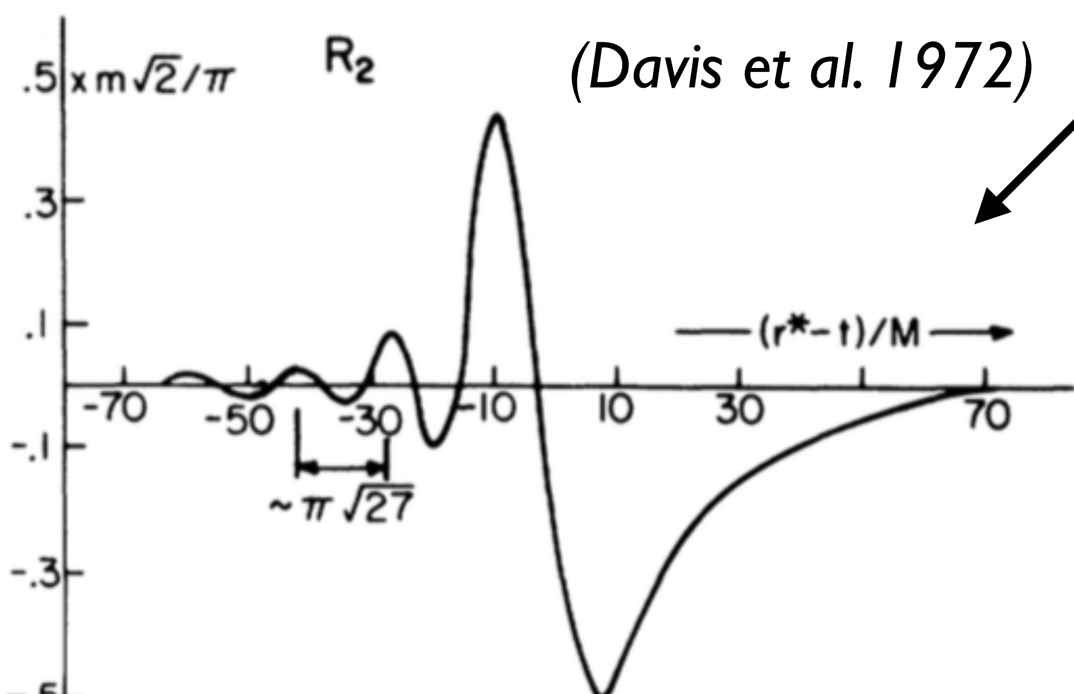
Can we get insights on accuracy from PN equations of motion?

$$
\begin{aligned}
&+\frac{180}{7}(n_{12}v_2)(v_1v_2)^2-\frac{534}{35}(n_{12}v_1)v_1^2v_2^2+\frac{90}{7}(n_{12}v_2)v_1^2v_2^2\\
&+\frac{984}{35}(n_{12}v_1)(v_1v_2)v_2^2-\frac{732}{35}(n_{12}v_2)(v_1v_2)v_2^2-\frac{204}{35}(n_{12}v_1)v_2^4\\
&\left.\left.+\frac{24}{7}(n_{12}v_2)v_2^4\right)\right]\boldsymbol{n}_{12}\\
+\Bigg[&-\frac{184}{21}\frac{G^4m_1^3m_2}{r_{12}^5}+\frac{6224}{105}\frac{G^4m_1^2m_2^2}{r_{12}^6}+\frac{6388}{105}\frac{G^4m_1m_2^3}{r_{12}^6}\\
&+\frac{G^3m_1^2m_2}{r_{12}^4}\left(\frac{52}{15}(n_{12}v_1)^2-\frac{56}{15}(n_{12}v_1)(n_{12}v_2)-\frac{44}{15}(n_{12}v_2)^2-\frac{132}{35}v_1^2+\frac{152}{35}(v_1v_2)\right.\\
&\left.\qquad-\frac{48}{35}v_2^2\right)\\
&+\frac{G^3m_1m_2^2}{r_{12}^4}\left(\frac{454}{15}(n_{12}v_1)^2-\frac{372}{5}(n_{12}v_1)(n_{12}v_2)+\frac{854}{15}(n_{12}v_2)^2-\frac{152}{21}v_1^2\right.\\
&\left.\qquad+\frac{2864}{105}(v_1v_2)-\frac{1768}{105}v_2^2\right)\\
&+\frac{G^2m_1m_2}{r_{12}^3}\left(60(n_{12}v_{12})^4-\frac{348}{5}(n_{12}v_1)^2v_{12}^2+\frac{684}{5}(n_{12}v_1)(n_{12}v_2)v_{12}^2\right.\\
&\qquad-66(n_{12}v_2)^2v_{12}^2+\frac{334}{35}v_1^4-\frac{1336}{35}v_1^2(v_1v_2)+\frac{1308}{35}(v_1v_2)^2+\frac{654}{35}v_1^2v_2^2\\
&\left.\left.\left.\qquad-\frac{1252}{35}(v_1v_2)v_2^2+\frac{292}{35}v_2^4\right)\right]\boldsymbol{v}_{12}\right\}+\mathcal{O}\left(\frac{1}{c^8}\right).
\end{aligned}
$$

Merger-ringdown waveform in small-mass ratio limit

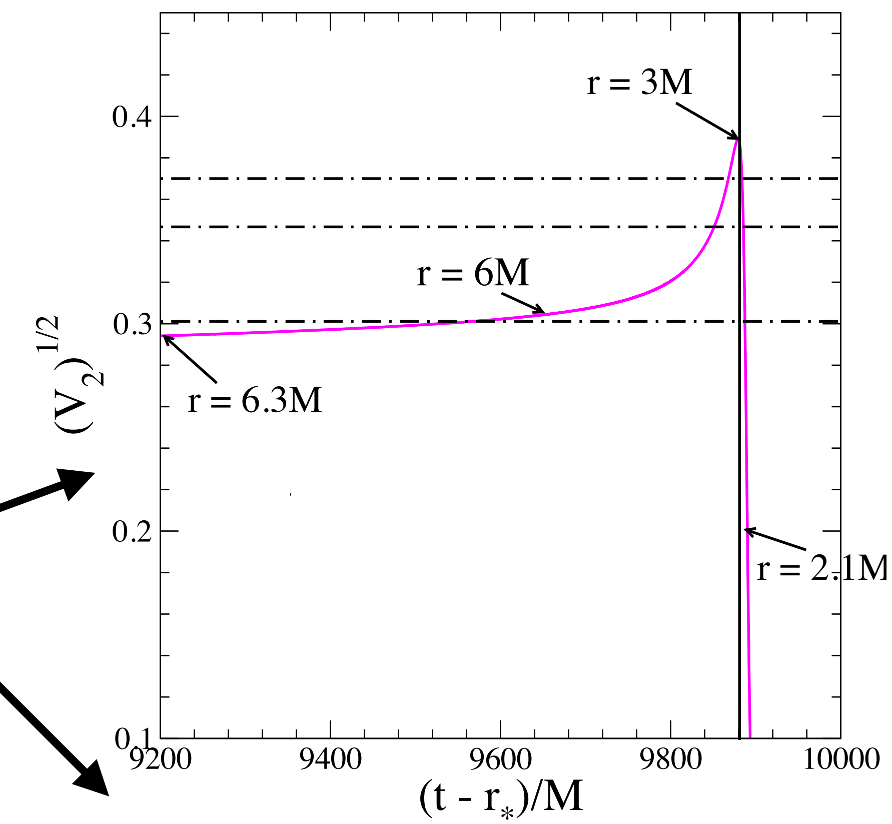
$$-\frac{\partial^2 \Psi}{\partial r_*^2} + (V_{\ell m} - \omega_{\ell m}^2)\Psi = S_{\ell m}$$

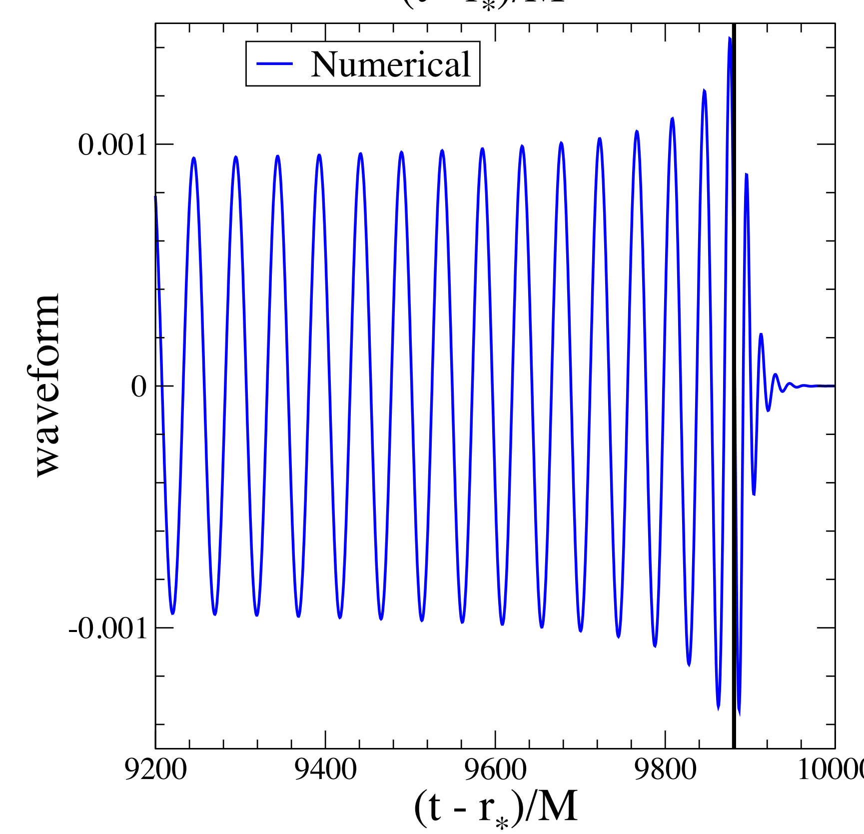




(Davis et al. 1972)

- Quasi-circular inspiral
- Radial infall





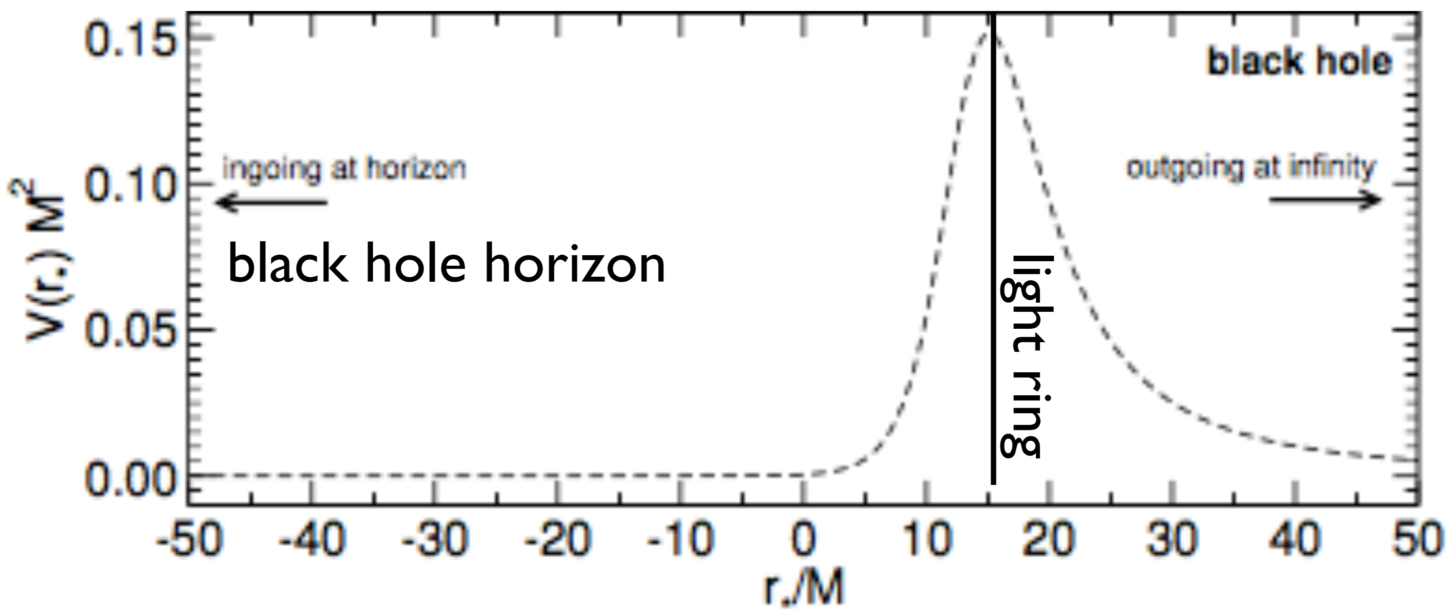
(Barausse, AB et al. 11, see also Damour & Nagar 07)

On the simplicity of merger signal in small-mass ratio limit

- Equation of gravitational perturbations in black-hole spacetime

(Regge & Wheeler 56, Zerilli 70, Teukolsky 72)

$$\frac{\partial^2 \Psi}{\partial t^2} - \frac{\partial^2 \Psi}{\partial r_\star^2} + V_{\ell m} \Psi = S_{\ell m}$$



- **Peak** of black-hole potential **close to "light ring".**

- Once particle is inside potential, **direct gravitational radiation** from its motion is **strongly filtered** by potential barrier (**high-pass filter**).

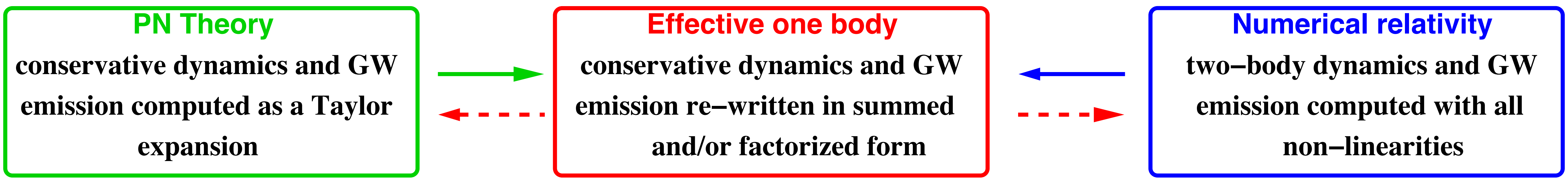
- Only **black-hole spacetime vibrations** (quasi-normal modes) **leaks out** black-hole potential.

(Goebel 1972, Davis et al. 1972, Ferrari & Mashhoon 1984)

The effective-one-body (EOB) approach

- **EOB** approach **introduced before NR** breakthrough

(AB, Pan, Taracchini, Barausse, Bohe', Shao, Hinderer, Steinhoff, Vines; Damour, Nagar, Bernuzzi, Bini, Balmelli, Messina; Iyer, Sathyaprakash; Jaranowski, Schaefer)

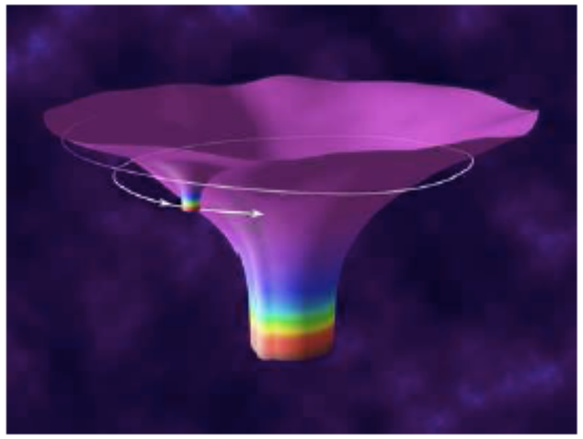


- **EOB** model uses best information available in PN theory, but **resums PN terms** in suitable way to describe accurately dynamics and radiation during inspiral and plunge (going beyond quasi-circular adiabatic motion).

- **EOB** assumes **comparable-mass** description is **smooth deformation of test-particle limit**. It employs non-perturbative ingredients and **models analytically merger-ringdown** signal.

One-body problem: test-particle orbiting non-spinning BH

- Schwarzschild metric



$$ds^2 = -\left(1-\frac{2GM}{rc^2}\right)c^2\,dt^2 + \left(1-\frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2\,d\Omega^2$$

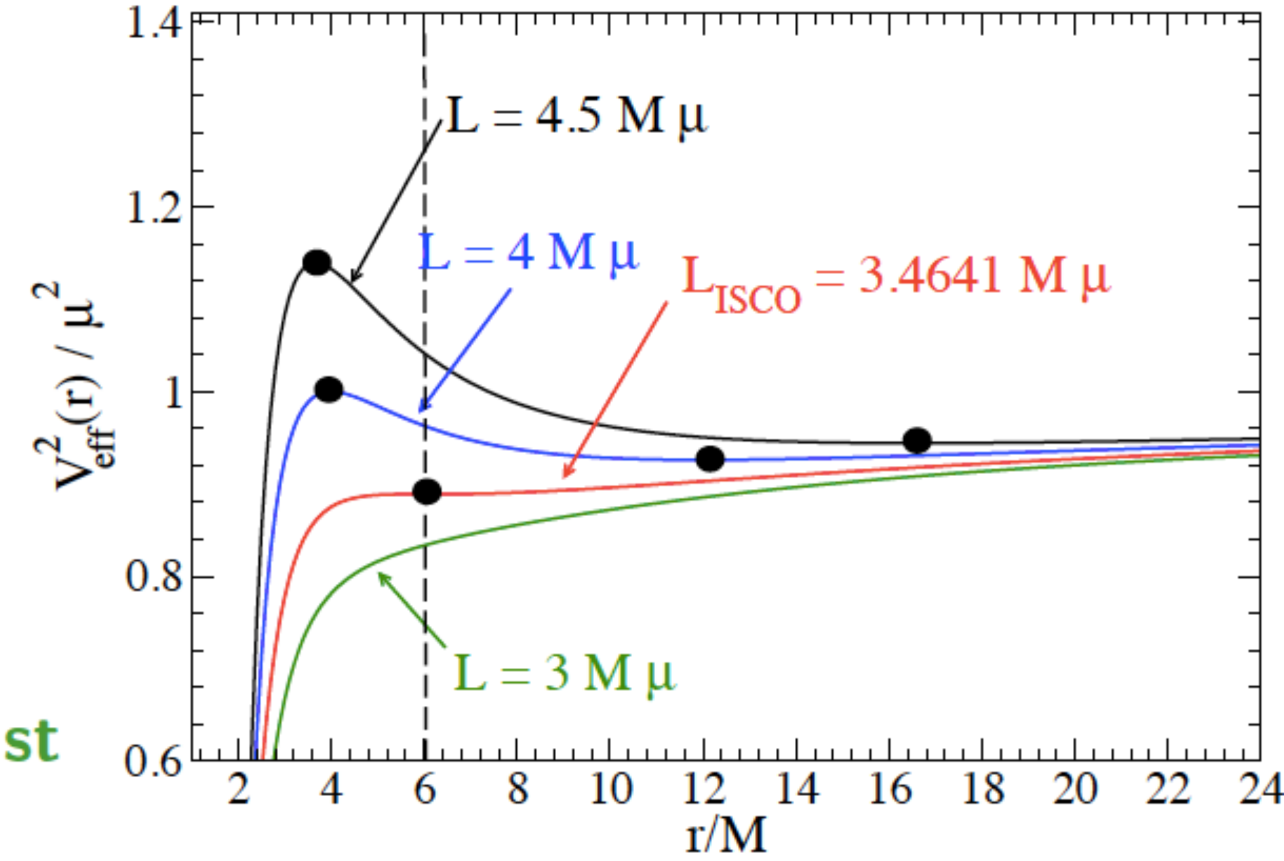
$$H_{\rm Schw}(\mathbf{r},\mathbf{p}) = \mu\sqrt{\left(1-\frac{2M}{r}\right)\left[1+\frac{\mathbf{p}^2}{\mu^2}-\frac{2M}{r}\frac{p_r^2}{\mu^2}\right]}$$

- $H_{\rm Schw}(\mathbf{r},\mathbf{p})$ describes a test-particle of mass μ orbiting a black hole of mass M

- Effective radial potential:

$$\frac{V_{\rm eff}^2(r)}{\mu^2} = \left(1-\frac{2M}{r}\right)\left(1+\frac{L^2}{\mu^2 r^2}\right)$$

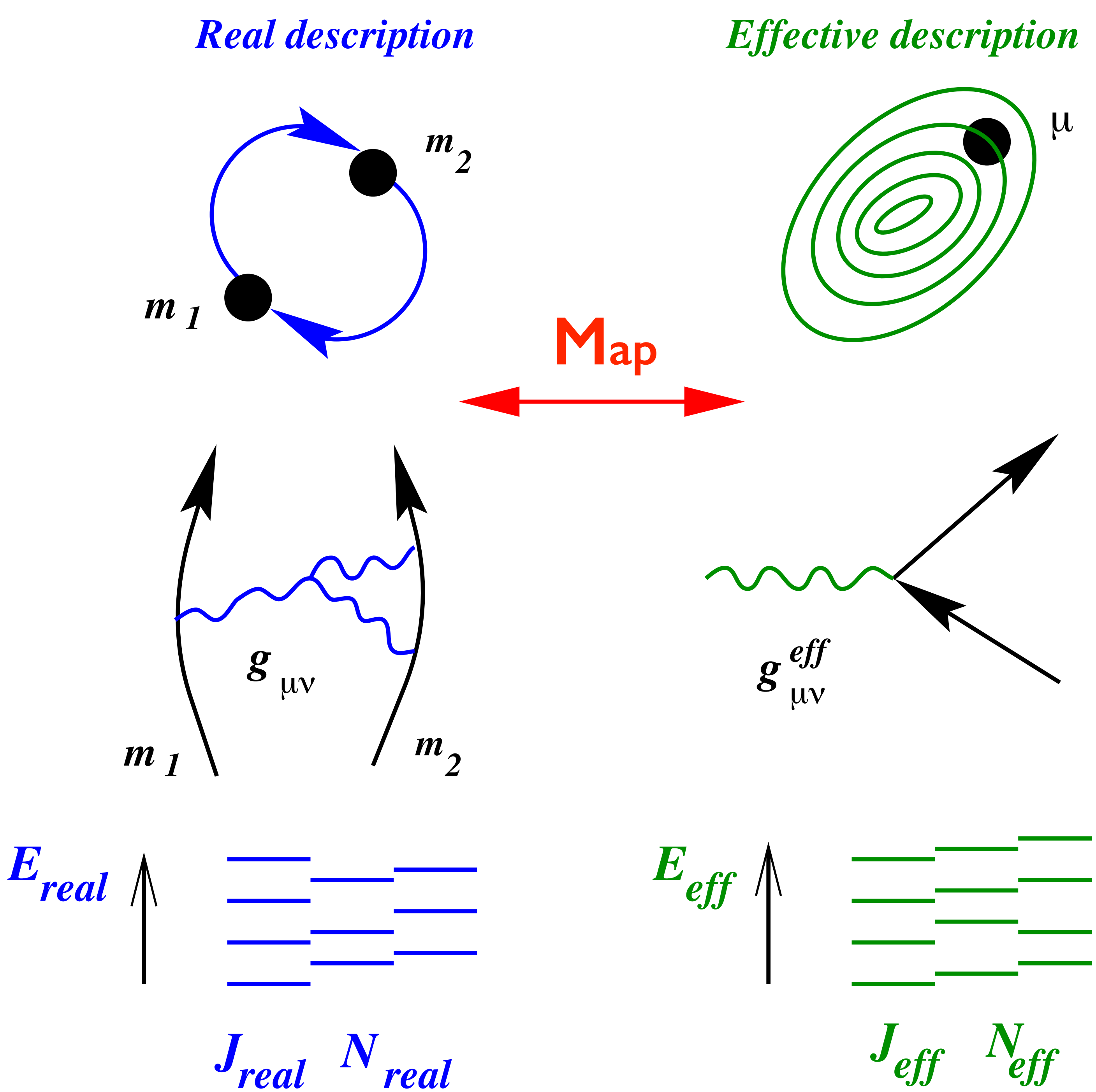
- For $L < L_{\rm ISCO}$ circular orbits no longer exist



The effective-one-body approach in a nutshell

$$\nu = \frac{\mu}{M} \qquad 0 \leq \nu \leq 1/4$$

$$\mu = \frac{m_1 m_2}{M} \qquad M = m_1 + m_2$$



- **Two-body dynamics is mapped** into dynamics of **one-effective body** moving **in deformed black-hole spacetime**, deformation being the mass ratio.

- Some key **ideas** of EOB model were **inspired by quantum** field **theory** when describing energy of comparable-mass charged bodies.

(AB & Damour 1998)

Finding the energy for comparable-mass binary black holes

- **Thinking "quantum mechanically"** (à la Wheeler): **N & J are classical action variables,** and are "quantized" in integers. Natural to require that **"quantum numbers" (N & J) between real and effective** descriptions be **the same**.

- ***Real description:***

$$E_{\mathrm{real}}(N,J) = Mc^2 - \frac{1}{2}\frac{\mu\alpha^2}{N^2}\left[1 + \frac{\alpha^2}{c^2}\left(\frac{6}{NJ} - \frac{1}{4}\frac{15-4\nu}{N^2}\right) + \cdots\right], \ \alpha = GM\mu$$

principal quantum number

- ***Effective description:***

$$E_{\mathrm{eff}}(N,J) = \mu\, c^2 - \frac{1}{2}\frac{\mu\alpha^2}{N^2}\left[1 + \frac{\alpha^2}{c^2}\left(\frac{C_{3,1}}{NJ} + \frac{C_{4,0}}{N^2}\right) + \cdots\right], \ \alpha = GM\mu$$

- Allow transformation of energy axis:

$$E_{\mathrm{eff}}^{\mathrm{NR}} = E_{\mathrm{real}}^{\mathrm{NR}}\left[1 + \alpha_1\frac{E_{\mathrm{real}}^{\mathrm{NR}}}{\mu c^2} + \alpha_2\left(\frac{E_{\mathrm{real}}^{\mathrm{NR}}}{\mu c^2}\right)^2 + \cdots\right]$$

(AB & Damour 1998)

Energy for comparable-mass bodies

- Classical gravity: *(AB & Damour 99)*

$$E_{\rm real}^2 = m_1^2 + m_2^2 + 2m_1 m_2 \left(\frac{E_{\rm eff}}{\mu}\right)$$

- Quantum electrodynamics: *(Brezin, Itzykson & Zinn-Justin 1970)*

$$E_{\rm real}^2 = m_1^2 + m_2^2 + 2m_1 m_2 \frac{1}{\sqrt{1 + Z^2\,\alpha^2/(n-\epsilon_j)^2}}$$

- Considering scattering states:

$$\varphi(s) \equiv \frac{s - m_1^2 - m_2^2}{2m_2 m_2} = \frac{-(p_1+p_2)^2 - m_1^2 - m_2^2}{2m_2 m_2} = -\frac{p_1 \cdot p_2}{m_1 m_2}$$

Hamilton-Jacobi formalism: “real” description

The two-body dynamics can be summarized in a coordinate-invariant manner by evaluating the “energy-levels” of the system

$$\mathcal{H}_{\rm real} \sim \mathcal{H}_{\rm New} + \frac{1}{c^2}\mathcal{H}_{\rm 1PN} + \frac{1}{c^4}\mathcal{H}_{\rm 2PN} + \cdots\,, \qquad \mathcal{H}^{\rm NR}_{\rm real} = \mathcal{H}_{\rm real} - Mc^2$$

$$S = -E^{\rm NR}_{\rm real}\, t + J_{\rm real}\,\phi + S_R(R, E^{\rm NR}_{\rm real}, J_{\rm real})\,, \quad \mathcal{H}^{\rm NR}_{\rm real}(\boldsymbol{P},\boldsymbol{Q}) = E^{\rm NR}_{\rm real}\,, \quad \boldsymbol{P} = \partial S/\partial \boldsymbol{Q}$$

$$S_R(R, E^{\rm NR}_{\rm real}, J_{\rm real}) = \int dR\,\sqrt{\mathcal{R}(R, E^{\rm NR}_{\rm real}, J_{\rm real})}$$

radial action variable: $I_R(R, E^{\rm NR}_{\rm real}, J_{\rm real}) = \frac{2}{2\pi}\int_{R_{\rm min}}^{R_{\rm max}} dR\,\sqrt{\mathcal{R}(R, E^{\rm NR}_{\rm real}, J_{\rm real})}$

$$E_{\rm real}(\mathcal{N}_{\rm real}, J_{\rm real}) = M\,c^2 - \frac{1}{2}\frac{\mu\,\alpha^2}{\mathcal{N}^2_{\rm real}}\,\left[1 + \mathcal{O}\left(\frac{1}{c^2}\right)\cdots\right]$$

$$\alpha = G\,m_1\,m_2 \qquad\qquad \mathcal{N}_{\rm real} = I_R + J_{\rm real}$$

(AB & Damour 1998)

Hamilton-Jacobi formalism: "effective" description

Effective one-body dynamics in external spacetime

$$\mathcal{S}_{\text{eff}} = -\int m_0\, c\, ds_{\text{eff}}$$

$$ds^2_{eff} = -A_\nu(r)\, c^2\, dt^2 + \frac{D_\nu(r)}{A_\nu(r)}\, dr^2 + r^2\, d\Omega^2$$

$$A_\nu(r) = \sum_{n=0}^{3} a_n^\nu \left(\frac{GM_0}{c^2 r}\right)^n \qquad D_\nu(r) = \sum_{n=0}^{2} d_n^\nu \left(\frac{GM_0}{c^2 r}\right)^n$$

Hamilton-Jacobi equation: $g_{\text{eff}}^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} + m_0^2\, c^2 = 0$

$$S = -E_{\text{eff}}^{\text{NR}}\, t + J_{\text{eff}}\, \phi + S_R(R, E_{\text{eff}}^{\text{NR}}, J_{\text{eff}})$$

$$E_{\text{eff}}(\mathcal{N}_{\text{eff}}, J_{\text{eff}}) = m_0\, c^2 - \frac{1}{2} \frac{m_0\, \alpha_{\text{eff}}^2}{\mathcal{N}_{\text{eff}}^2} \left[1 + \mathcal{O}\left(\frac{1}{c^2}\right) + \cdots\right]\,, \;\; \alpha = G\, m_0\, M_0\,, \;\; \mathcal{N}_{\text{eff}} = I_R + J_{\text{eff}}$$

(AB & Damour 1998)

Mapping between energies through canonical transformation

- **Mapping between real and effective Hamiltonians can be obtained also through canonical transformation.**

$$\boldsymbol{q} = \boldsymbol{Q} + \frac{\partial G(\boldsymbol{Q},\boldsymbol{p})}{\partial \boldsymbol{p}} \qquad \boldsymbol{P} = \boldsymbol{p} - \frac{\partial G(\boldsymbol{Q},\boldsymbol{p})}{\partial \boldsymbol{Q}} \qquad G \to \text{generating function}$$

$$1 + \frac{\mathcal{H}_{\rm real}^{\rm NR}(\boldsymbol{Q},\boldsymbol{P})}{c^2}\left(1 + \frac{\nu}{2}\frac{\mathcal{H}_{\rm real}^{\rm NR}(\boldsymbol{Q},\boldsymbol{P})}{c^2}\right) = \frac{\mathcal{H}_{\rm eff}\,(\boldsymbol{q}(\boldsymbol{Q},\boldsymbol{P}),\boldsymbol{p}(\boldsymbol{Q},\boldsymbol{P}))}{c^2}$$

$$\Rightarrow G = G_{\rm New} + \frac{1}{c^2}G_{\rm 1PN} + \frac{1}{c^4}G_{\rm 2PN} + \cdots \Rightarrow \boldsymbol{q} = \mathcal{Q}(\boldsymbol{Q},\boldsymbol{P}) \qquad \boldsymbol{p} = \mathcal{P}(\boldsymbol{Q},\boldsymbol{P})$$

(AB & Damour 1998)

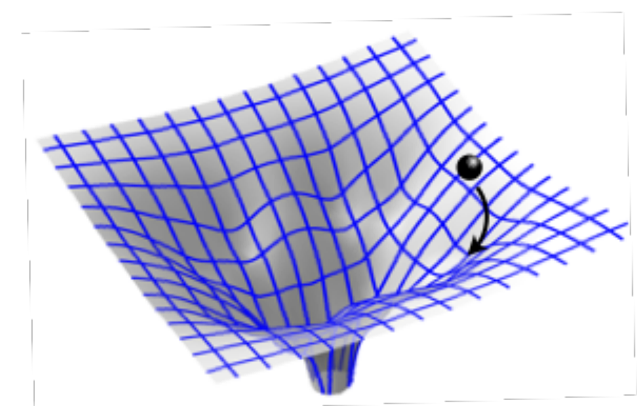
EOB Hamiltonian: resummed conservative dynamics (@2PN)

- Real Hamiltonian → - Effective Hamiltonian

$$H_{\rm real}^{\rm PN} = H_{\rm Newt} + H_{\rm 1PN} + H_{\rm 2PN} + \cdots$$

$$H_{\rm eff}^{\nu} = \mu\sqrt{A_\nu(r)\left[1+\frac{\mathbf{p}^2}{\mu^2}+\left(\frac{1}{B_\nu(r)}-1\right)\frac{p_r^2}{\mu^2}\right]}$$

$$ds^2_{\rm eff} = -A_\nu(r)dt^2 + B_\nu(r)dr^2 + r^2 d\Omega^2$$



(credit: Hinderer)

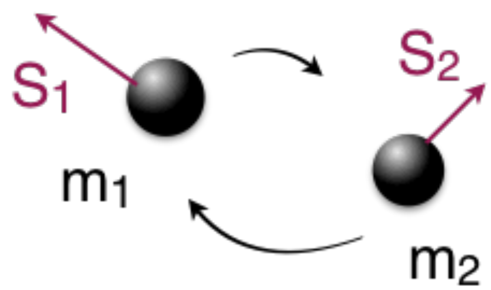
- EOB Hamiltonian: $H_{\rm real}^{\rm EOB} = M\sqrt{1+2\nu\left(\frac{H_{\rm eff}{}^{\nu}}{\mu}-1\right)}$

- Dynamics condensed in $A_\nu(r)$ and $B_\nu(r)$

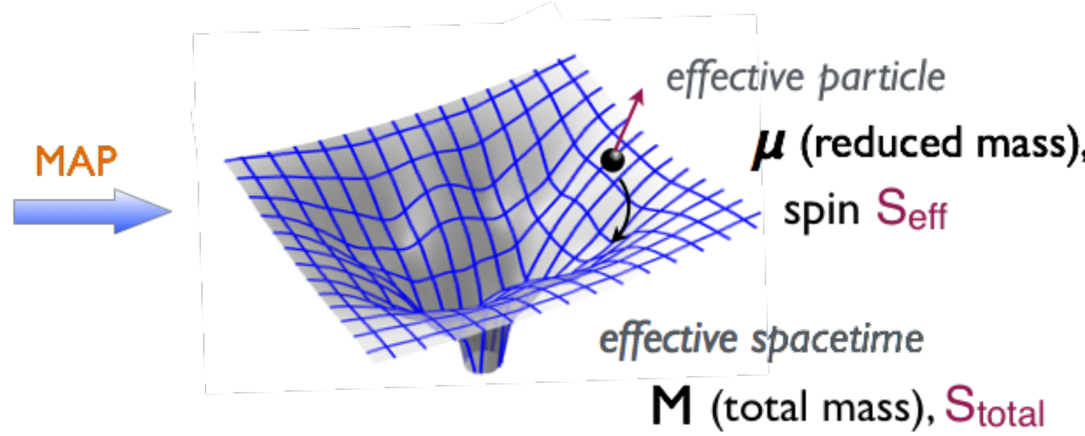
- $A_\nu(r)$, which encodes the energetics of circular orbits, is quite simple:

$$A_\nu(r) = 1 - \frac{2M}{r} + \frac{2M^3\nu}{r^3} + \left(\frac{94}{3}-\frac{41}{32}\pi^2\right)\frac{M^4\nu}{r^4} + \frac{a_5(\nu)+a_5^{\log}(\nu)\log(r)}{r^5} + \frac{a_6(\nu)}{r^6} + \cdots$$

EOB resummed spin dynamics & waveforms



(credit: Hinderer)



$$H_{\rm real}^{\rm EOB} = M\sqrt{1+2\nu\left(\frac{H_{\rm eff}{}^{\nu}}{\mu}-1\right)}$$

- ${H_{\rm eff}}^{\nu}$ extended to include spins

  (Barausse, Racine & AB 09; Barausse & AB 10, 11)

(Damour 01, Damour, Jaranowski & Schäfer 08; Damour & Nagar 14)

- EOB equations of motion *(AB et al. 00, 05; Damour et al. 09)*:

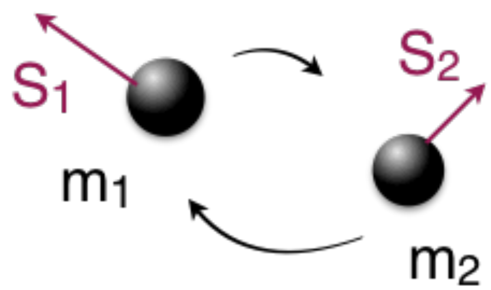
$$\dot{\mathbf{r}} = \frac{\partial H_{\rm real}^{\rm EOB}}{\partial \mathbf{p}} \qquad F \propto \frac{dE}{dt}, \quad \frac{dE}{dt} \propto \sum_{\ell m} |h_{\ell m}|^2$$

$$\dot{\mathbf{p}} = -\frac{\partial H_{\rm real}^{\rm EOB}}{\partial \mathbf{r}} + \mathbf{F} \qquad \dot{\mathbf{S}} = \{\mathbf{S}, H_{\rm real}^{\rm EOB}\}$$

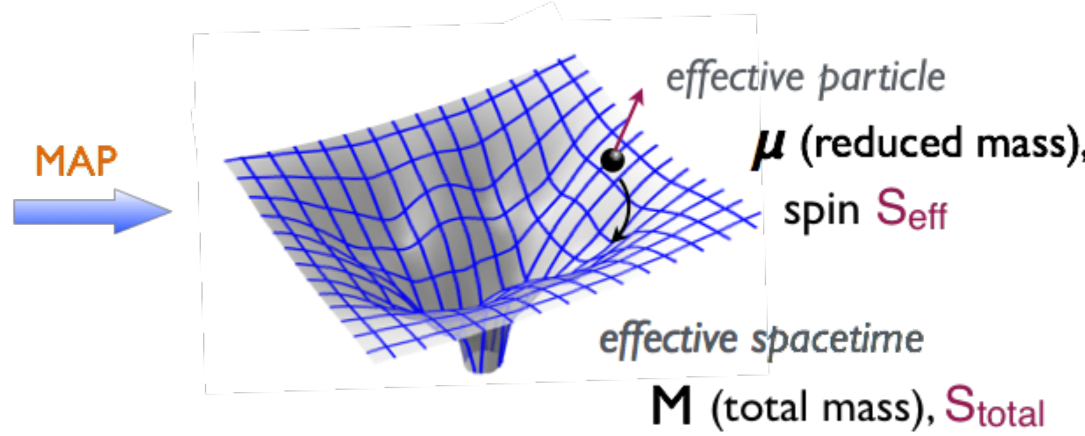
- EOB waveforms *(AB et al. 00; Damour et al. 09; Pan et al. 11)*:

$$h_{\ell m}^{\rm insp-plunge} = h_{\ell m}^{\rm Newt}\, e^{-im\Phi}\, S_{\rm eff}\, T_{\ell m}\, e^{i\delta_{\ell m}}\, (\rho_{\ell m})^{\ell}\, h_{\ell m}^{\rm NQC}$$

On EOB Hamiltonians with spins



(credit: Hinderer)



$$H_{\rm real}^{\rm EOB} = M\sqrt{1+2\nu\left(\frac{H_{\rm eff}{}^{\nu}}{\mu}-1\right)}$$

- $H_{\rm eff}{}^{\nu}$ extended to include spins

  (Barausse, Racine & AB 09; Barausse & AB 10, 11)

- Hamiltonian of spinning particle in Kerr *(Barausse, Racine & AB 09)*

- $H_{\rm eff}{}^{\nu} = H_{\rm NS}(m_1, m_2, \mathbf{S}_{\rm Kerr}) + H_{\rm S}(m_1, m_2, \mathbf{S}_{\rm Kerr}, \mathbf{S}_*) - \dfrac{\mu}{2M\, r^3} S_*^2$

Hamiltonian of non-spinning effective particle in deformed-Kerr.

Term linearly proportional to spin $\mathbf{S}_*$ of effective particle.

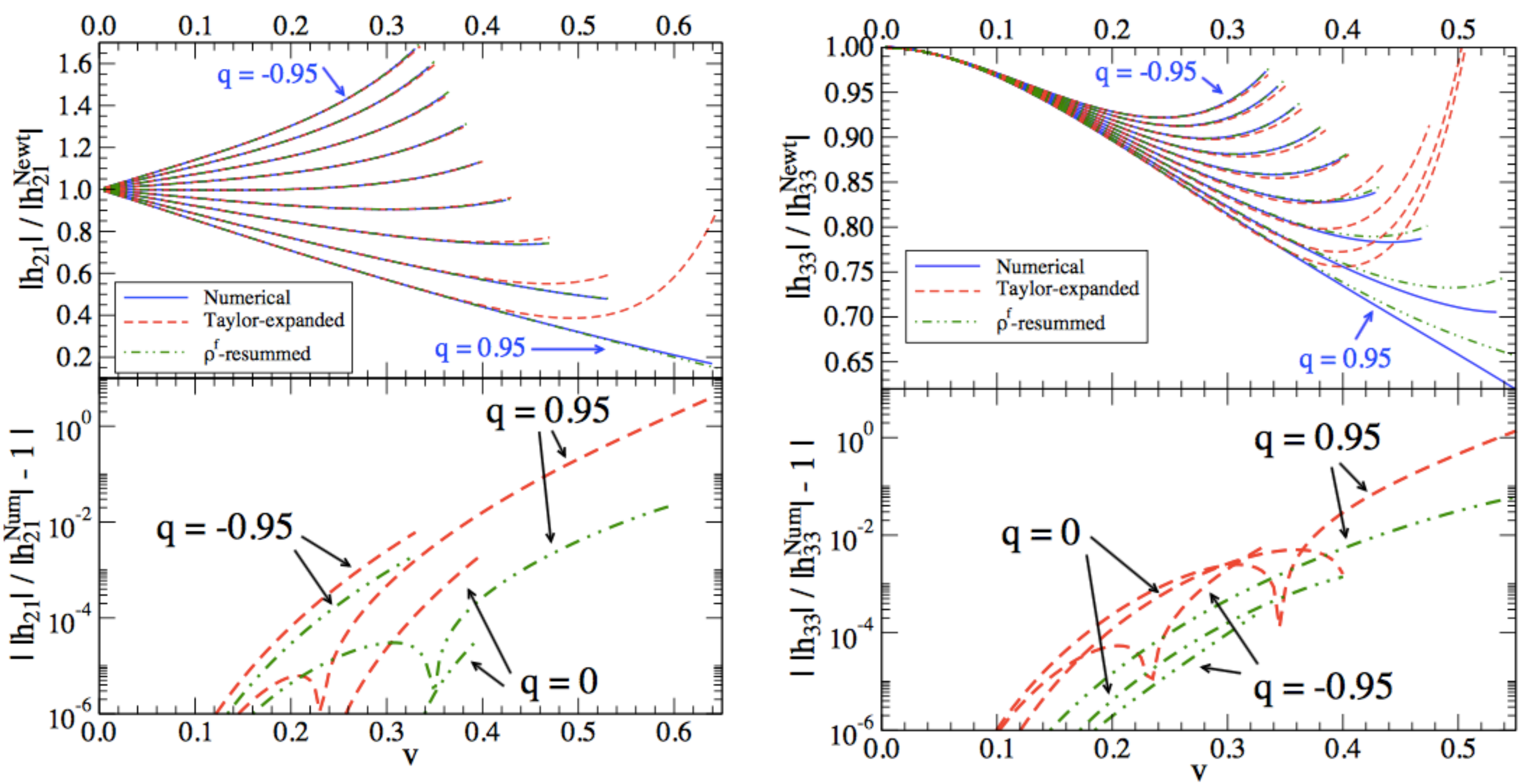
Term added to obtain full leading-order terms quadratic in spins.
(see also Vines et al. 16)

$$\mathbf{S}_{\rm Kerr} = \mathbf{S_1} + \mathbf{S_2} \qquad \mathbf{S}_* = \mathbf{S_1}\,\frac{m_2}{m_1} + \mathbf{S_2}\,\frac{m_1}{m_2} + \frac{1}{c^2}\,\mathbf{\Delta}_{\mathbf{S}_*}$$

(for a different EOB mapping in presence of spins see Damour 01, Damour, Jaranowski & Schäfer 08; Damour & Nagar 14, Balmelli & Damour 15)

EOB factorized waveforms with spins

- Gravitational **modes** for particle **orbiting Kerr BH** $q = a/M$



(Damour, Nagar & Iyer 09, Pan, AB, Fujita Racine & Tagoshi 10)

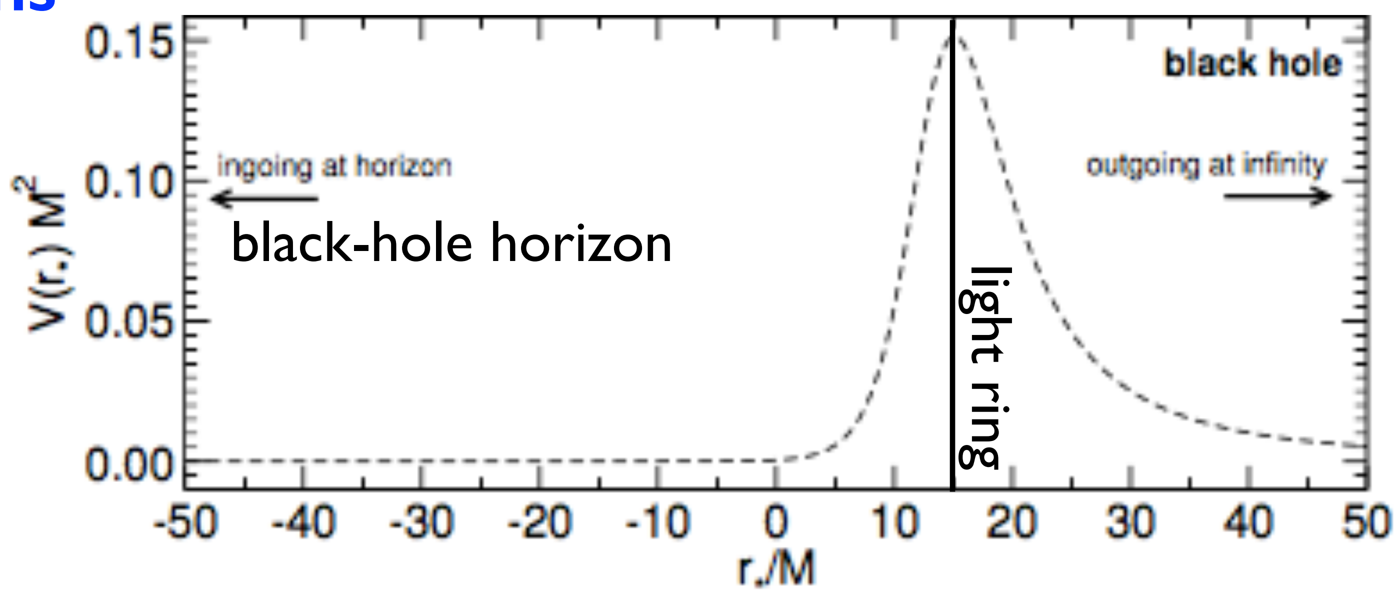
If perturbed, black holes ring or vibrate: quasi-normal modes

- Equation of **gravitational perturbations** in black-hole spacetime:

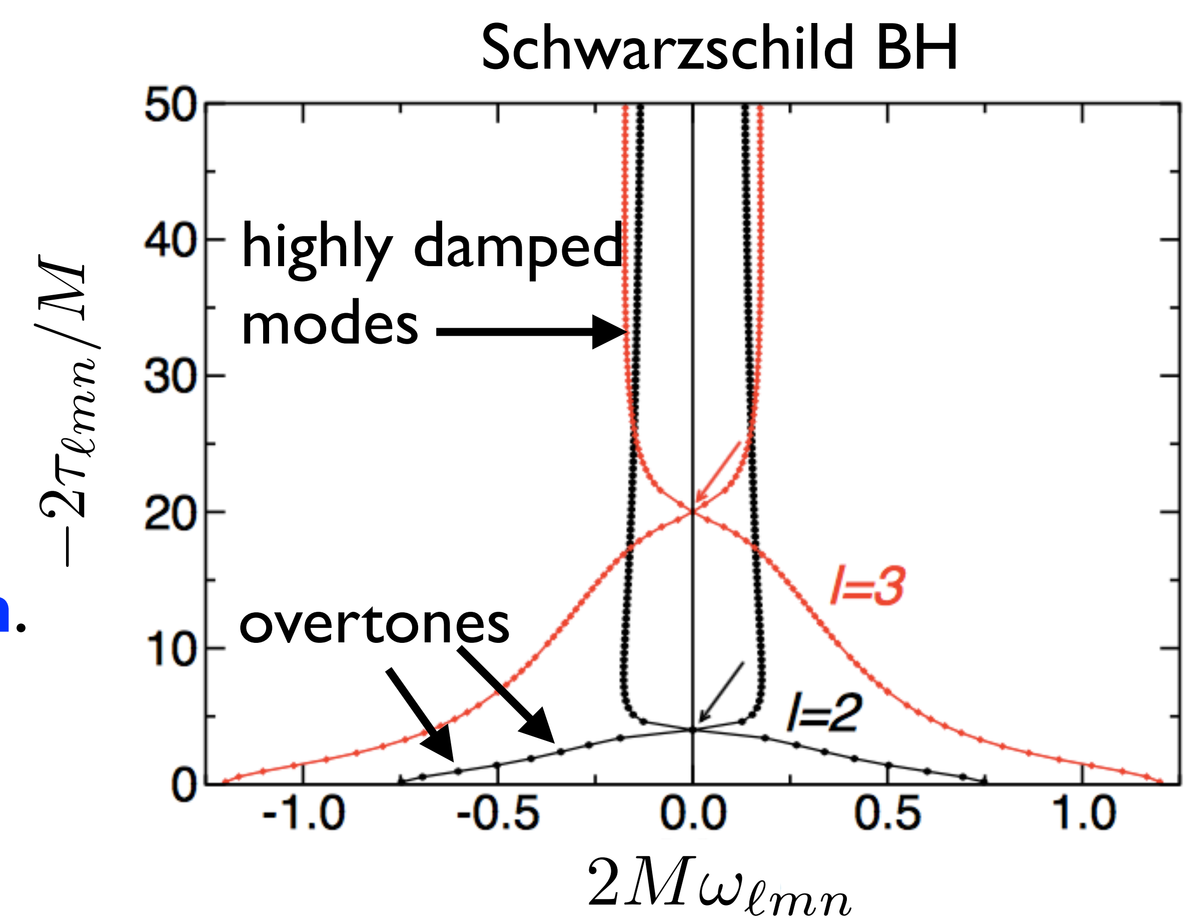
(Regge & Wheeler 56, Zerilli 70, Teukolsky 72)

$$\frac{\partial^2 \Psi}{\partial t^2} - \frac{\partial^2 \Psi}{\partial r_\star^2} + V_{\ell m} \Psi = 0$$

(Vishveshwara 70, Press 71, Chandrasekhar et al. 75)



- If black-hole's size is $R_{\rm BH} = 2GM/c^2$ and mass $M = 20 M_\odot \Rightarrow R_{\rm BH} \sim 60\,\mathrm{km}$
  travel time of spacetime vibration
  $\Rightarrow R_{\rm BH}/c \sim 0.2\,\mathrm{msec}.$



- For **each (l,m)**, infinite tower of **overtones n**.
- For astrophysical black holes (zero charge), QNM's **frequency** and **decay time** only depend on **mass and spin**.

If perturbed, black holes ring or vibrate: quasi-normal modes

- Equation of **gravitational perturbations** in black-hole spacetime:

(Regge & Wheeler 56, Zerilli 70, Teukolsky 72)

$$\frac{\partial^2 \Psi}{\partial t^2} - \frac{\partial^2 \Psi}{\partial r_\star^2} + V_{\ell m} \Psi = 0$$

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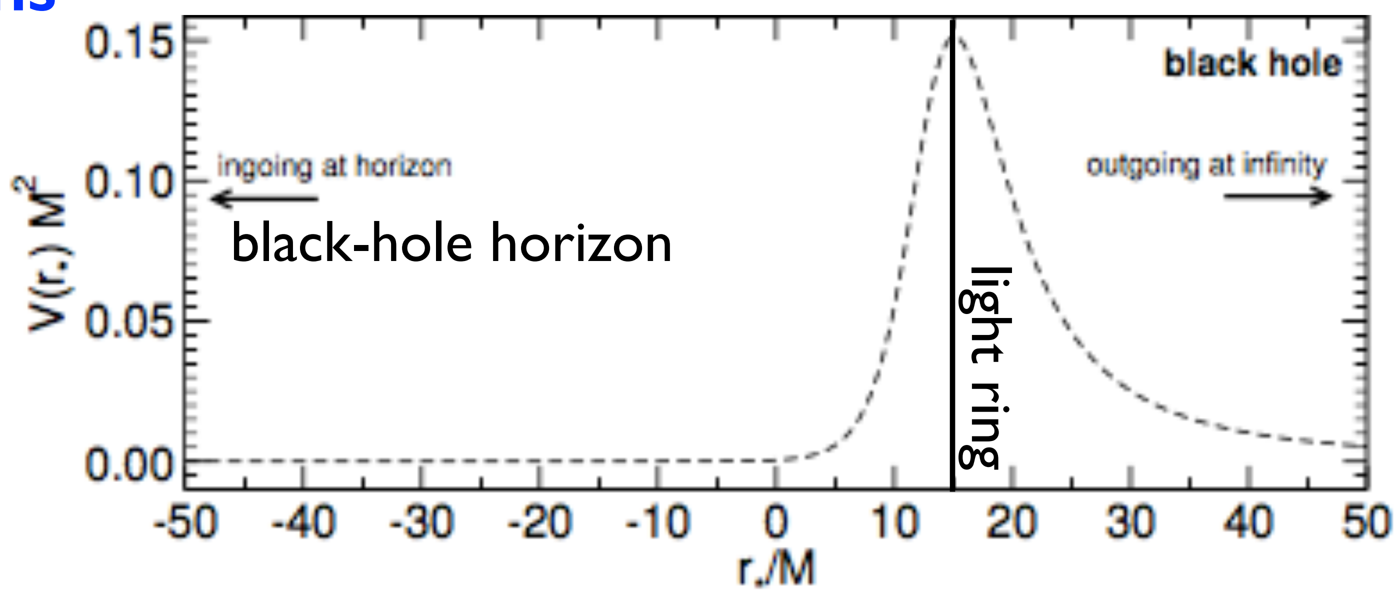
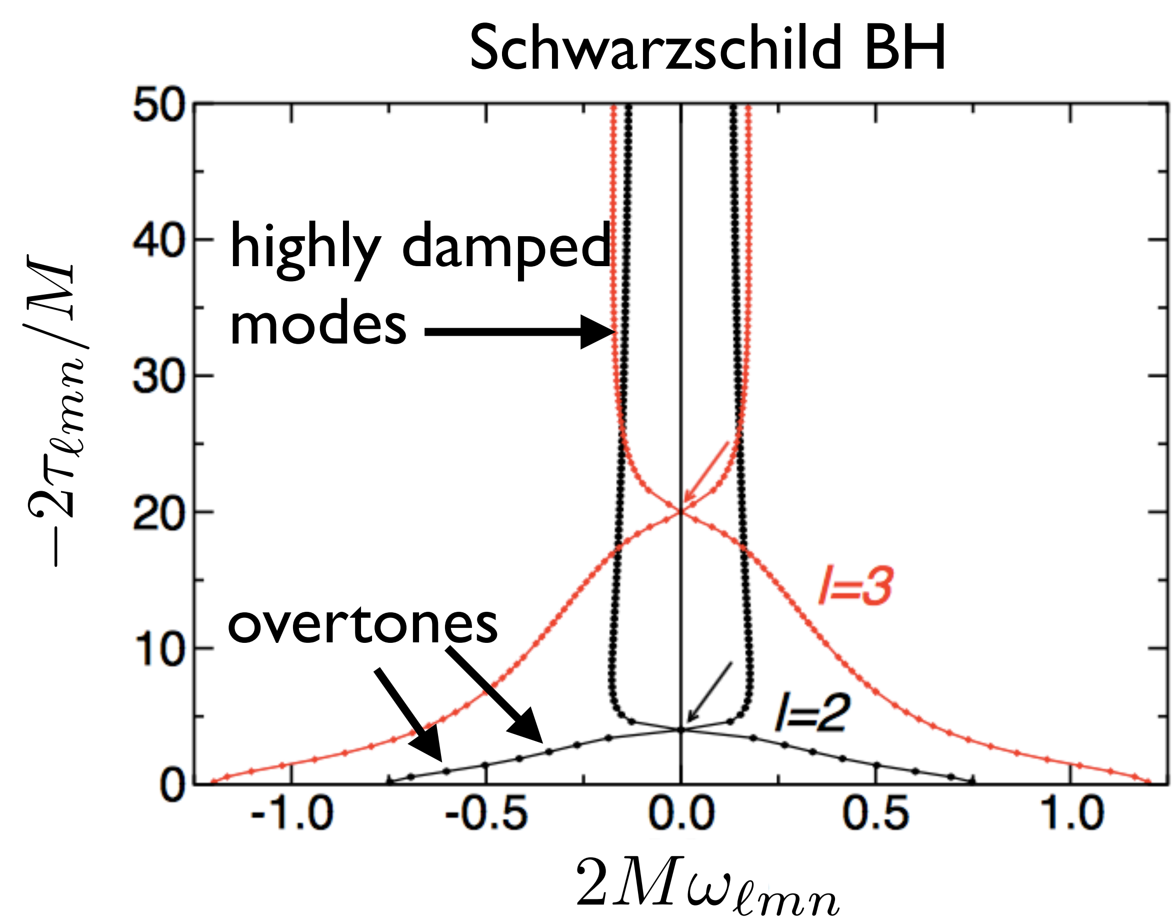
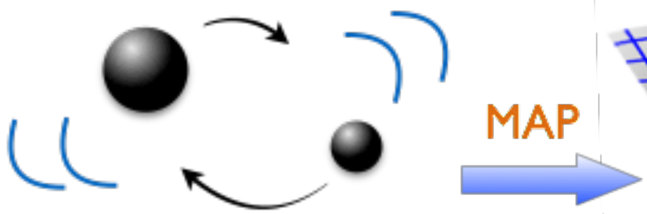


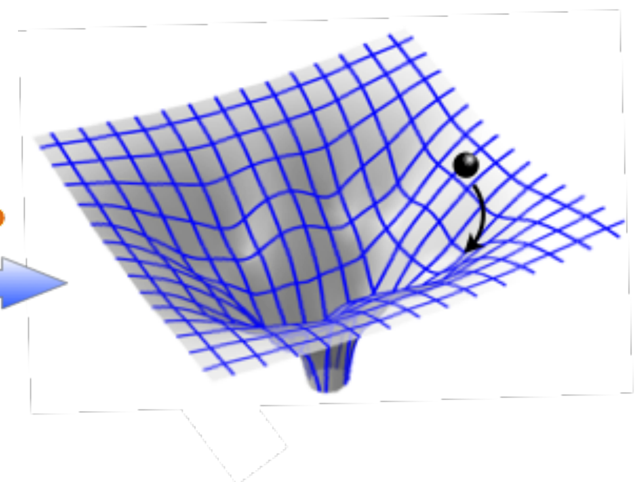
- For Schwarzschild BH of $M = 20 M_\odot$

$$f_{2m0} = 604\,\mathrm{Hz}, \tau_{2m0} = 1.10\,\mathrm{msec}$$
$$f_{2m1} = 560\,\mathrm{Hz}, \tau_{2m1} = 0.36\,\mathrm{msec}$$
$$f_{2m2} = 486\,\mathrm{Hz}, \tau_{2m2} = 0.20\,\mathrm{msec}$$



EOB inspiral-merger-ringdown analytic waveform



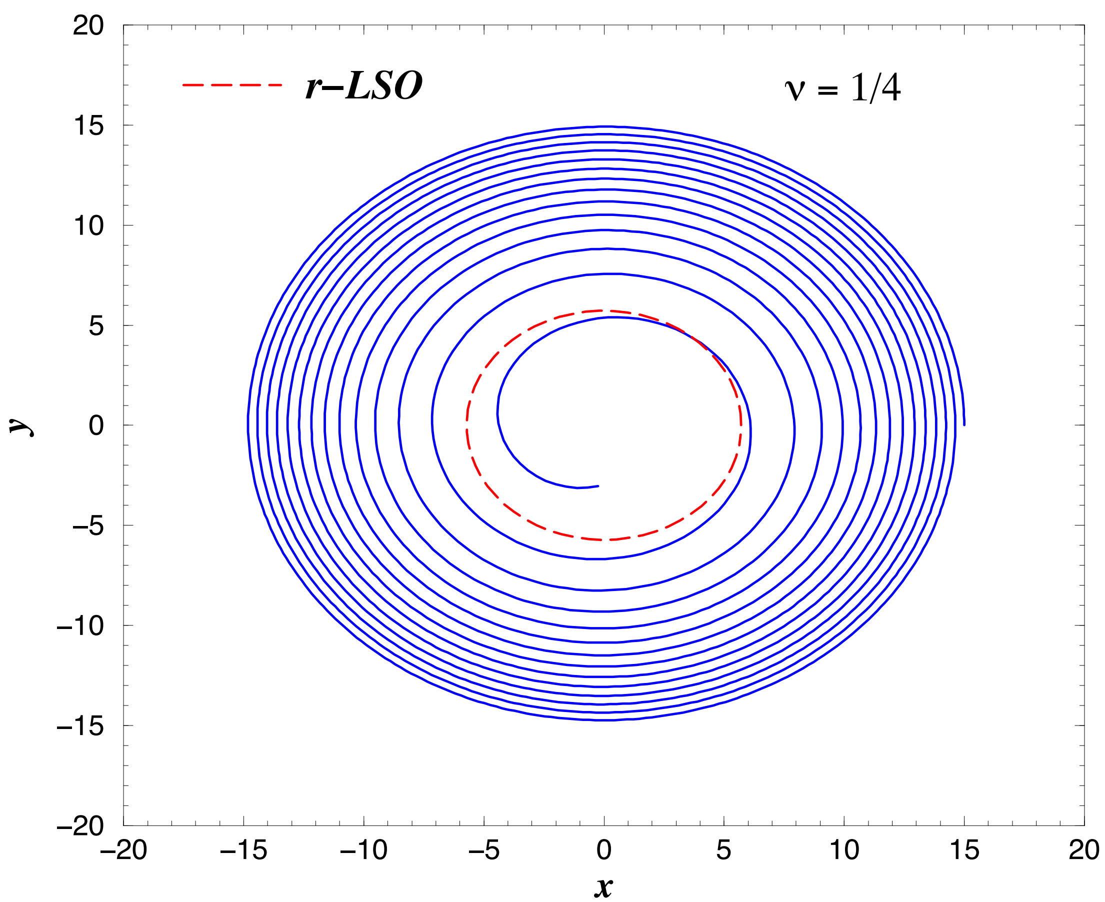


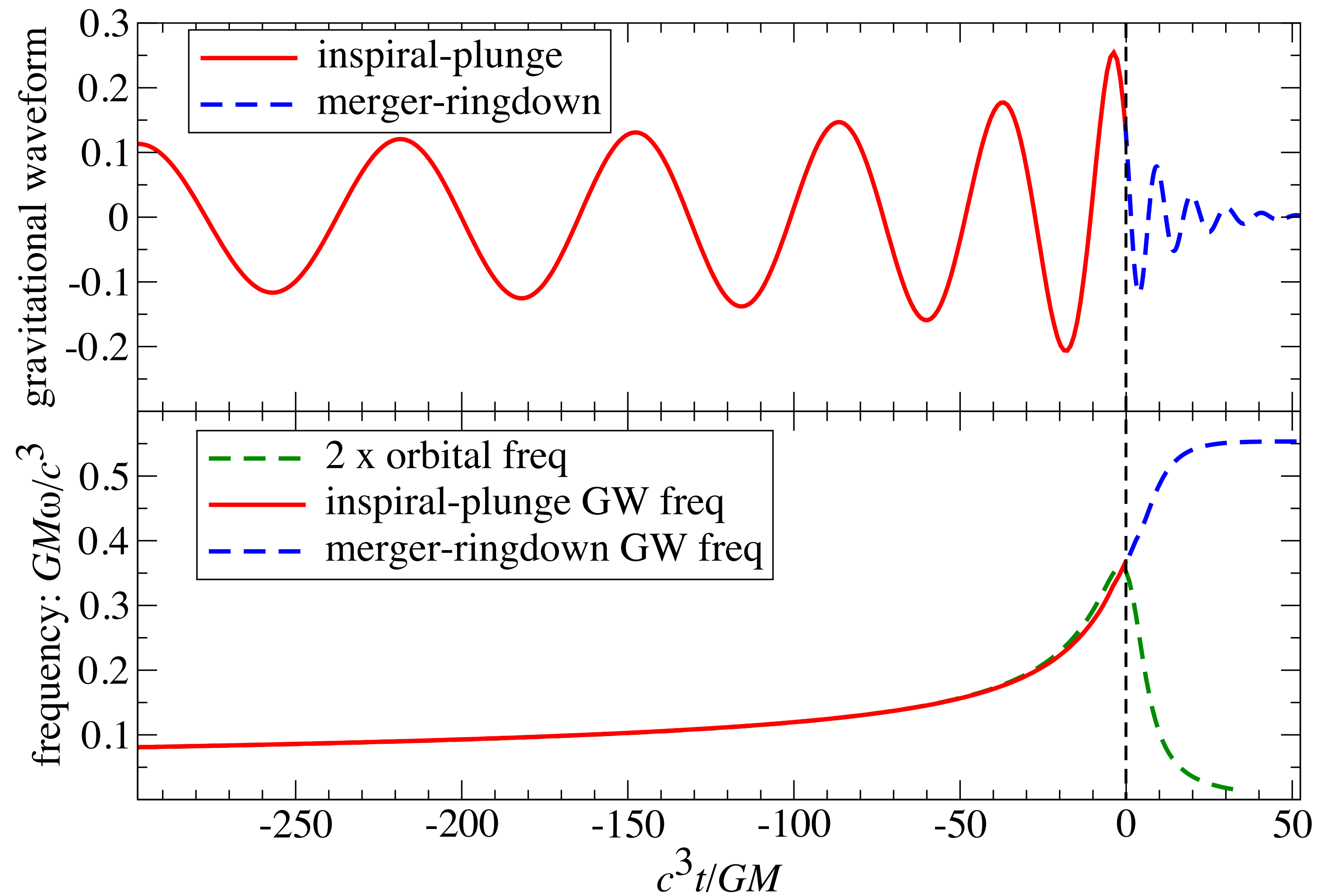
(credit: Hinderer)

$$H^2_{\rm real} = m_1^2 + m_2^2 + 2m_1 m_2 \left(\frac{H_{\rm eff}}{\mu}\right)$$

$$h^{\rm insp-plunge}(t) = \nu \left(\frac{GM}{c^2 D}\right) \frac{v^2}{c^2} \cos 2\Phi$$

$$\frac{v}{c} = \left(\frac{GM\omega}{c^3}\right)^{1/3} \qquad \nu = \mu/M$$





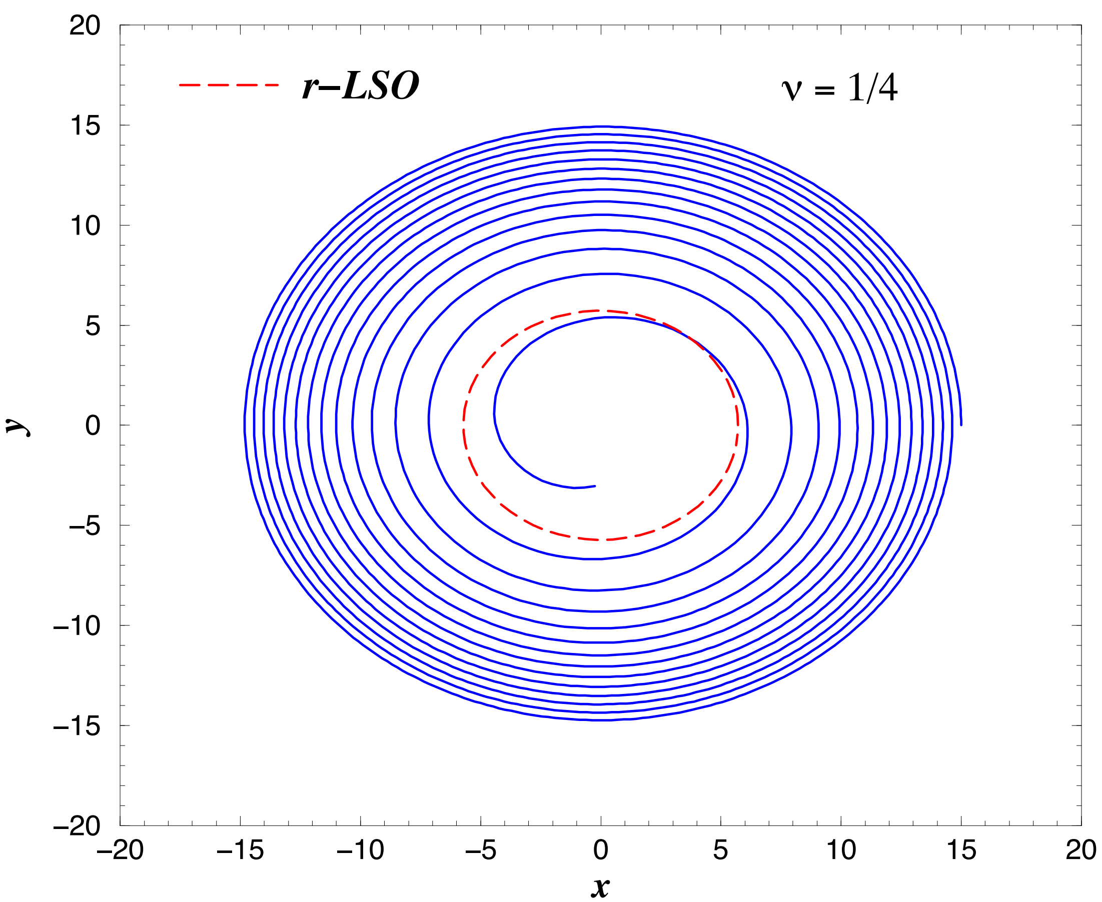
(AB & Damour 00)

EOB inspiral-merger-ringdown analytic waveform

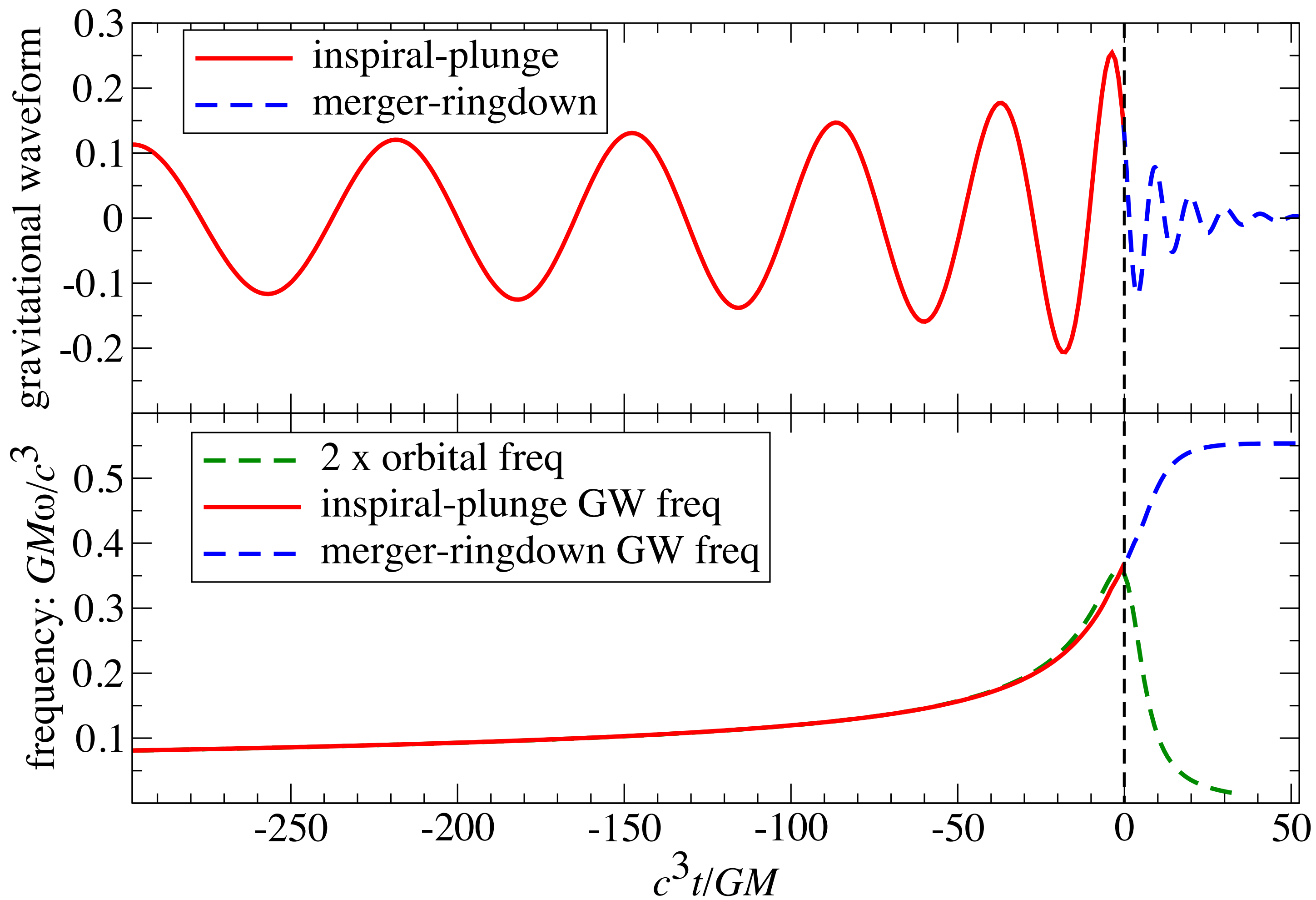
- The plunge (~ 1.5 GW cycles) is a smooth continuation of the inspiral phase
- The transition merger to ringdown was assumed *very short*
- One single **QNM matched using** $M_{\rm BH} = E_{\rm LR} = 0.976\,M$, $a_{\rm BH} = \boldsymbol{J}_{\rm LR}/E^2_{\rm LR} = 0.77$

$$h^{\rm merger-RD}(t) = A\, e^{-(t-t_{\rm match})/\tau_{\rm QNM}}\ \cos[\omega_{\rm QNM}(t-t_{\rm match})+B]$$

In 2005, NR breakthrough found 0.68 for equal-mass binary merger.

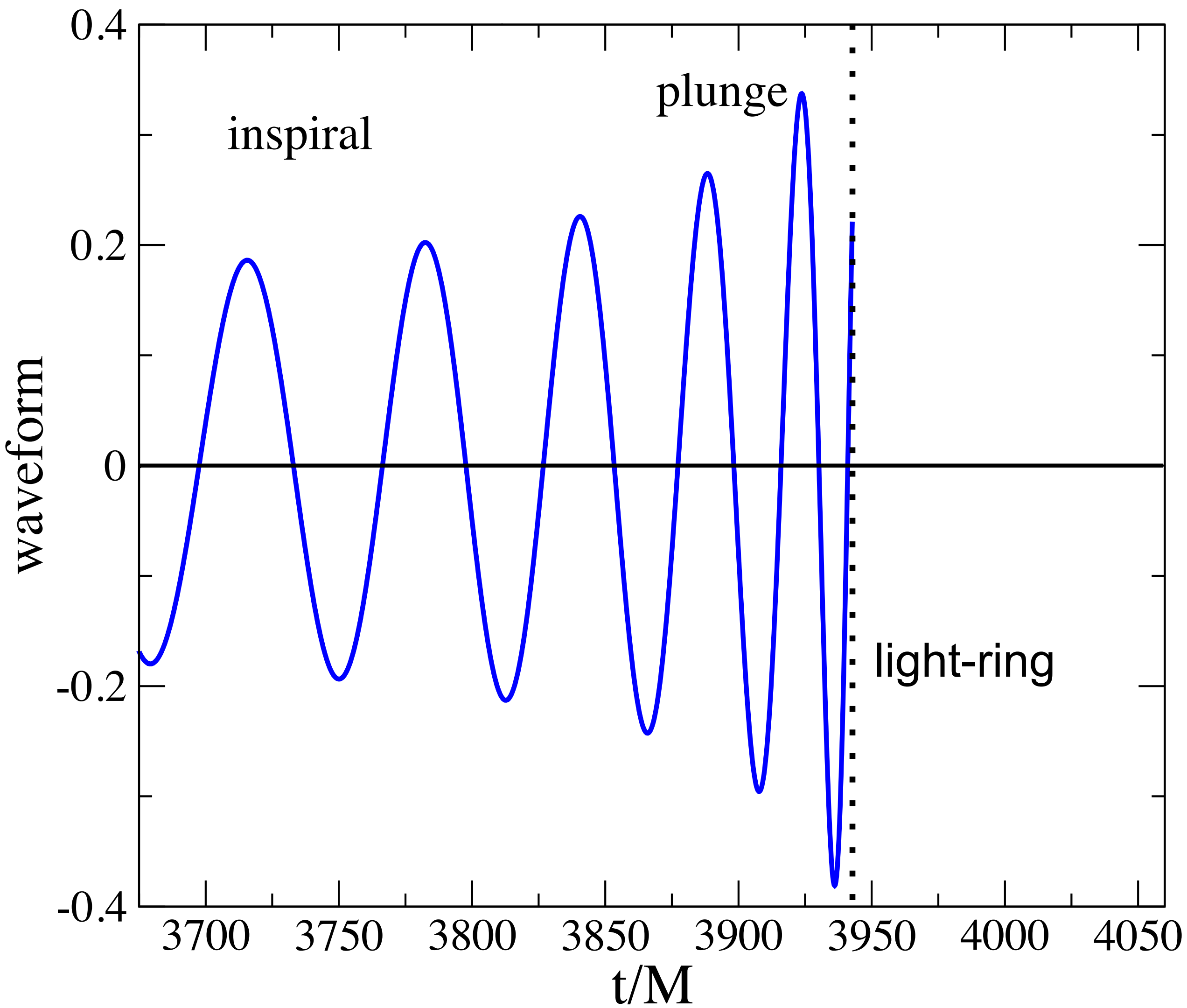


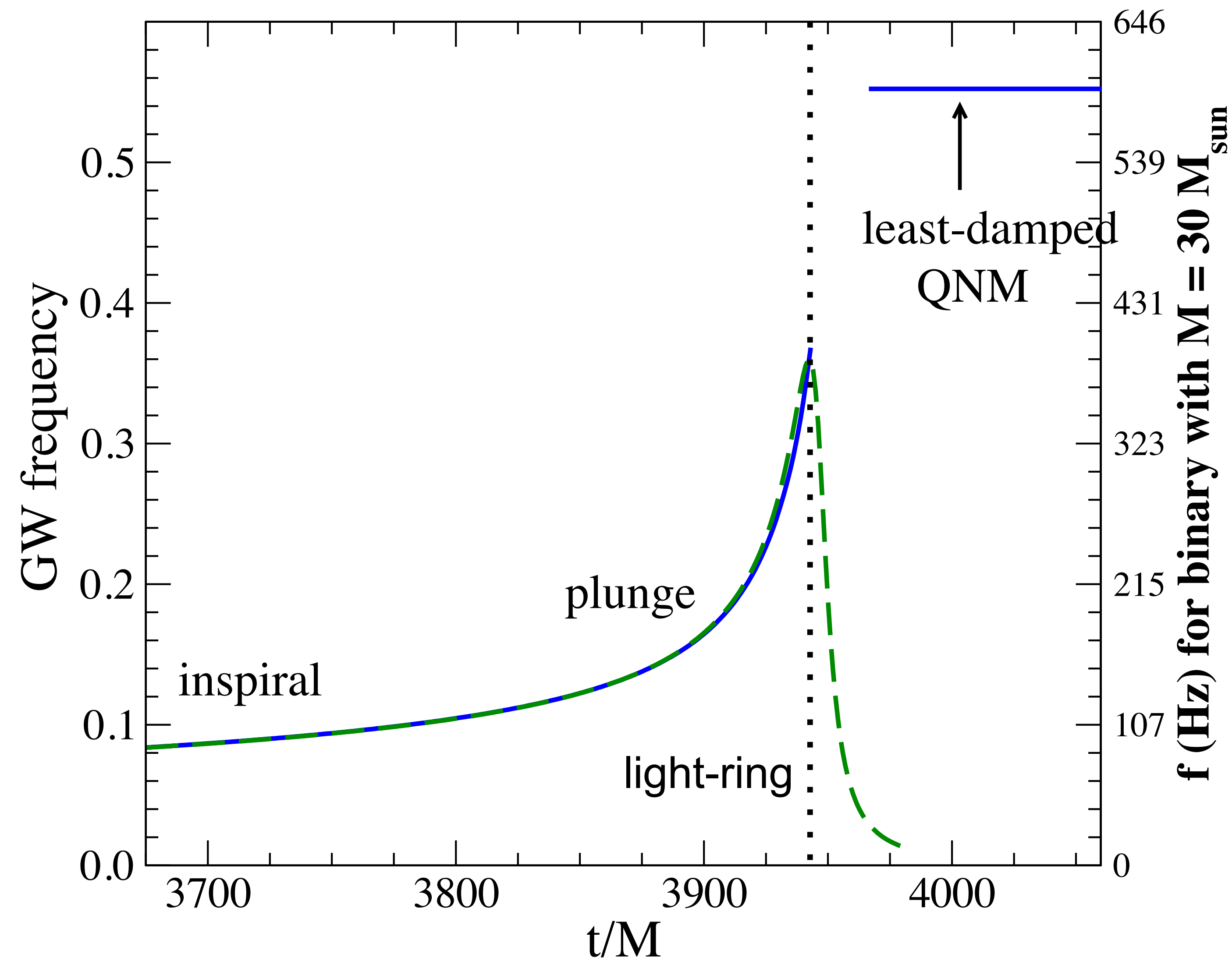
(AB & Damour 00)



On the full effective-one-body waveforms

- Evolve **two-body dynamics up to light ring** (or photon orbit) and then …



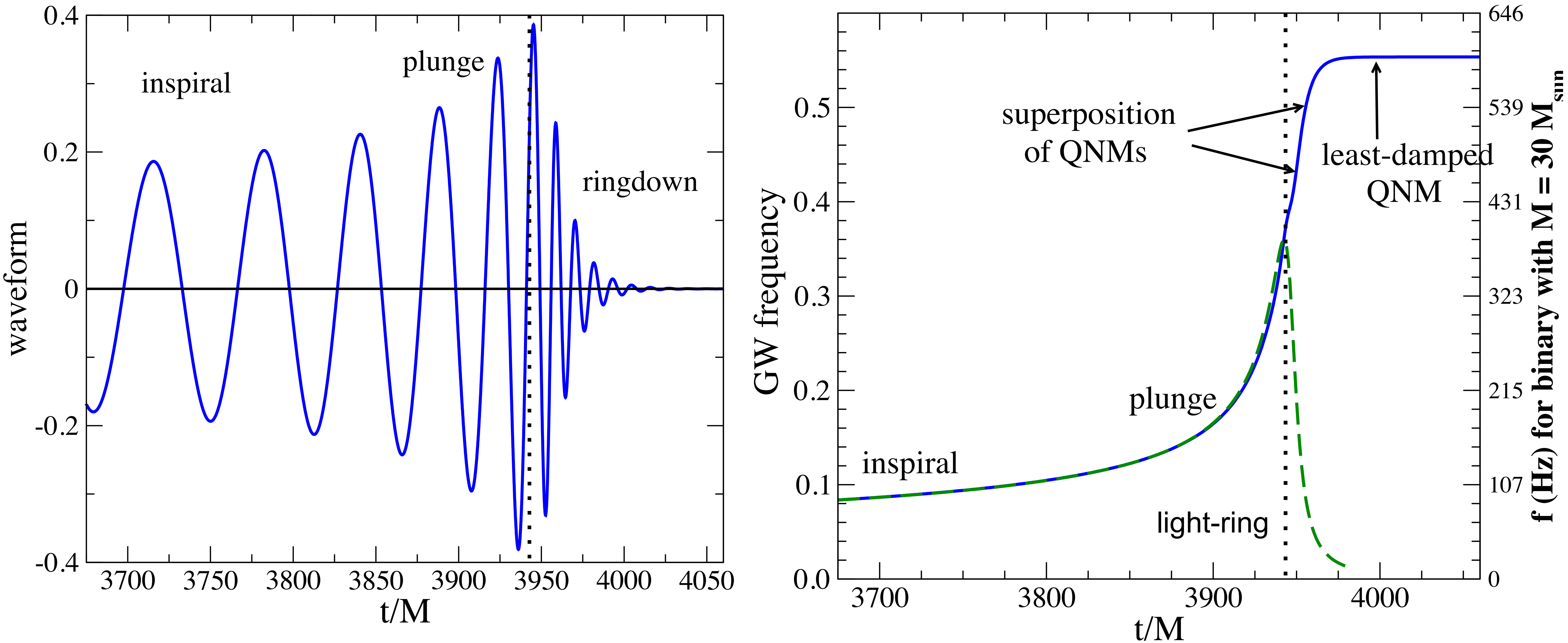


- **Quasi-normal modes** excited at **light-ring crossing**

(Goebel 1972, Davis et al. 1972, Ferrari et al. 1984, Damour et al. 07, Barausse et al. 11, Price et al. 15)

On the full effective-one-body waveforms (contd.)

… attach **superposition** of **quasi-normal modes** of **remnant** black hole.

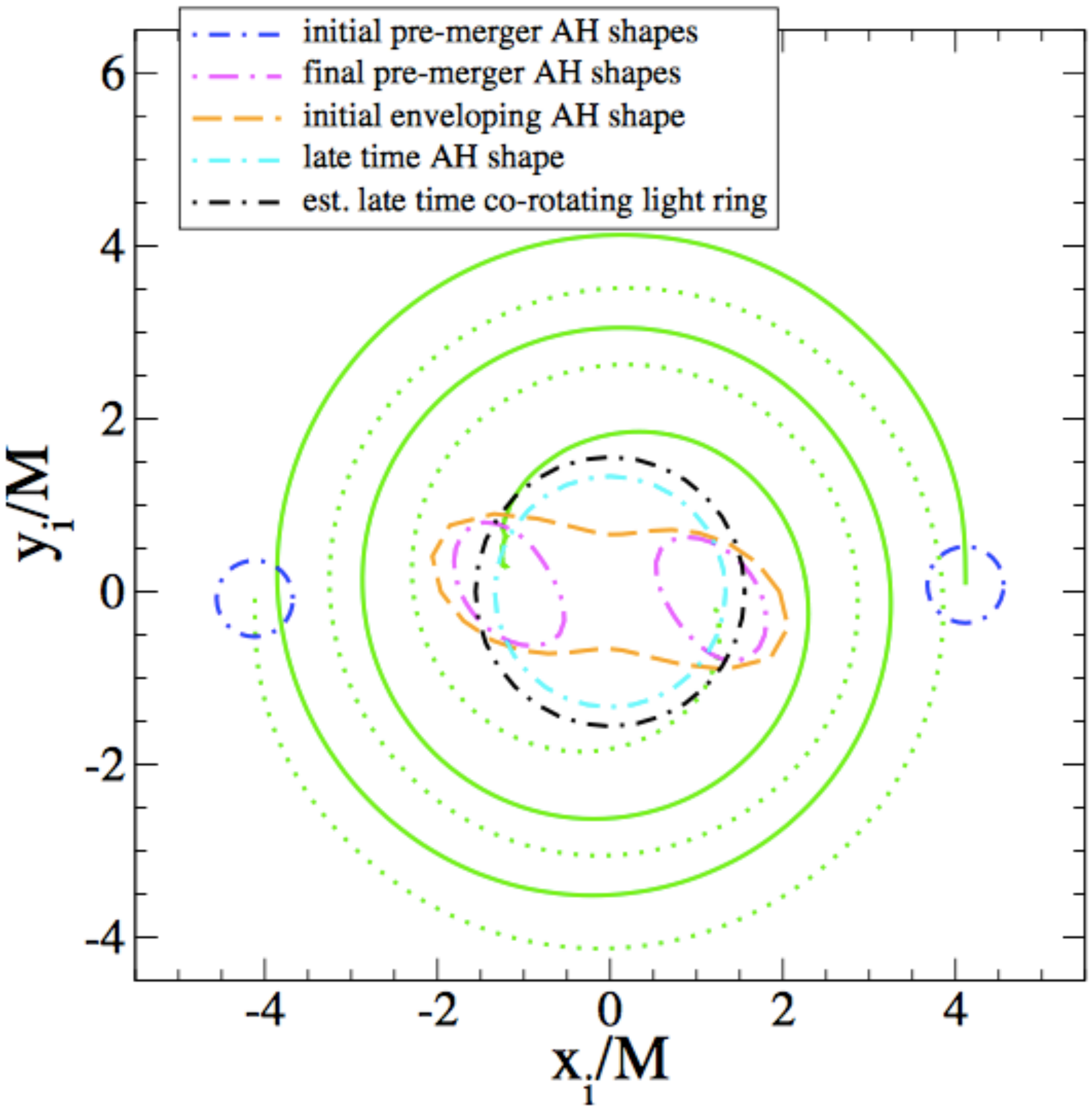


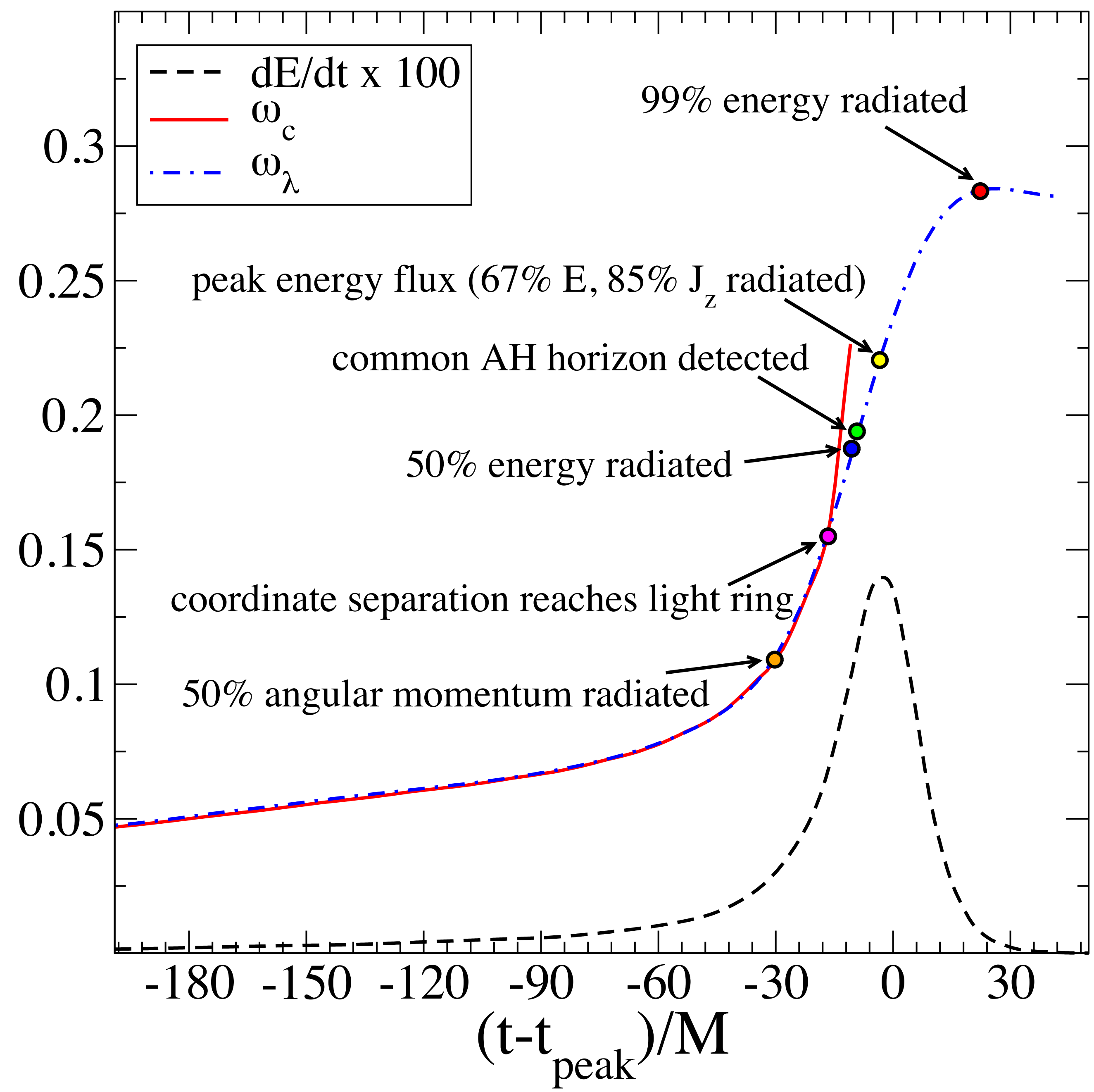
$$h_{22}^{\rm IMR}(t) = h_{22}^{\rm insp-plunge}\,\theta(t_{\rm match}-t) + h_{22}^{\rm merger-RD}\theta(t-t_{\rm match})$$

$$h_{22}^{\rm merger-RD}(t) = \sum_{n=0}^{N-1} A_n\, e^{-i\sigma_{22n}(t-t_{\rm match})} \qquad \sigma_{22n} \equiv \omega_{22n} - i/\tau_{22n}$$

The (plunge and) merger in first NR simulations

(AB, Cook & Pretorius 07)



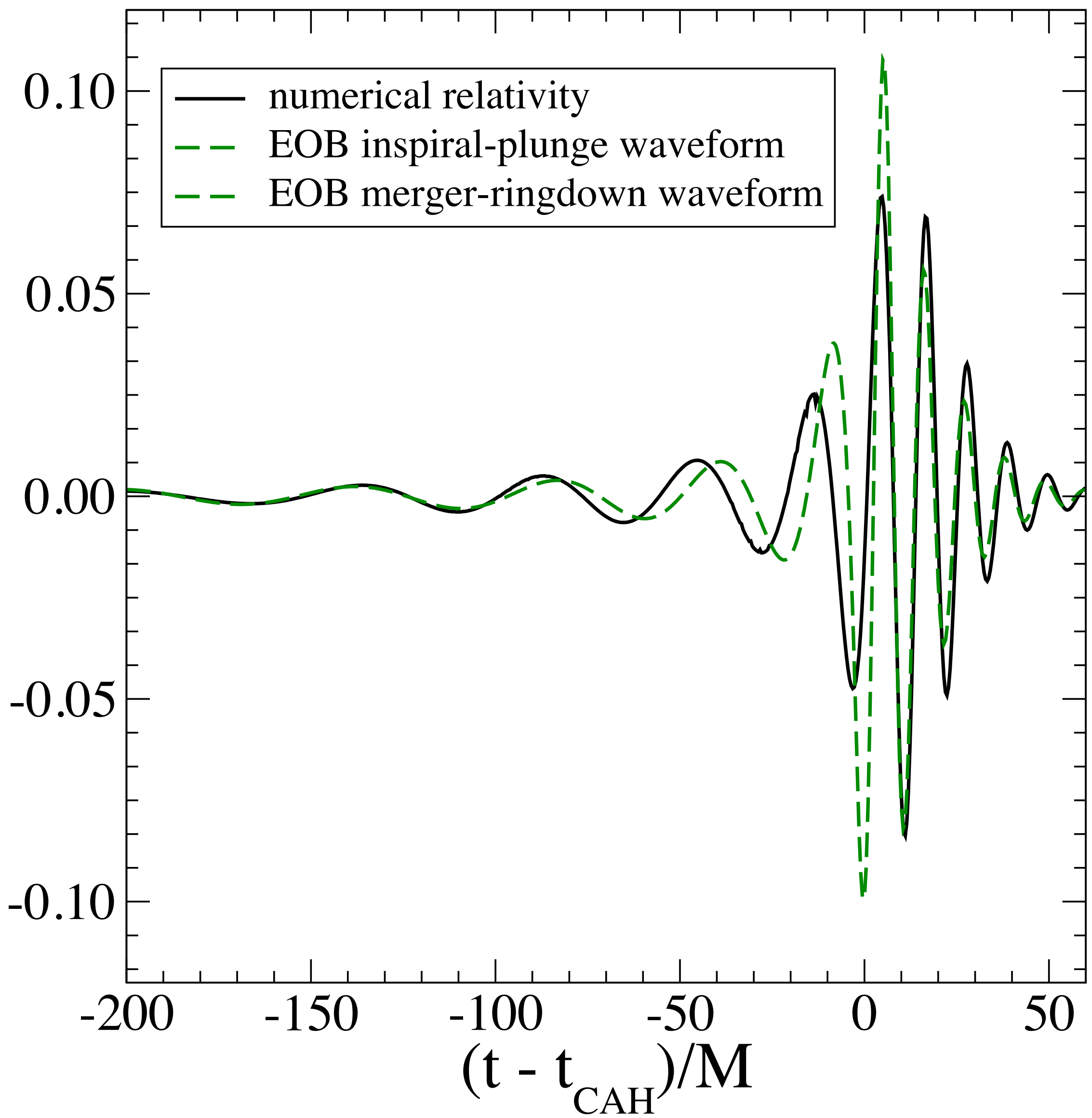


- Very **short/simple transition** plunge-merger-ringdown

- Energy **quickly released** during merger: 2%-12%M

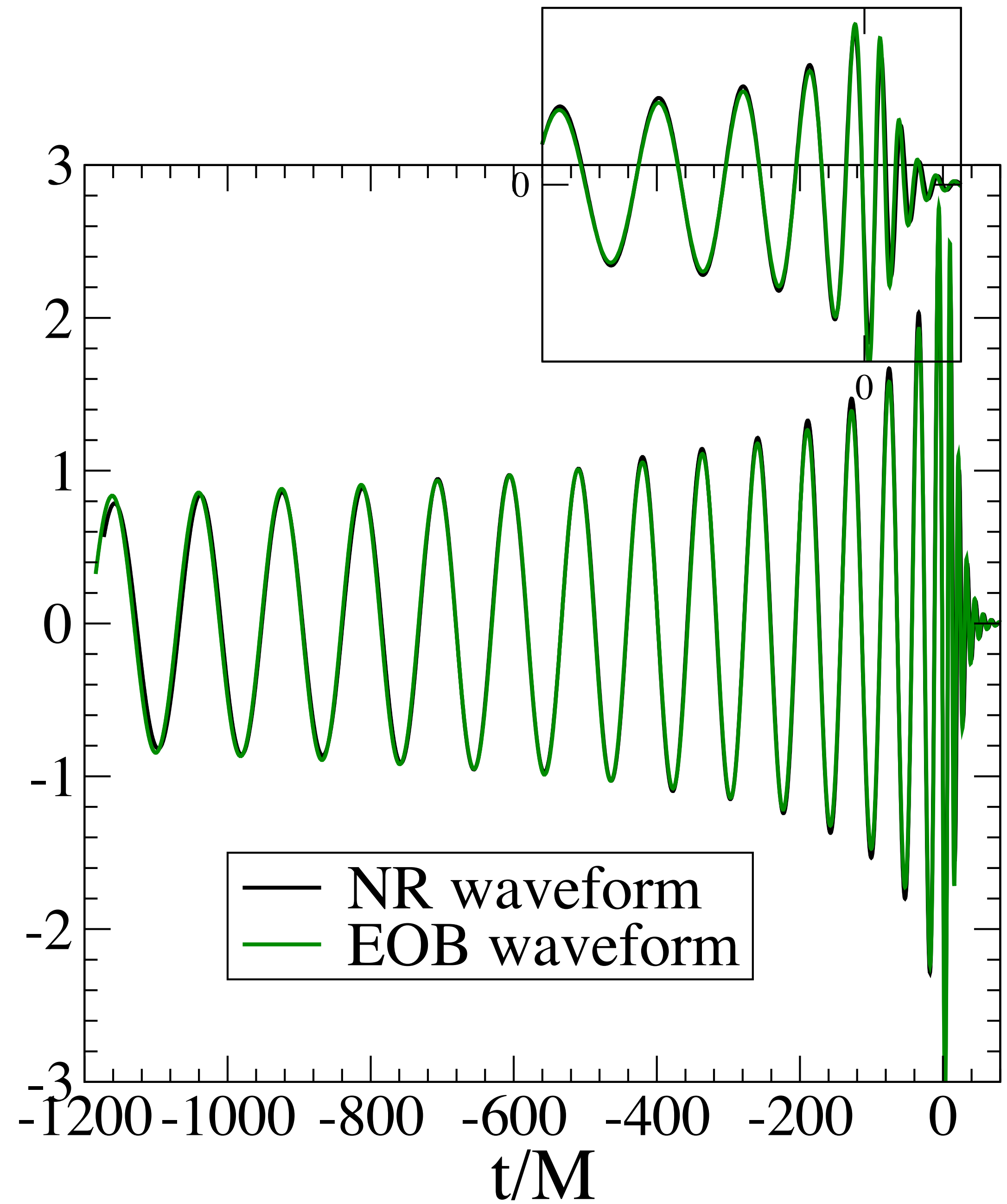
First comparison/calibration between NR and EOB model

- **Uncalibrated** EOB waveform at 3.5PN order



(AB, Cook & Pretorius 07)

- **Calibrated** EOBNR waveform



(AB, Pan, Baker, Centrella, Kelly at al. 08)

- EOBNRv1 waveforms used in **iLIGO BBH searches**

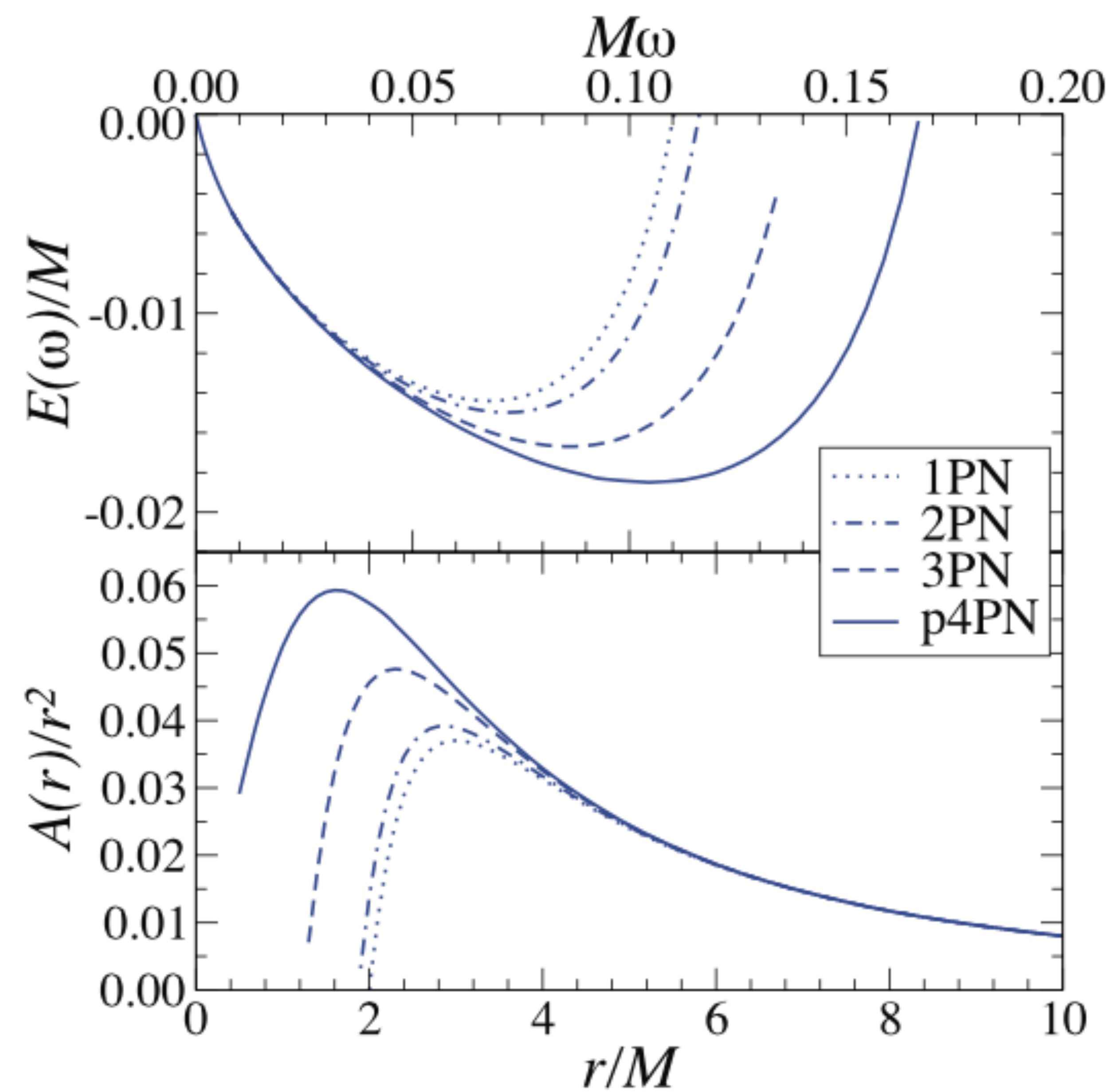
First calibration between NR and EOB model

$$\widehat{\omega} = \frac{\partial \widehat{H}}{\partial p_\varphi}(r, p_r, p_\varphi)$$

$$\omega(t) = \frac{A(r)}{r^2} \frac{p_\varphi}{\eta \hat{H}_{\rm real} \hat{H}_{\rm eff}}$$

$$\omega_{\rm plunge}(t) = \frac{A(r)}{r^2} \, {\rm const} \, ,$$
$${\rm const} = \left[\frac{p_\varphi}{\eta \hat{H}_{\rm real} \hat{H}_{\rm eff}} \right]_{\rm LSO}$$

- **Calibrated** EOBNR waveforms

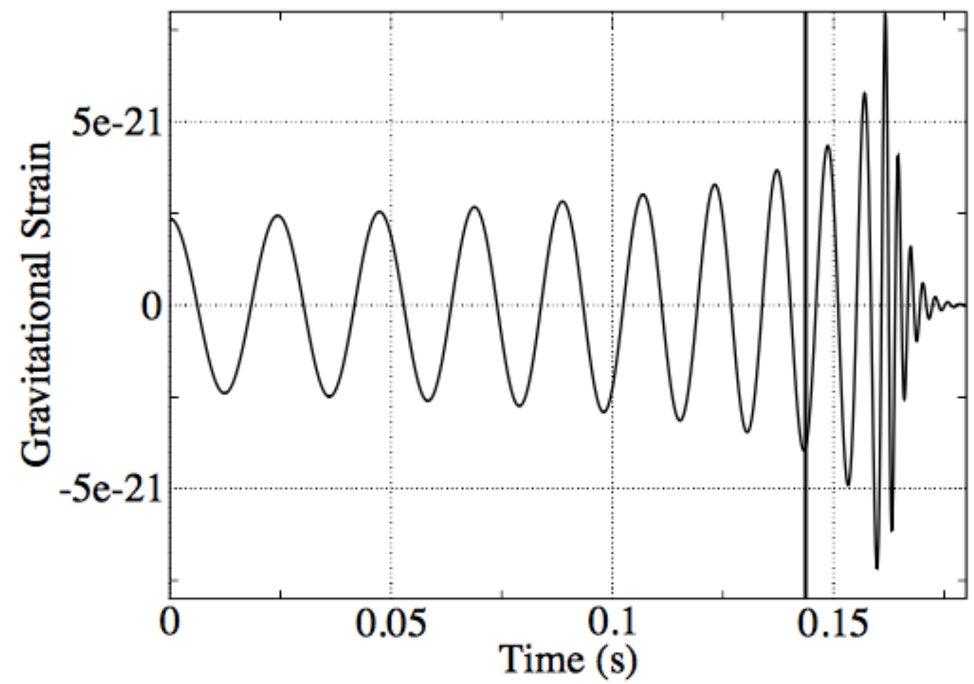


(AB, Pan, Baker, Centrella, Kelly at al. 08)

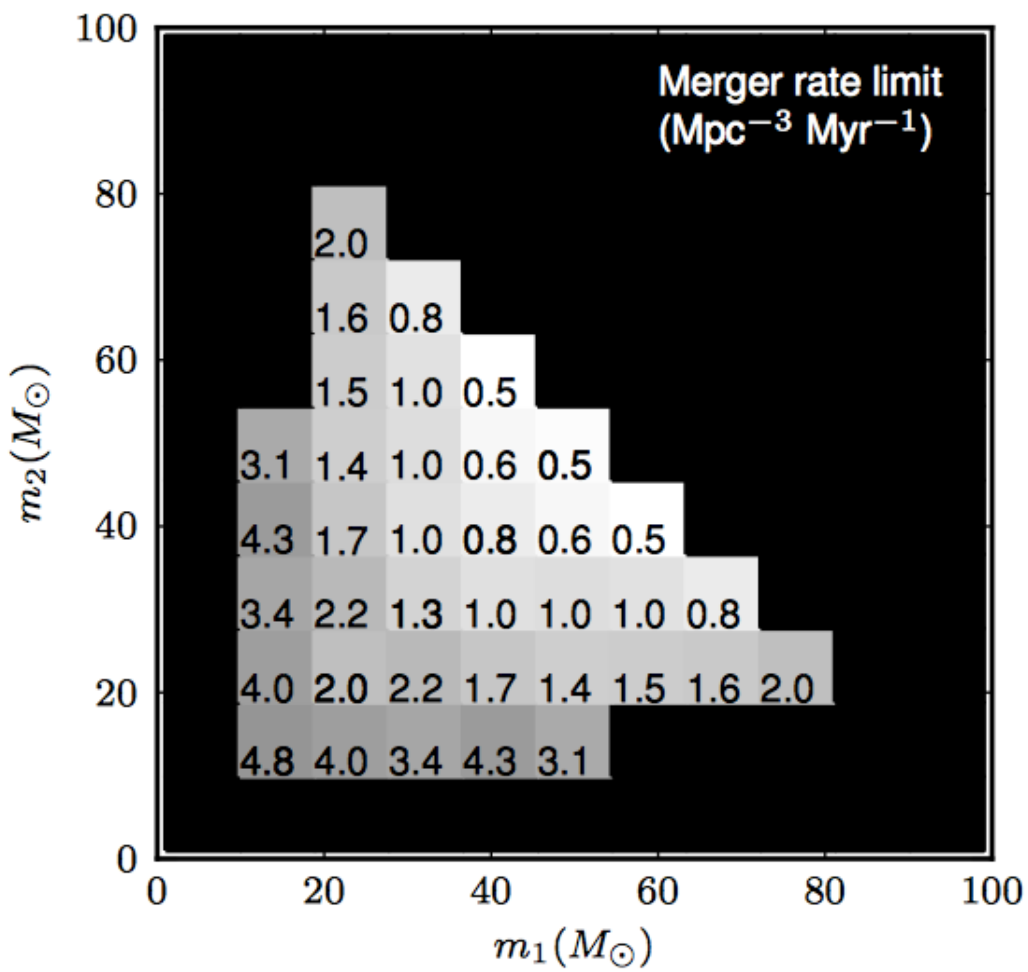
- EOBNRv1 waveforms used in **iLIGO BBH searches**

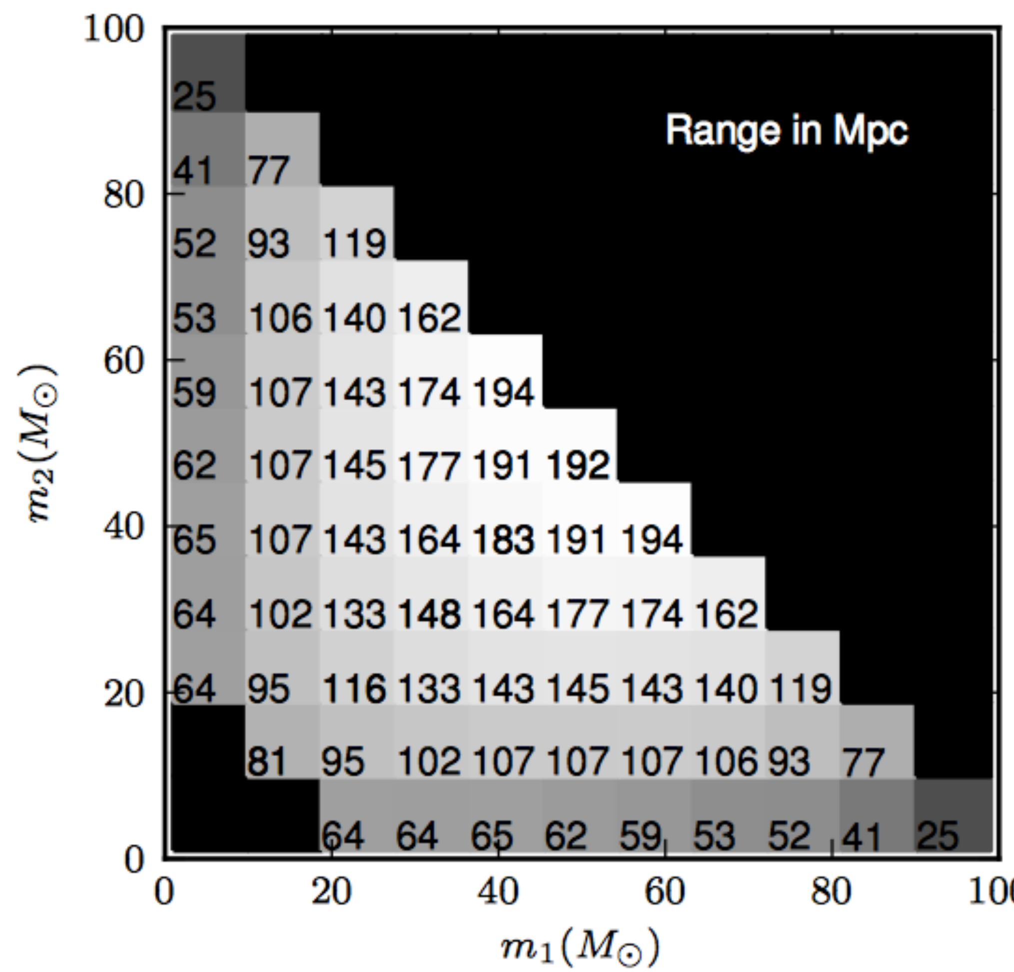
BBH searches in initial LIGO (S5 & S6 runs)

(Abadie et al. PRD83 (2011) 122005,
Abadie et al. PRD87 (2013) 022002)



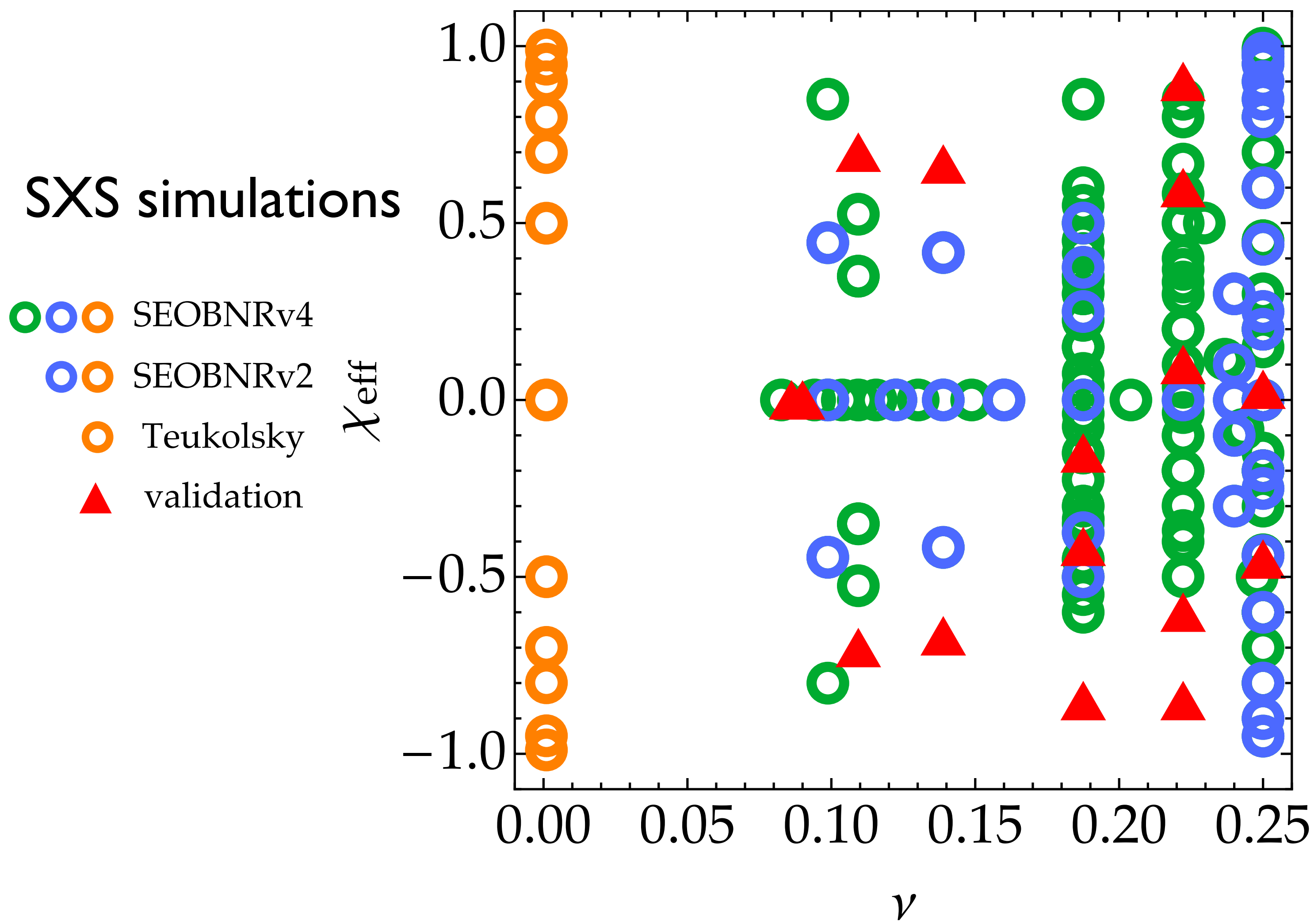
- First **upper limits** from iLIGO





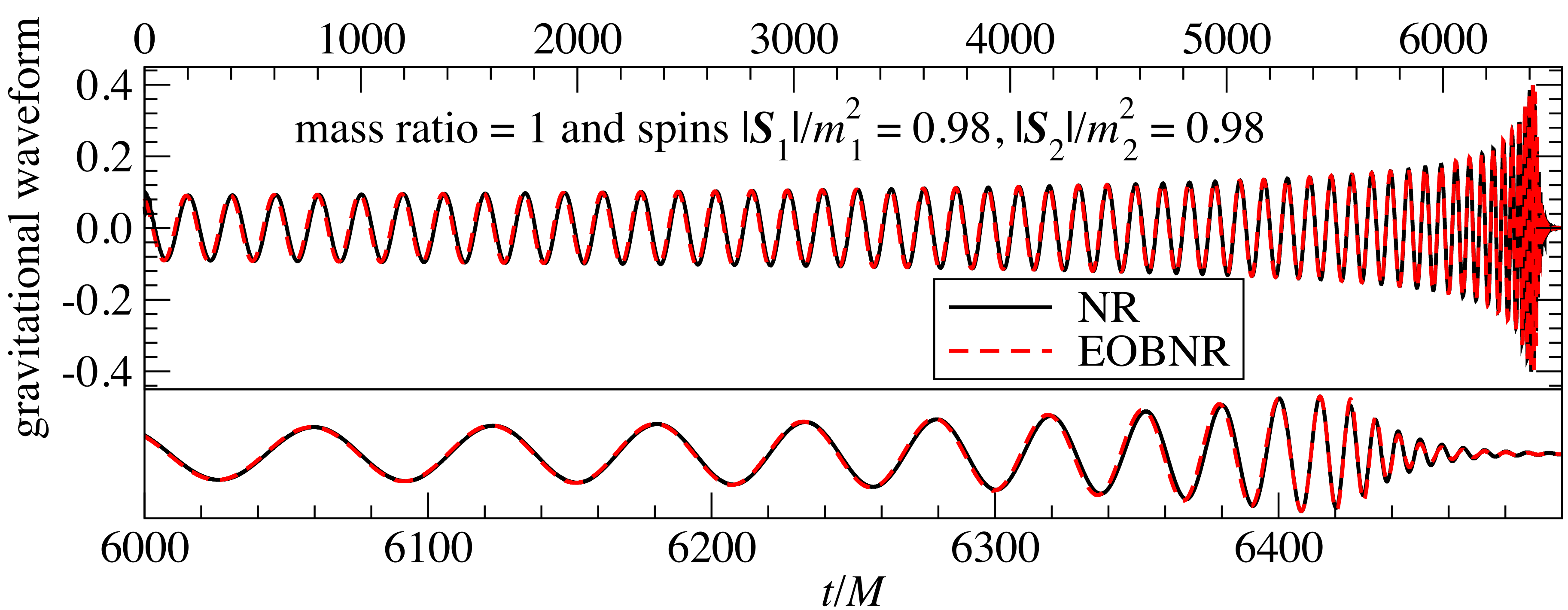
Calibration of EOBNR for O1 & O2 searches/follow-up analyses

141 SXS simulations



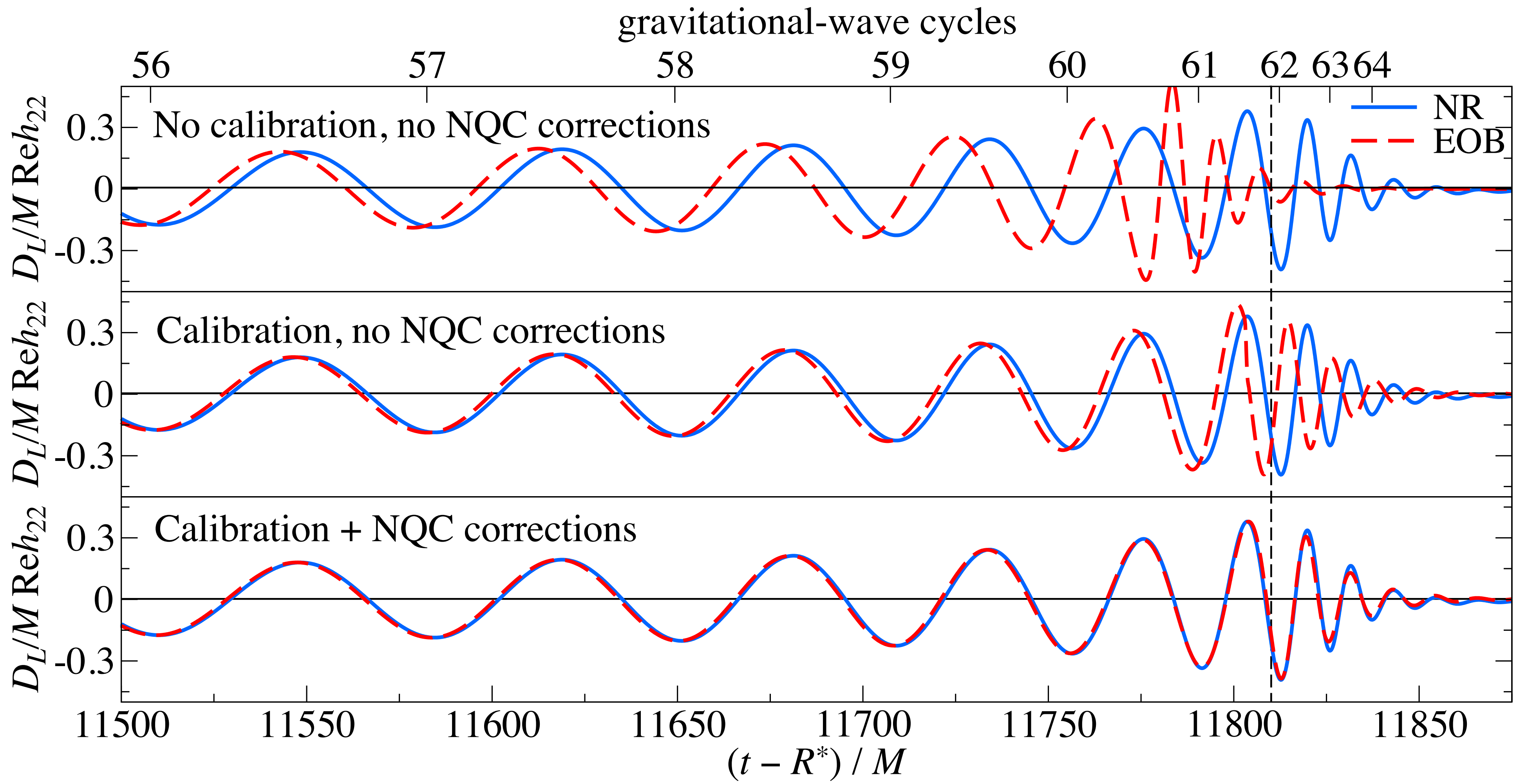
(Pan, AB et al. 13, Taracchini, AB, Pan, Hinderer & SXS 14, Puerrer 15)

(Bohe', Shao, Taracchini, AB & SXS 16, Babak et al. 16)



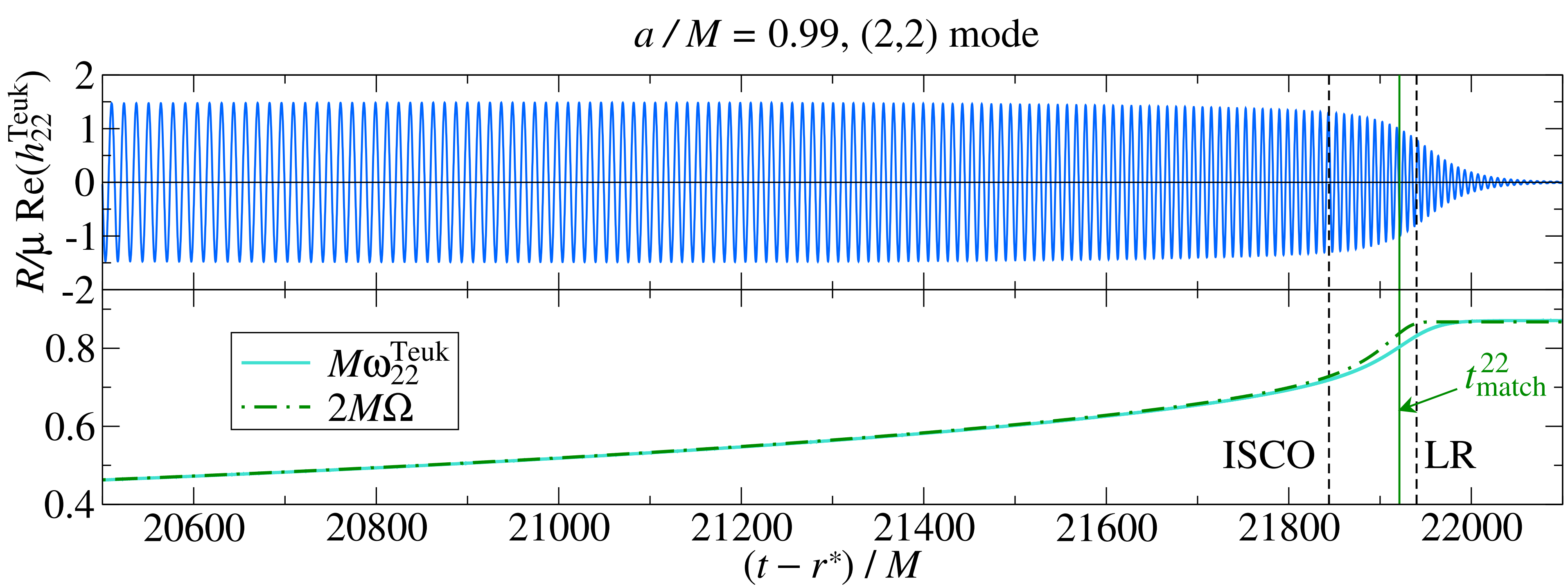
Completing EOB waveforms using NR/perturbation theory information

- We calibrate to inspiral-merger-ringdown **NR** waveforms.



(credit: Taracchini)

- We calibrate to merger-ringdown **Teukolsky-equation based** waveforms.



(credit: Taracchini)

Completing EOB waveforms using NR/perturbation theory information

$$A_\nu(r) = 1 - \frac{2M}{r} + \frac{2M^3\nu}{r^3} + \left(\frac{94}{3} - \frac{41}{32}\pi^2\right)\frac{M^4\nu}{r^4} + \frac{a_5(\nu) + a_5^{\log}(\nu)\log(r)}{r^5} + \frac{a_6(\nu)}{r^6} + \cdots$$

(Damour et al. 07-09, AB et al. 09, Pan et al. 09, Bernuzzi et al. 11, Pan et al. 11)

- **Once a_5 and a_6 are calibrated, the EOB light-ring (peak of orbital frequency) automatically occurs close to the time when $h^{\rm NR}$ reaches its peak**

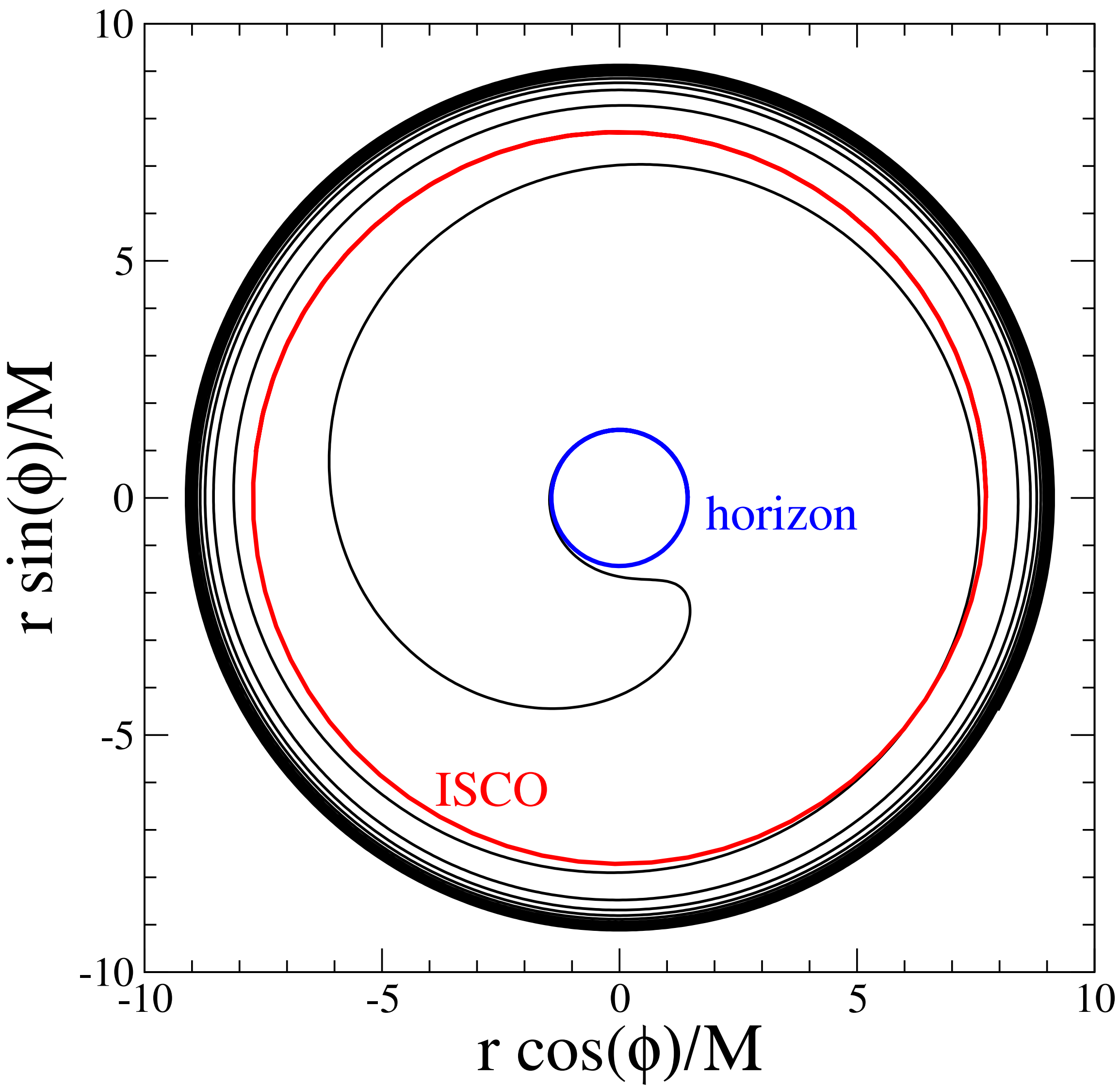
$$h^{\rm NQC} = \left[1 + \frac{p_{r*}^2}{(r\,\Omega)^2}\left(a_1 + a_2\frac{1}{r} + a_3\frac{1}{r^{3/2}}\right)\right]\exp\left[i\left(b_1\frac{p_{r*}}{r\,\Omega} + b_2\frac{p_{r*}^3}{r\,\Omega}\right)\right]$$

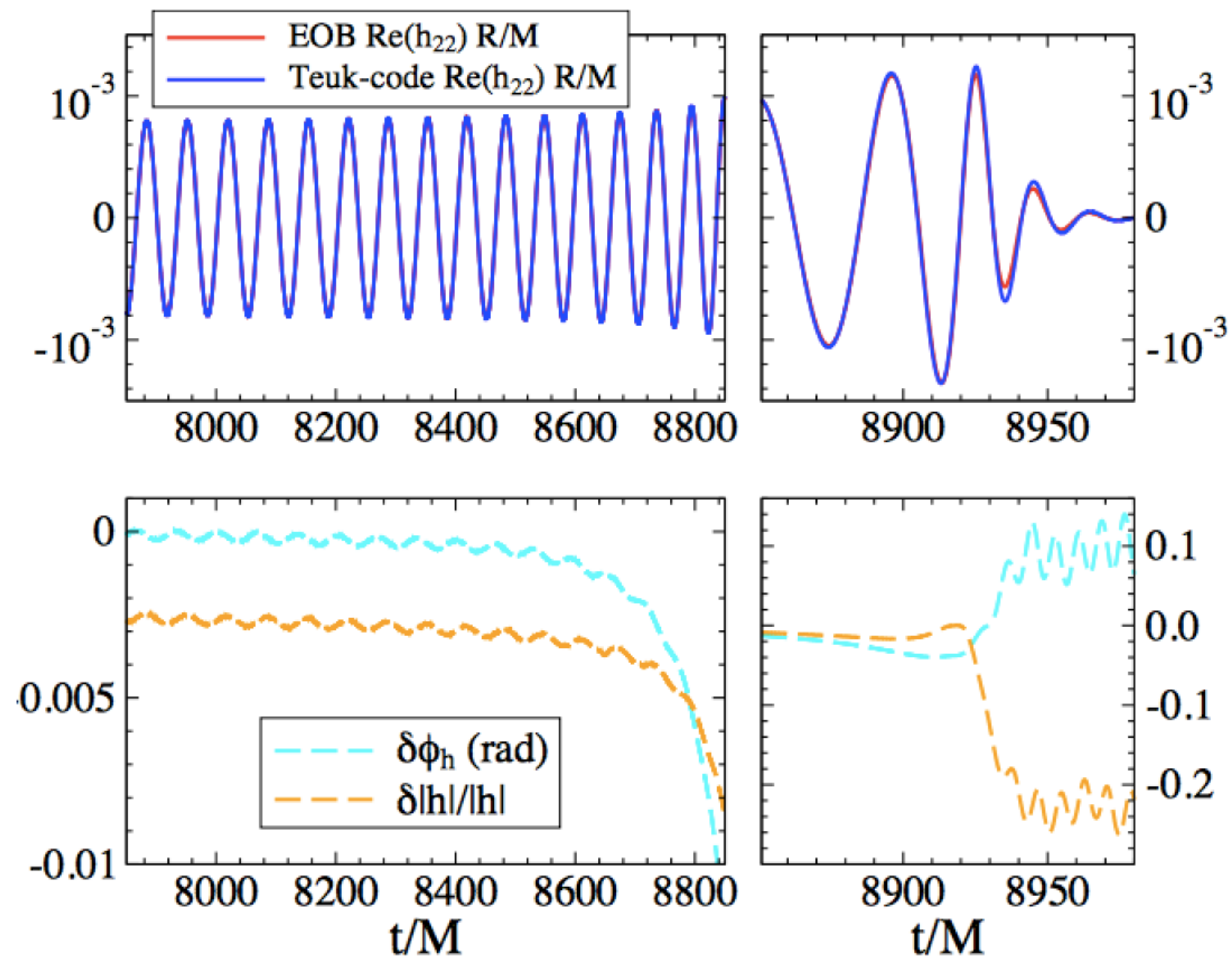
- **a_i, b_i are obtained imposing that the peak of $h^{\rm EOB}$ occurs at the EOB light-ring, its value and its second time derivative, $\omega^{\rm EOB}$, $\dot\omega^{\rm EOB}$, coincide with the NR ones**

$|h^{\rm NR}(t^{\rm peak})|, |\ddot{h}^{\rm NR}(t^{\rm peak})|, \omega^{\rm NR}(t^{\rm peak}), \dot\omega^{\rm NR}(t^{\rm peak}) \Rightarrow$ **modeled as polynomials in ν**

Calibrating EOB to Teukolsky-equation—based waveforms

- Solving **Teukolsky equation** for perturbations in **Kerr spacetime**





- **Retrograde** orbit: BH spin = -0.5

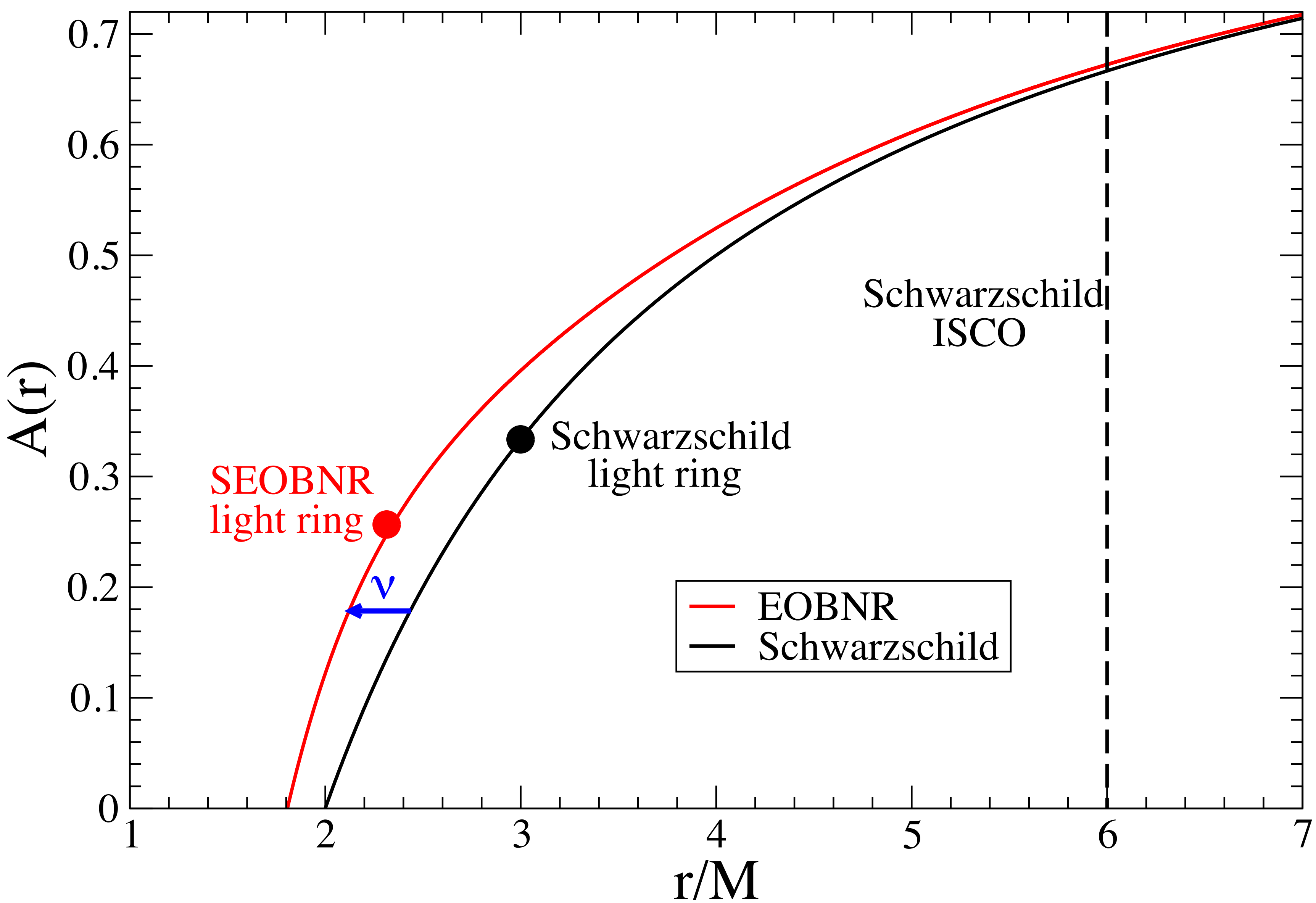
(*Taracchini, AB, Khanna & Hughes 13, Barausse, AB, Hughes & Khanna 11*)

(see also Damour & Nagar 07, *Bernuzzi et al. 10, 11, Harms et al. 14, 16*)

Strong-field effects in binary black holes included in EOB

Finite mass-ratio effects make **gravitational** interaction **less attractive**

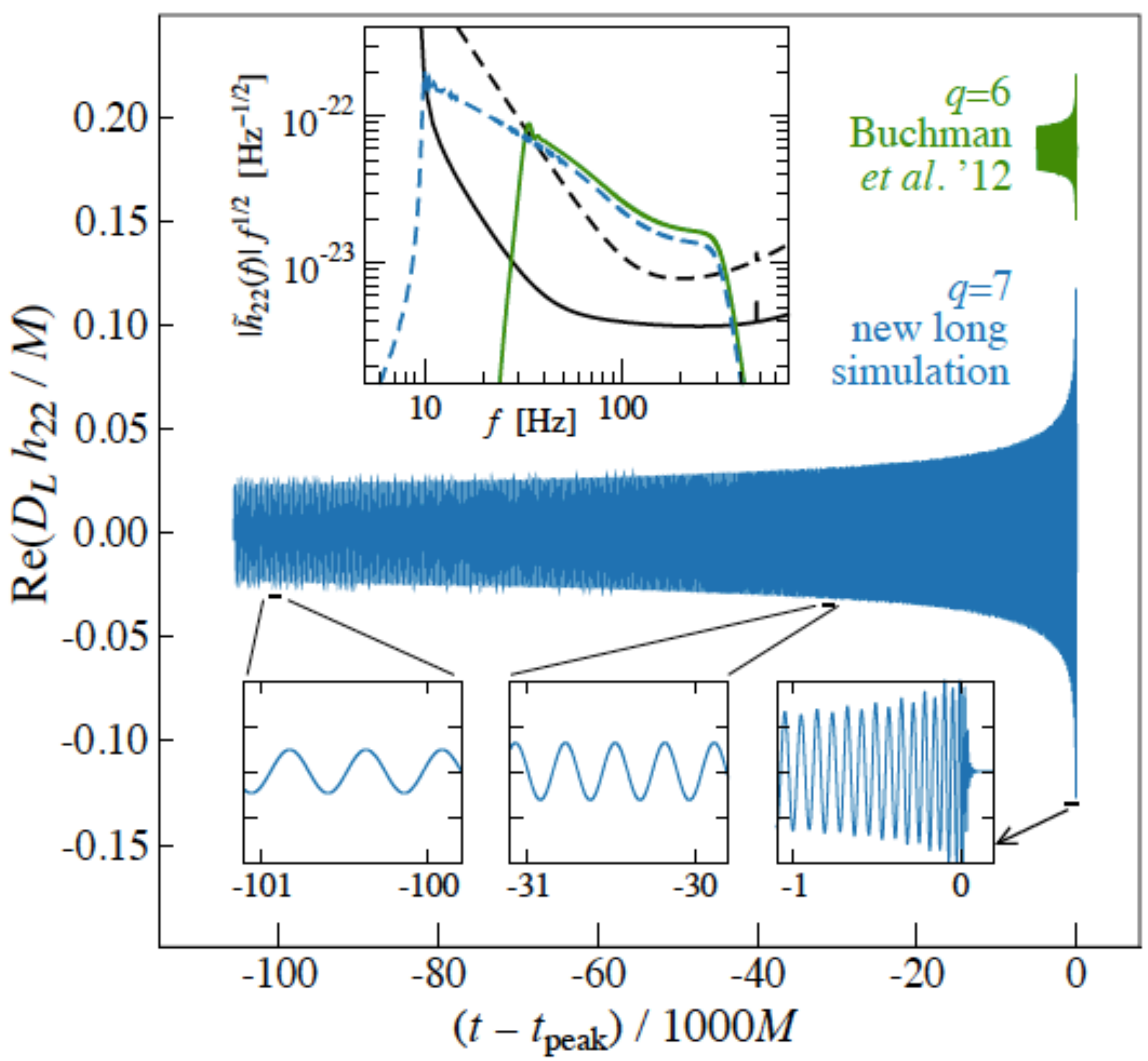
(Taracchini, AB, Pan, Hinderer & SXS 14)

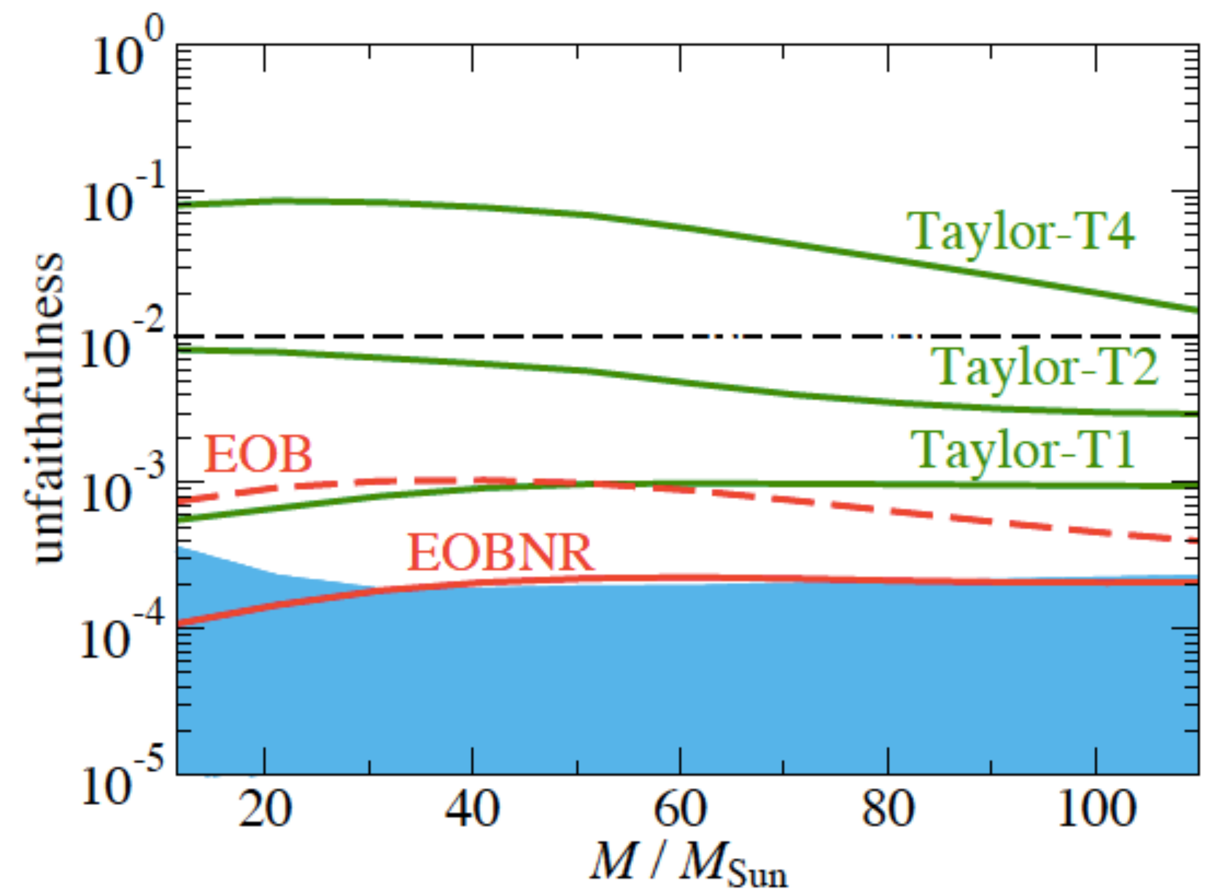


$$A_\nu(r) = 1 - \frac{2M}{r} + \frac{2M^3\nu}{r^3} + \left(\frac{94}{3} - \frac{41}{32}\pi^2\right)\frac{M^4\nu}{r^4} + \frac{a_5(\nu) + a_5^{\log}(\nu)\log(r)}{r^5} + \frac{a_6(\nu)}{r^6} + \cdots$$

Comparing uncalibrated EOB to ultralong NR waveform

- **376 GW cycles: 20 times longer than previously achieved**





- **NR waveform covers** *entire* **band for** $M > 45 M_\odot$
- **"EOBNR" means calibrated to much shorter NR waveforms with mass ratio** $\neq$ **7**

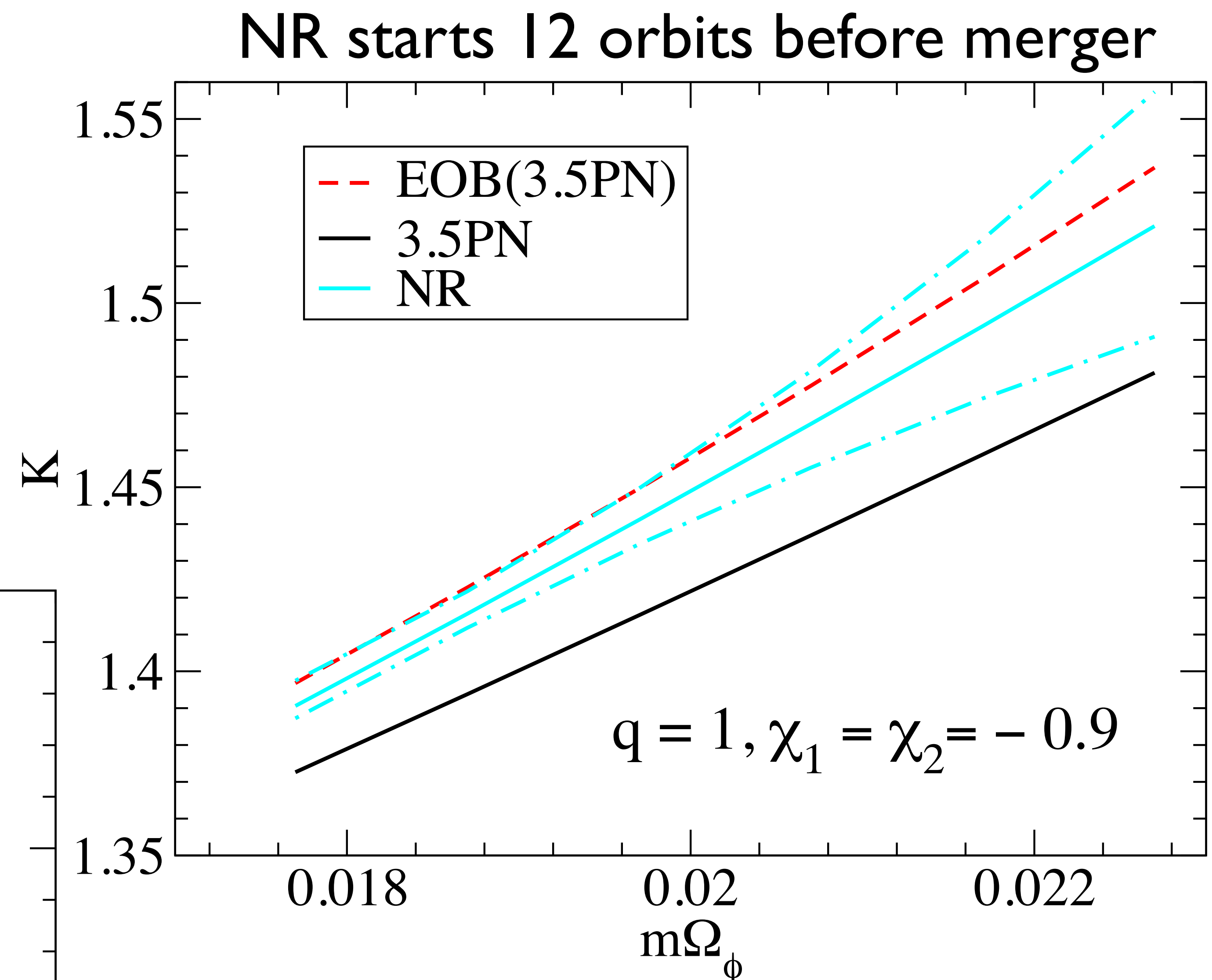
(Szilagyi, Blackman, AB, Taracchini et al. 15)

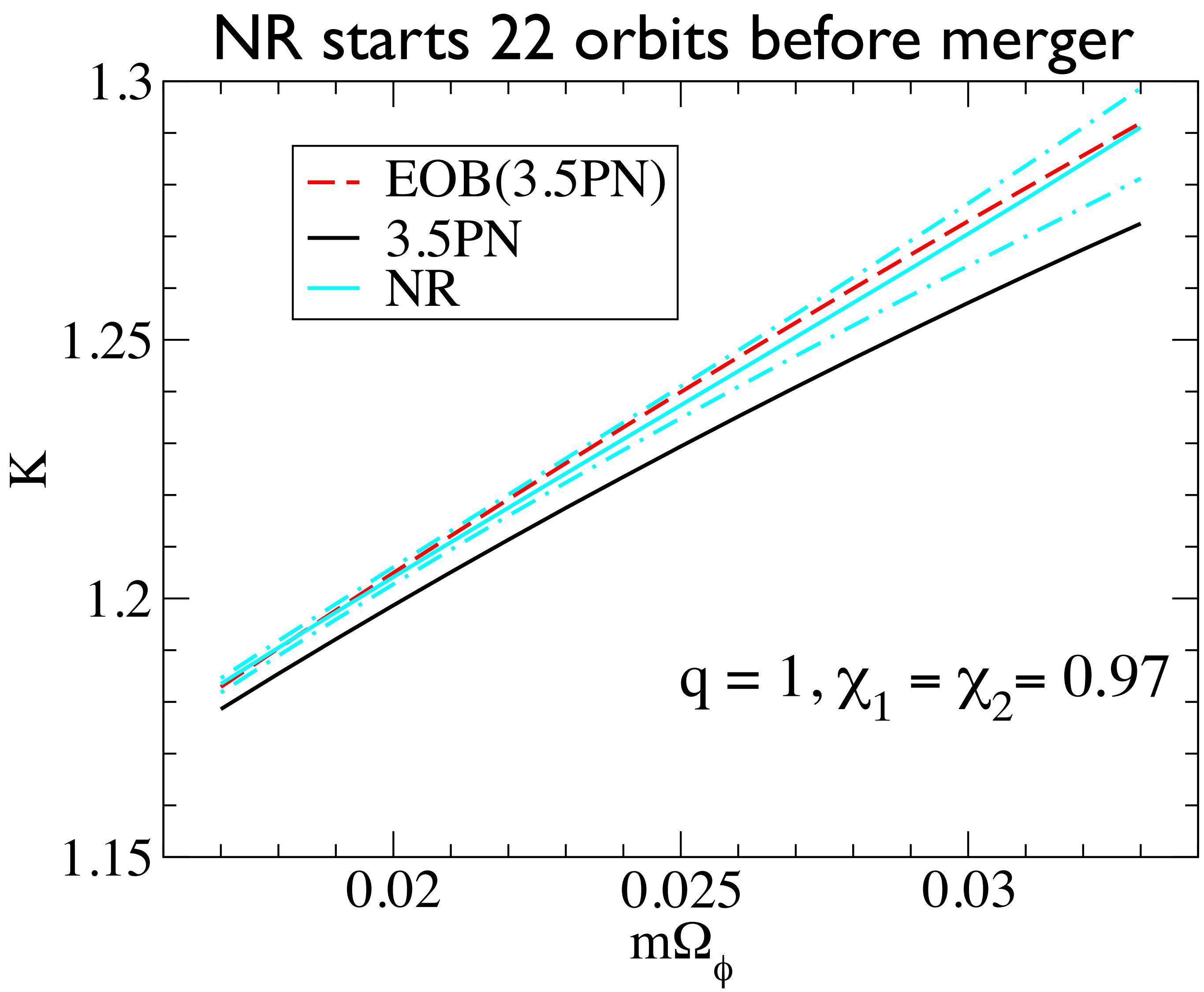
Comparing NR, PN & EOB beyond waveforms

- **Periastron advance**

$$K = 1 + \frac{\Delta\Phi}{2\pi} = \frac{\Omega_\phi}{\Omega_r}$$

(Hinderer et al. 13, Le Tiec et al. 11, 13)

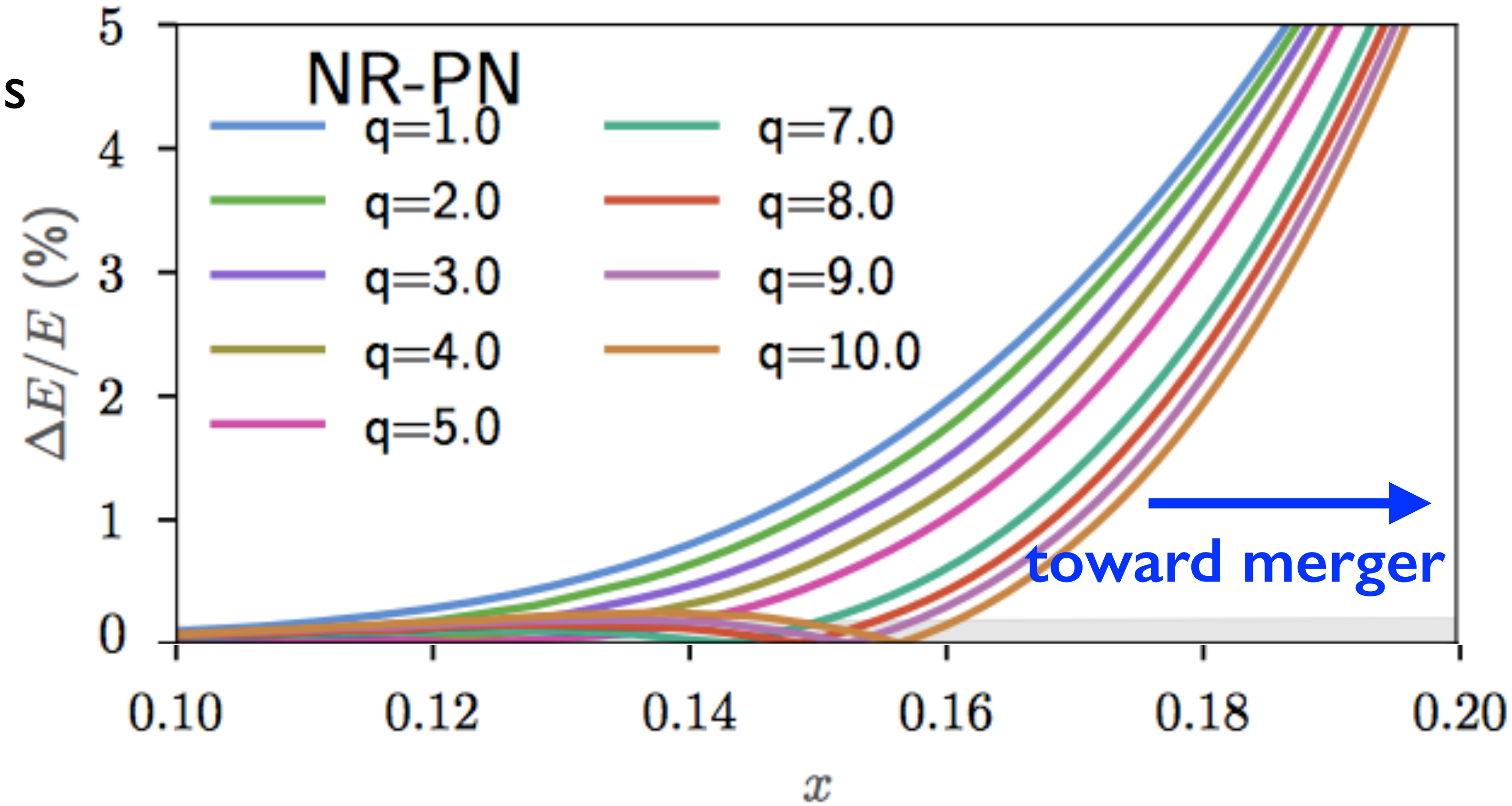




- **Energy/angular momentum** *(Damour et al. 11, Le Tiec et al. 11, Ossokine, Dietrich et al. 17)*
- **Scattering angle** of hyperbolic encounters *(Damour et al. 14)*

Comparing the energetics of NR against PN

non-spinning BHs

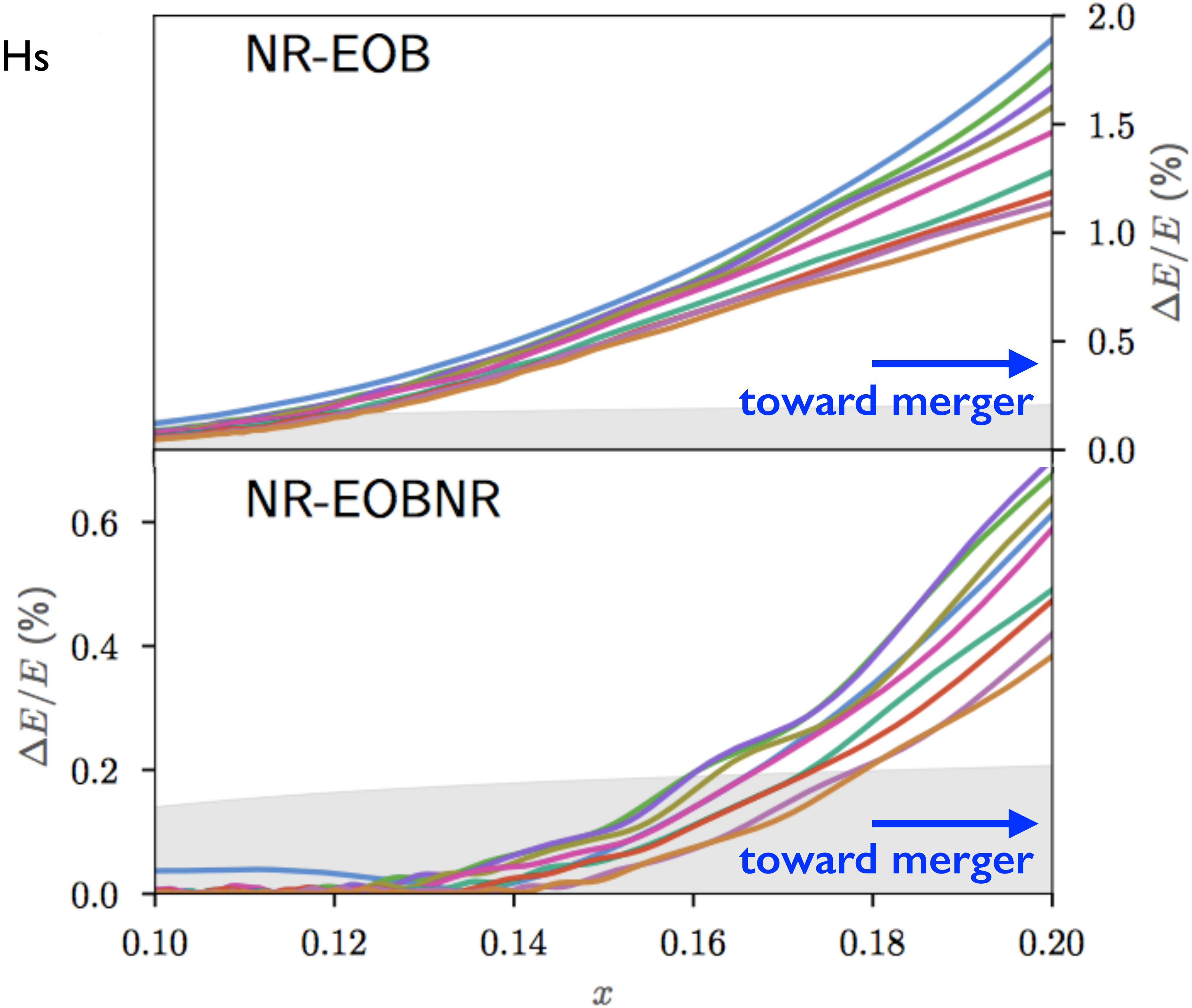


(Ossokine, Dietrich et al. 17)

$$x = (M\Omega)^{1/3}$$

Comparing the energetics of NR against EOB & EOBNR

non-spinning BHs



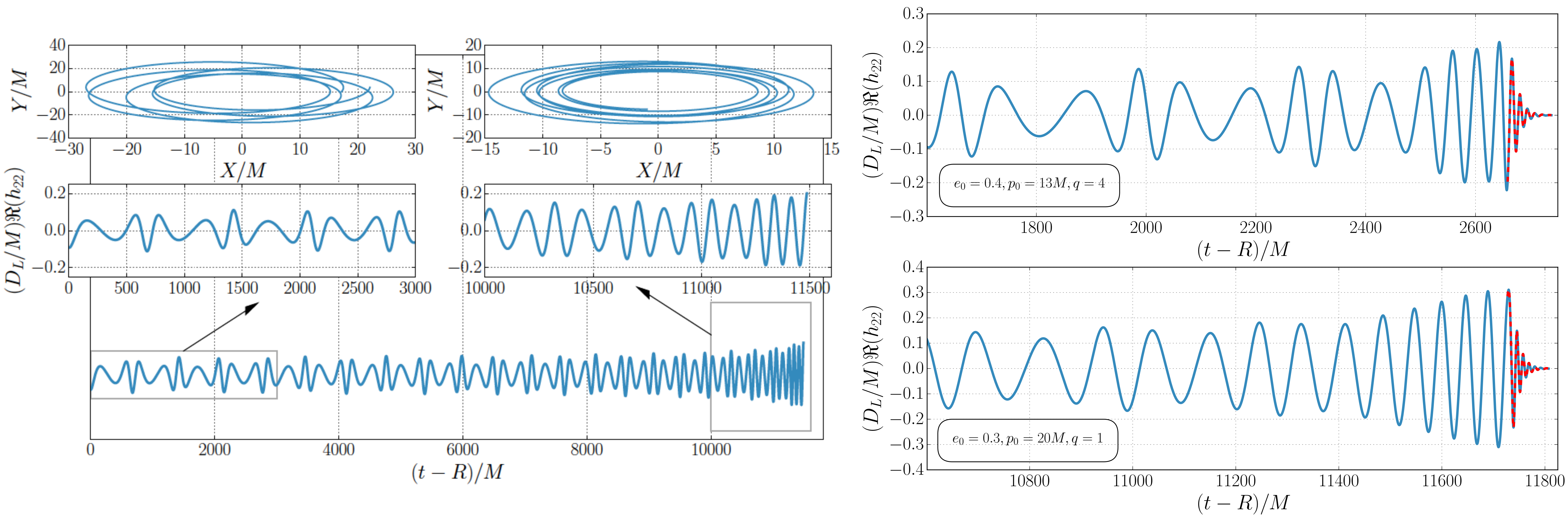
(Ossokine, Dietrich et al. 17)

$$x = (M\Omega)^{1/3}$$

Eccentric waveform models

- **EOB dynamics** & **waveform** extended to ***any* eccentricity value** for **nonspinning binaries.**
- Binary's **degrees of freedom** are divided into a **set of phase variables**, and a **set of quantities that are constant** in the absence of radiation reaction.

(Hinderer & Babak 17)



(for eccentricity modeling see also Huerta et al. 14, 16; Hinder et al. 17; Loutrel & Yunes 16, 17)