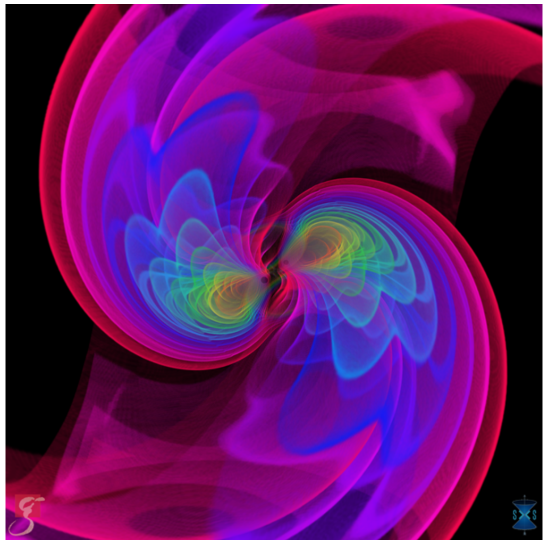

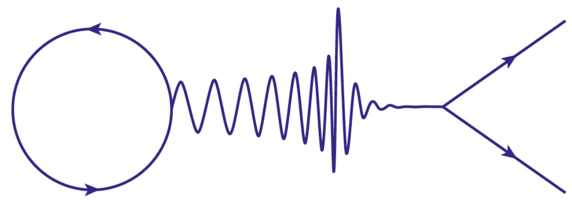

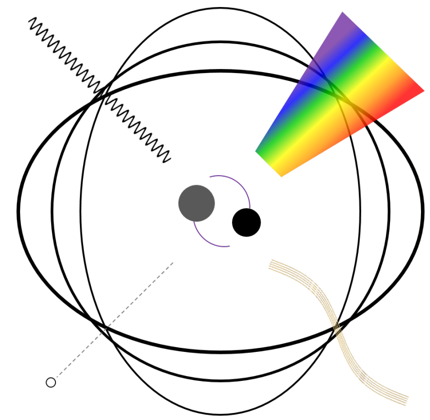

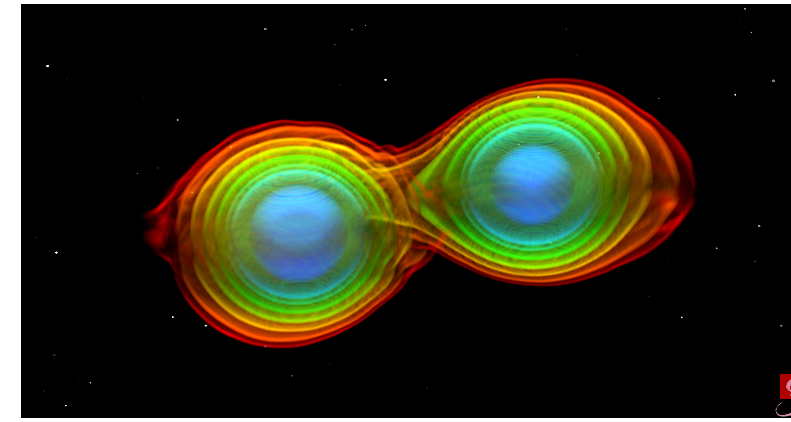

Effective-One-Body Theory & Applications to Gravitational-Wave Observations

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Outline

- **Lectures I-II:** Basics of gravitational waves
- **Lectures III-IV:** Motivations and development of effective-one-body (EOB) theory (two-body dynamics and waveforms)
- **Lecture V:** Using waveform models to infer astrophysical and cosmological information of gravitational-wave observations
- **Lecture VI:** Using waveform models to probe dynamical gravity and extreme matter with gravitational-wave observations

(NR simulation: Ossokine, AB & SXS @AEI)

(visualization credit: Benger @ Airborne Hydro Mapping Software & Haas @AEI)

References:

- **M. Maggiore's books**: "Gravitational Waves Volume 1: Theory and Experiments" (2007) & "Gravitational Waves Volume II: Astrophysics and Cosmology" (2018).
- **E. Poisson & C. Will's book**: "Gravity" (2015).
- **E.E. Flanagan & S.A. Hughes' review**: arXiv:0501041.
- **AB's Les Houches School Proceedings**: arXiv:0709.4682.
- **AB & B. Sathyaprakash's review**: arXiv:1410.7832.
- **UMD/AEI graduate course on GW Physics & Astrophysics** taught in Winter-Spring 2017: http://www.aei.mpg.de/2000472.

Einstein equations and notations

- Einstein-Hilbert action:

$$S_g = \frac{c^3}{16\pi G}\int d^4x\sqrt{-g}\,R, \qquad d^4x = c\,dt\,d^3x, \quad g = \det(g_{\mu\nu})$$

$$\eta_{\mu\nu} = (-,+,+,+) \qquad x^\mu = (x^0, \mathbf{x}) = (c\,t, \mathbf{x})$$

$$\mu,\nu = 0,\cdots 3\,, \quad i,j = 1,2,3 \qquad \partial_\mu = (\partial_0, \partial_i)$$

partial derivative denoted also with colon
covariant derivative denoted with semicolon

- Matter action/energy-momentum tensor:

$$\delta S_{\rm matter} = \frac{1}{2c}\int d^4x\sqrt{-g}\,T^{\mu\nu}\,\delta g_{\mu\nu}$$

$$S = S_g + S_{\rm matter}$$

Einstein equations and notations (contd.)

- Curvature tensor:

$$R^{\nu}{}_{\mu\rho\sigma} = \partial_\rho \Gamma^{\nu}_{\mu\sigma} - \partial_\sigma \Gamma^{\nu}_{\mu\rho} + \Gamma^{\nu}_{\lambda\rho} \Gamma^{\lambda}_{\mu\sigma} - \Gamma^{\nu}_{\lambda\sigma} \Gamma^{\lambda}_{\mu\rho}$$

- Affine connection:

$$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2} g^{\mu\lambda} \left(\partial_\rho g_{\lambda\nu} + \partial_\nu g_{\lambda\rho} - \partial_\lambda g_{\nu\rho} \right)$$

- Ricci tensor and scalar:

$$R_{\mu\nu} = g^{\rho\sigma} R_{\rho\mu\sigma\nu} \qquad R = g^{\mu\nu} R_{\mu\nu}$$

- Bianchi identity:

$$R^{\lambda}{}_{\mu\nu\rho;\sigma} + R^{\lambda}{}_{\mu\sigma\nu;\rho} + R^{\lambda}{}_{\mu\rho\sigma;\nu} = 0$$

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho}\,, \quad R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}\,, \quad R_{\mu\nu\rho\sigma} + R_{\mu\sigma\nu\rho} + R_{\mu\rho\sigma\nu} = 0$$

Einstein equations and notations (contd.)

- Varying total action with respect to $g_{\mu\nu}$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu}$$

(Einstein 1915)

Non-linear equations with well posed initial value structure, i.e., they determine future values of $g_{\mu\nu}$ from given initial values.

(Choquet-Bruhat 1952)

Ten differential equations, using symmetry of tensors $R_{\mu\nu}$, $T_{\mu\nu}$.

Using Bianchi's identity, we have six differential equations.

- GR invariant under coordinate transformations $x^\mu \rightarrow x'^\mu(x)$

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x)$$

Gravitational waves: signature of dynamical spacetime

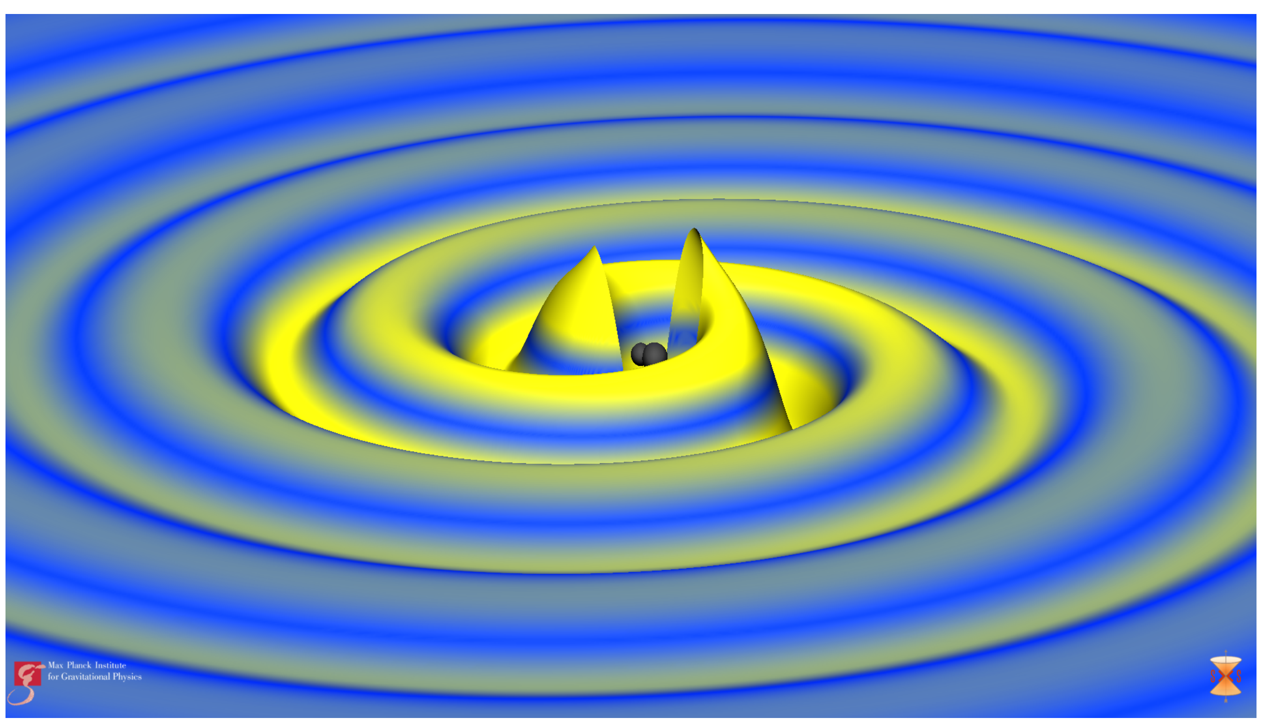
- In 1916 Einstein predicted **existence of gravitational waves:**

Linearized gravity (weak field): $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ $\quad |h_{\mu\nu}| \ll 1$

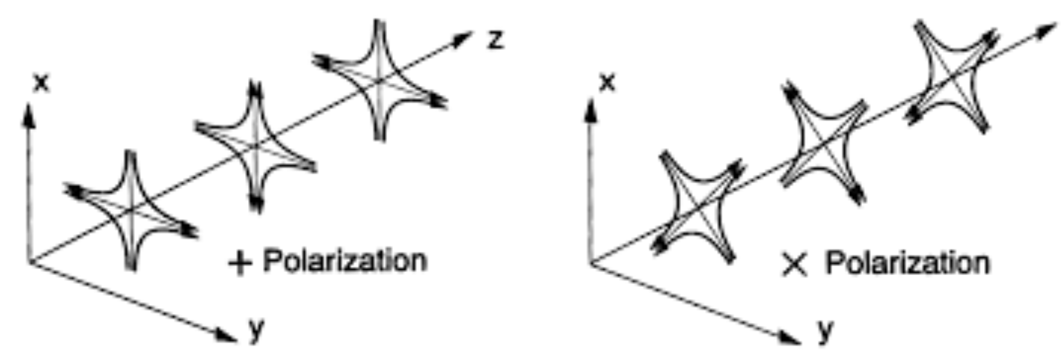
$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G}{c^4} T_{\mu\nu} \longrightarrow \Box \bar{h}_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

- Distribution of **mass deforms spacetime** geometry in its neighborhood. **Deformations propagate** away at speed of light **in form of waves** whose oscillations reflect temporal variation of matter distribution.

(visualization: Dietrich @ AEI)



Ripples in the curvature of spacetime



Two radiative degrees of freedom

First paper by Einstein on gravitational waves: 1916

Approximative Integration of the Field Equations of Gravitation

by A. Einstein

For the treatment of the special (not basic) problems in gravitational theory one can be satisfied with a first approximation of the $g_{\mu\nu}$. The same reasons as in the special theory of relativity make it advantageous to use the imaginary time variable $x_4 = it$. By "first approximation" we mean that the quantities $\gamma_{\mu\nu}$, defined by the equation

$$g_{\mu\nu} = -\delta_{\mu\nu} + \gamma_{\mu\nu}, \tag{1}$$

$$A = \frac{\kappa}{24\pi}\sum_{\alpha\beta}\left(\frac{\partial^3 J_{\alpha\beta}}{\partial t^3}\right)^2. \tag{21}$$

This expression would get an additional factor $\frac{1}{c^4}$ if we would measure time in seconds and energy in Erg. Considering furthermore that $\kappa = 1.87 \cdot 10^{-27}$, it is obvious that A has, in all imaginable cases, a practically vanishing value.

Nevertheless, due to the inneratomic movement of electrons, atoms would have to radiate not only electromagnetic but also gravitational energy, if only in tiny amounts. As this is hardly true in nature, it appears that quantum theory would have to modify not only MAXWELLIAN electrodynamics, but also the new theory of gravitation.

Second paper by Einstein on gravitational waves: 1918

The important question of how gravitational fields propagate was treated by me in an academy paper one and a half years ago.[1] However, I have to return to the subject matter since my former presentation is not sufficiently transparent and, furthermore, is marred by a regrettable error in calculation.

If one forms the mean value of S over all directions of space for a fixed value of $A_{\mu\nu}$, one obtains the mean density $\bar{S}$ of the radiation. Finally, $\bar{S}$ multiplied by $4\pi R^2$ is the energy loss (per time unit) of the mechanical system due to gravitational waves. The calculation finds

$$4\pi R^2 \bar{S} = \frac{\kappa}{80\pi}\left[\sum_{\mu\nu} \dddot{\mathfrak{J}}_{\mu\nu}^{2} - \frac{1}{3}\left(\sum_{\mu} \dddot{\mathfrak{J}}_{\mu\mu}\right)^2\right]. \qquad (30)$$

wrong by a factor 2!

This result shows that a mechanical system which permanently retains spherical symmetry cannot radiate; this is in contrast to the result of the previous paper, marred by an error in calculation.

Linearization of Einstein equations

- We assume there is a coordinate frame such that:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \qquad |h_{\mu\nu}| \ll 1$$

- By choosing particular frame we broke invariance under coordinate transformations, however, residual gauge symmetry remains:

$$x^\mu \to x'^\mu = x^\mu + \xi^\mu(x), \qquad |\partial_\mu \xi_\nu| \leq |h_{\mu\nu}|$$

$$g'_{\mu\nu} = \eta_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu + h_{\mu\nu} + \mathcal{O}(\partial\xi^2)$$

$$h'_{\mu\nu} = h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu} \qquad g'_{\mu\nu} = \eta_{\mu\nu} + h'_{\mu\nu}\,, \qquad |h'_{\mu\nu}| \ll 1$$

- $h_{\mu\nu}$ tensor under Poincare' group.

Linearization of Einstein equations (contd.)

- At linear order in $h_{\mu\nu}$

$$R^{\nu}_{\mu\rho\sigma} = \partial_\rho \Gamma^{\nu}_{\mu\sigma} - \partial_\sigma \Gamma^{\nu}_{\mu\rho} + \mathcal{O}(h^2)$$

$$\Gamma^{\nu}_{\mu\rho} = \frac{1}{2}\eta^{\nu\lambda}\left(\partial_\rho h_{\lambda\mu} + \partial_\mu h_{\lambda\rho} - \partial_\lambda h_{\mu\rho}\right)$$

$$R_{\mu\nu\rho\sigma} = \frac{1}{2}\left(\partial_{\rho\nu} h_{\mu\sigma} + \partial_{\sigma\mu} h_{\nu\rho} - \partial_{\rho\mu} h_{\nu\sigma} - \partial_{\sigma\nu} h_{\mu\rho}\right)$$

- Linearized Riemann tensor is invariant under $h'_{\mu\nu} = h_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}$

- Trace-reverse tensor:

$$\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\, h \qquad h = \eta_{\alpha\beta}\, h^{\alpha\beta} \qquad \bar{h} = -h$$

Linearization of Einstein equations (contd.)

- Linearized Einstein equations:

$$\Box \bar{h}_{\nu\sigma} + \eta_{\nu\sigma}\,\partial^\rho\,\partial^\lambda \bar{h}_{\rho\lambda} - \partial^\rho\,\partial_\nu \bar{h}_{\rho\sigma} - \partial^\rho\,\partial_\sigma \bar{h}_{\rho\nu} + \mathcal{O}(h^2) = -\frac{16\pi G}{c^4} T_{\nu\sigma}$$

$$\Box = \eta_{\rho\sigma}\,\partial^\rho\,\partial^\sigma$$

- Imposing Lorenz gauge (harmonic gauge):

$$\boxed{\partial_\nu \bar{h}^{\mu\nu} = 0}$$

$$\boxed{\Box \bar{h}_{\nu\sigma} = -\frac{16\pi G}{c^4} T_{\nu\sigma}} \qquad \partial_\mu T^{\mu\nu} = 0$$

- 6 independent components, **2 physical radiative** degrees of freedom, **4 physical non-radiative** degrees of freedom. *(Flanagan & Hughes 05)*

Lorenz gauge can always be imposed

$x^{\mu}_{\mathrm{new}} = x^{\mu} + \xi^{\mu}(x)$ with ξ^{μ} an arbitrary and infinitesimal vector field

$$g^{\mathrm{new}}_{\mu\nu} = g_{\mu\nu} - \xi_{\mu,\nu} - \xi_{\nu,\mu}$$

$$\bar{h}^{\mu\nu}_{\mathrm{new}} = \bar{h}^{\mu\nu} - \eta^{\mu\rho}\,\xi^{\nu}{}_{,\rho} - \eta^{\lambda\nu}\,\xi^{\mu}{}_{,\lambda} + \eta^{\mu\nu}\xi^{\rho}{}_{,\rho}$$

$$\bar{h}^{\mu\nu}_{\mathrm{new},\nu} = \bar{h}^{\mu\nu}{}_{,\nu} - \eta^{\lambda\nu}\,\xi^{\mu}{}_{,\lambda\nu} = 0 \Rightarrow \Box\,\xi^{\mu} = \bar{h}^{\mu\nu}{}_{,\nu}$$

- ξ^{μ} exists for any well behaved $\bar{h}^{\mu\nu}$
- ξ^{μ} is not unique, we can always add to it q^{μ} such that $\Box q^{\mu} = 0$

Propagation of GW in vacuum (far from source)

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\bar{h}^{\mu\nu} = 0 \quad \text{with} \quad \partial_\nu \bar{h}^{\mu\nu} = 0$$

- Plane wave solution:

$$\bar{h}^{\mu\nu} = \mathcal{A}\,\epsilon^{\mu\nu}(k)\,e^{ik_\alpha x^\alpha} \quad \text{with} \quad k_\nu\,\epsilon^{\mu\nu} = 0 \quad \epsilon^{\mu\nu} \rightarrow \text{polarization tensor}$$

- General solution:

$$\bar{h}^{\mu\nu} = \mathrm{Re}\left[\int d^3k\,\mathcal{A}_{\mu\nu}(k)\,e^{ik_\alpha x^\alpha}\right] \text{ with } k^\mu = (\omega, \vec{k}) \text{ and } k^\mu\,\mathcal{A}_{\mu\nu} = 0$$

Using the freedom within Lorenz gauge $\Rightarrow$ we can determine the *only* physical radiative components in $\bar{h}^{\mu\nu} \Rightarrow \bar{h}^{\mu\nu}_{\mathrm{TT}}$

Imposing transverse-traceless gauge

$$\Box \bar{h}_{\mu\nu} = 0 \qquad \partial_\nu \bar{h}^{\mu\nu} = 0$$

(GWs propagate at speed of light)

- Within Lorenz gauge, we can consider transformations such that

$$\Box \xi_\mu = 0$$

$$\overline{h'}_{\mu\nu} = \bar{h}_{\mu\nu} + \xi_{\mu\nu} \qquad \xi_{\mu\nu} = \eta_{\mu\nu}\,\partial_\rho \xi^\rho - \xi_{\mu,\nu} - \xi_{\nu,\mu}$$

- Using $\Box \xi_\mu = 0$ we can subtract 4 of the 6 components of $\bar{h}_{\mu\nu}$

- We choose ξ^0 such that $\bar{h} = 0$, and ξ^i such that $h^{i0} = 0$, then from Lorenz gauge, we have $\partial_0 h^{00} = 0$, and being GWs time-dependent, we have $h^{00} = 0$.

$$h^{00} = 0\,, \quad h^{0i} = 0\,, \quad \partial_i h^{ij} = 0\,, \quad h^{ii} = 0$$

Imposing transverse-traceless gauge (contd.)

$$\mathbf{n} = \mathbf{k}/k \qquad n^i \, h_{ij}^{\mathrm{TT}} = 0$$

- Assuming plane wave propagates along z-axis:

$$h_{ij}^{\mathrm{TT}}(t,z) = \begin{pmatrix} h_+ & h_\times & 0 \\ h_\times & -h_+ & 0 \\ 0 & 0 & 0 \end{pmatrix} \cos\left[\omega\left(t - \frac{z}{c}\right)\right]$$

plus and cross polarizations

- Along generic direction of propagation:

$$P_{ij}(\mathbf{n}) = \delta_{ij} - n_i \, n_j$$

$$P_{ij} = P_{ji}\,, \quad n^i \, P_{ij} = 0\,, \quad P_{ij}\, P^{jk} = P_i^k\,, \quad P_{ii} = 2$$

$$\Lambda_{ij,kl}(\mathbf{n}) = P_{ik}\, P_{jl} - \frac{1}{2} P_{ij}\, P_{kl}$$

$$h_{ij}^{\mathrm{TT}} = \Lambda_{ij,kl}\, h_{kl}$$

Linearly polarized waves in EM & GR

- In EM theory linearly polarized vectors are:

$\mathbf{e}_x$ and $\mathbf{e}_y$

- In GW theory linearly polarized tensors are:

$$\mathbf{e}_+ = \mathbf{e}_x \times \mathbf{e}_x - \mathbf{e}_y \times \mathbf{e}_y \qquad \text{and} \qquad \mathbf{e}_\times = \mathbf{e}_x \times \mathbf{e}_y + \mathbf{e}_y \times \mathbf{e}_x$$

$$(\mathbf{u} \times \mathbf{v})(\lambda, \mathbf{q}) = (\lambda \cdot \mathbf{u})(\mathbf{q} \cdot \mathbf{v})$$

$$\mathbf{e}_+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{e}_\times = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Circularly polarized waves in EM & GR

- In EM theory circularly polarized vectors are:

$$\mathbf{e}_R = \frac{1}{\sqrt{2}}(\mathbf{e}_x + i\mathbf{e}_y) \qquad \text{and} \qquad \mathbf{e}_L = \frac{1}{\sqrt{2}}(\mathbf{e}_x - i\mathbf{e}_y)$$

- In GW theory circularly polarized tensors are:

$$\mathbf{e}_R = \frac{1}{\sqrt{2}}(\mathbf{e}_+ + i\mathbf{e}_\times) \qquad \text{and} \qquad \mathbf{e}_L = \frac{1}{\sqrt{2}}(\mathbf{e}_+ - i\mathbf{e}_\times)$$

$$\mathbf{e}_+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \mathbf{e}_\times = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Gravitational waves have helicity 2

Any plane wave ψ which is transformed by a rotation of any angle θ around the direction of propagation into $\psi' = e^{i\,h\,\theta}\,\psi$ is said to have **helicity** h

Let us rotate the coordinate system around z by θ

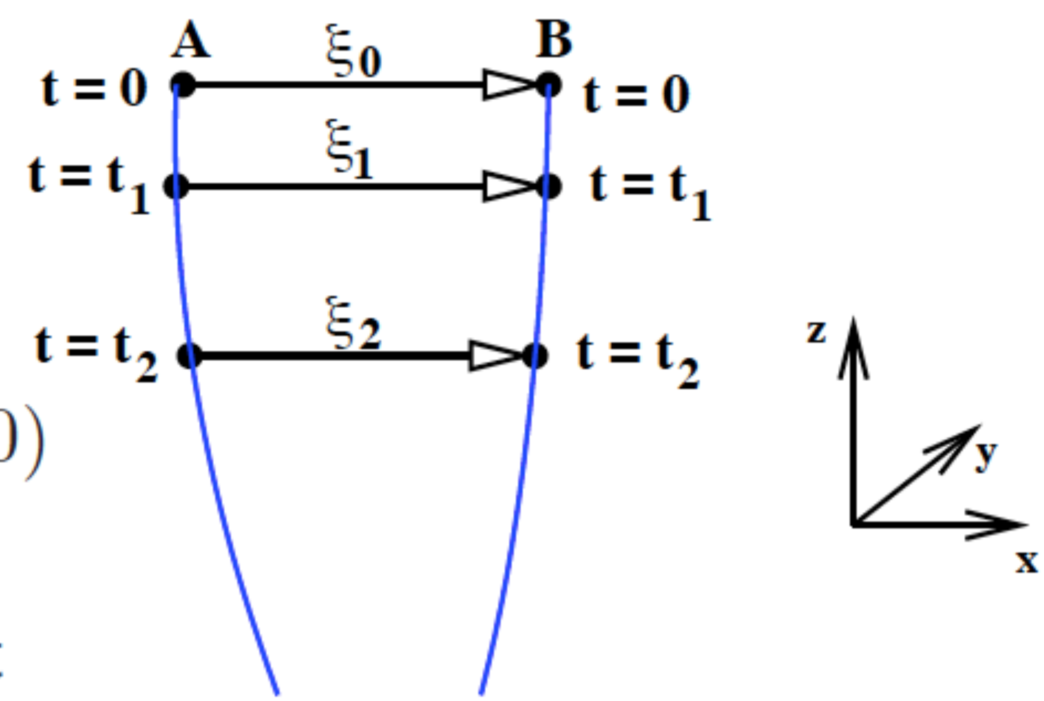
$$x' = x\,\cos\theta + y\,\sin\theta \qquad y' = y\,\cos\theta - x\,\sin\theta$$

$$\mathbf{e}_{x'} = \mathbf{e}_x\,\cos\theta + \mathbf{e}_y\,\sin\theta \qquad \mathbf{e}_{y'} = -\mathbf{e}_x\,\sin\theta + \mathbf{e}_y\,\cos\theta$$

$$\mathbf{e}_{+'} = \mathbf{e}_+\,\cos 2\theta + \mathbf{e}_\times\,\sin 2\theta \qquad \mathbf{e}_{\times'} = -\mathbf{e}_+\,\sin 2\theta + \mathbf{e}_\times\,\cos 2\theta$$

Newtonian description of tidal gravity

- Two point particles A and B falling freely under the action of external Newtonian potential Φ



- A and B at time $t = 0$ are separated by small distance ξ and have equal velocity $\mathbf{v}_{\rm A}(0) = \mathbf{v}_{\rm B}(0)$

- For $t > 0$, A and B experience slightly different gravitational potential and accelerations $\mathbf{g} = -\nabla\Phi$

$$\xi^i = x^i_{\rm A} - x^i_{\rm B} \qquad \dot{\xi}^i = \dot{x}^i_{\rm A} - \dot{x}^i_{\rm B}$$

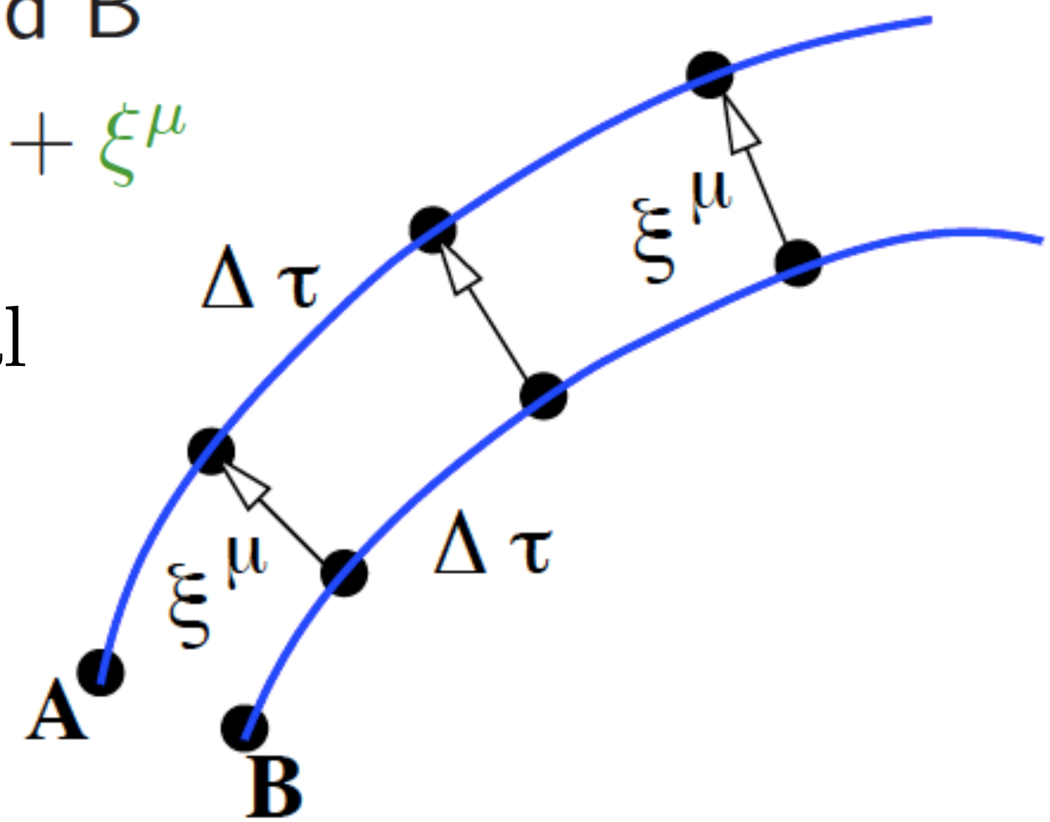
Assuming $|\xi^i|$ is much smaller than typical scale of variation of gravitational potential

$$\ddot{\xi}^i = \ddot{x}^i_{\rm A} - \ddot{x}^i_{\rm B} = -\left(\frac{\partial \Phi}{\partial x^i}\right)_{\rm B} + \left(\frac{\partial \Phi}{\partial x^i}\right)_{\rm A} \simeq -\left(\frac{\partial^2 \Phi}{\partial x^i\,\partial x^j}\right)_{\rm A} \xi^j$$

$$\epsilon_{ij} = -\left(\frac{\partial^2 \Phi}{\partial x^i\,\partial x^j}\right) \Rightarrow \text{Newtonian tidal gravitational field}$$

Equation of geodesic deviation

Pair of nearby freely-falling particles A and B traveling on trajectories $x^\mu(\tau)$ and $x^\mu(\tau) + \xi^\mu$



Assuming $|\xi^\mu|$ is much smaller than typical scale of variation of gravitational field

$$0 = \frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\lambda} \frac{dx^\mu}{d\tau} \frac{dx^\lambda}{d\tau}$$

$$0 = \frac{d^2 (x^\mu + \xi^\mu)}{d\tau^2} + \Gamma^\mu_{\nu\lambda}(x + \xi) \frac{d(x^\mu + \xi^\mu)}{d\tau} \frac{d(x^\lambda + \xi^\lambda)}{d\tau}$$

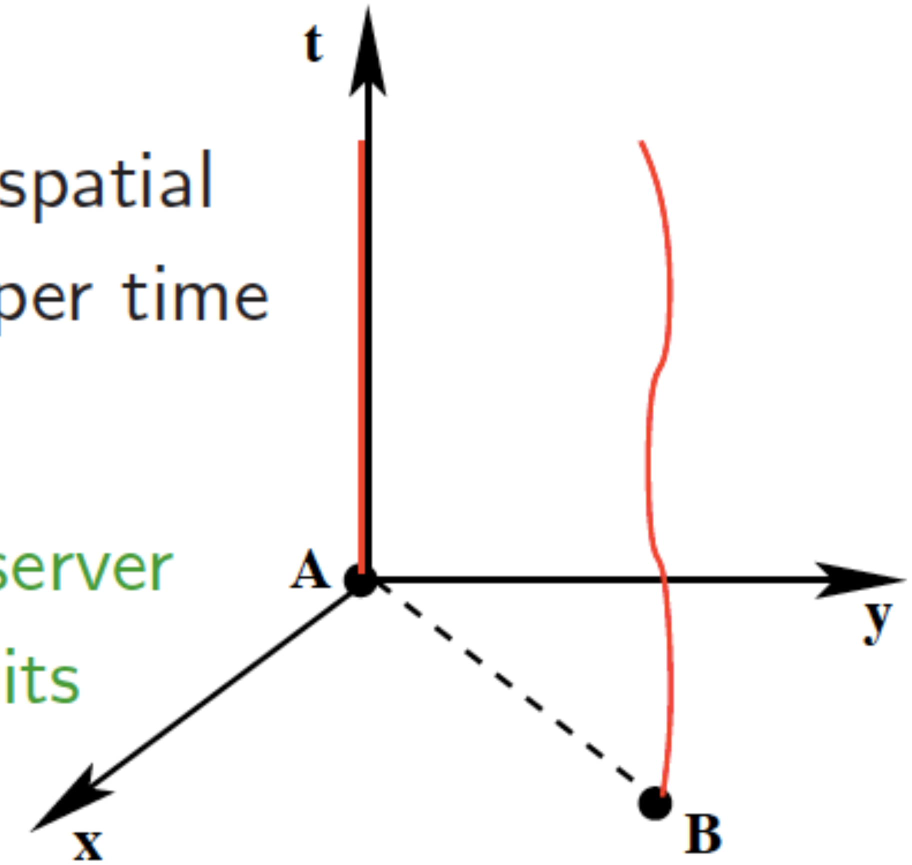
Taking the difference and limiting to first order in ξ

$$\nabla_U \nabla_U \xi^\lambda = -R^\lambda_{\nu\mu\rho} \xi^\mu U^\nu U^\rho \qquad U^\alpha = \frac{dx^\alpha}{d\tau}$$

$$\nabla_U \xi^\lambda = U^\nu \nabla_\nu \xi^\lambda = U^\nu (\partial_\nu \xi^\lambda + \Gamma^\lambda_{\nu\sigma} \xi^\sigma) = \frac{d\xi^\lambda}{d\tau} + \Gamma^\lambda_{\nu\sigma} \xi^\sigma \frac{dx^\nu}{d\tau}$$

Interaction of GWs with free-falling particles in local Lorentz frame

- Two test particles A and B initially at rest one respect to the other in absence of GWs
- Local Lorentz frame attached to particle A, with spatial origin at $x^j = 0$ and coordinate time equal to proper time $x^0 = t$
- By definition of LLF, the metric $g_{\mu\nu}$ of a LLF observer reduces to Minkowski metric at the origin and all its first derivatives must vanish at the origin



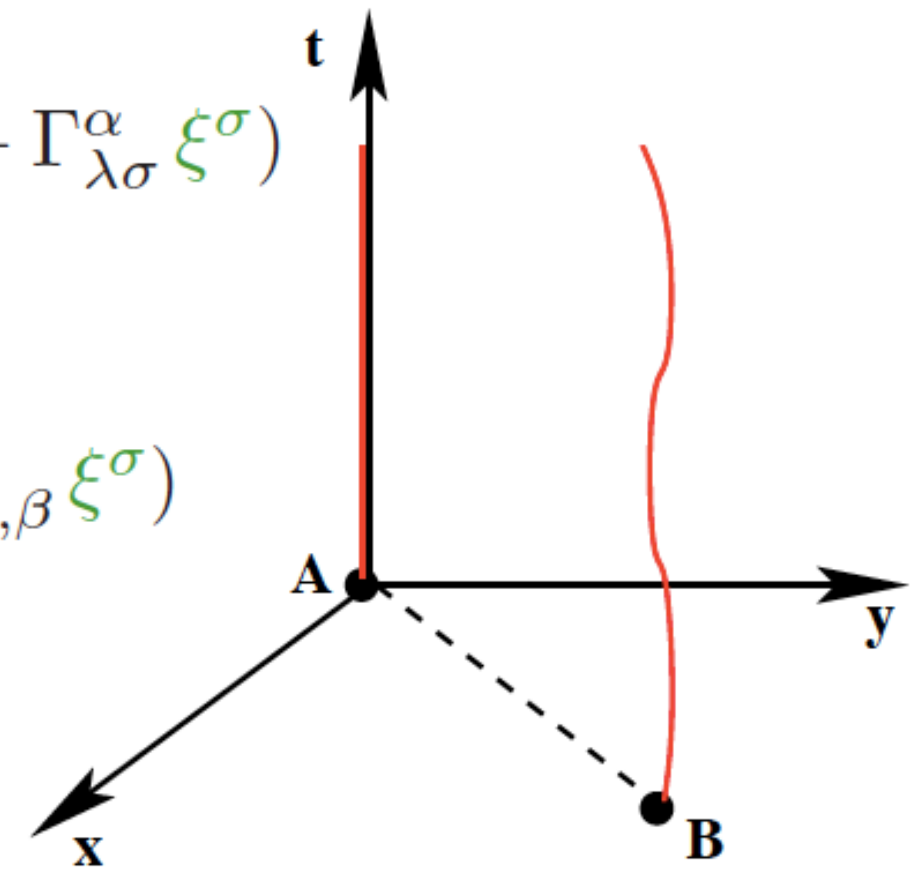
$$ds^2 = -dt^2 + d\mathbf{x}^2 + \mathcal{O}\left(\frac{|\mathbf{x}|^2}{\mathcal{R}}\right)$$

$\mathcal{R}$ being the curvature radius: $\mathcal{R}^{-2} = |R_{\mu\nu\rho\sigma}|$

Geodesic deviation equation in local-Lorentz frame (LLF)

$$\nabla_U \nabla_U \xi^\alpha = -R^\alpha_{\nu\lambda\rho}\, \xi^\lambda \frac{dx^\nu}{dt}\frac{dx^\rho}{dt}$$

$$\nabla_U \nabla_U \xi^\lambda = U^\beta \nabla_\beta (U^\lambda \nabla_\lambda \xi^\alpha) = U^\beta U^\lambda \nabla_\beta (\xi^\alpha{}_{,\lambda} + \Gamma^\alpha_{\lambda\sigma}\,\xi^\sigma)$$



In the LLF of particle A:

$$\Gamma^\sigma_{\alpha\beta} = 0 \quad \Rightarrow \quad \nabla_U \nabla_U \xi^\lambda = U^\beta U^\lambda \,(\xi^\alpha{}_{,\lambda\beta} + \Gamma^\alpha_{\lambda\sigma,\beta}\,\xi^\sigma)$$

$$U^\alpha = \delta^\alpha_0 \quad \Rightarrow \quad \nabla_U \nabla_U \xi^\alpha = \ddot{\xi}^\alpha + \Gamma^\alpha_{0\sigma,0}\,\xi^\sigma$$

$$\Gamma^\alpha_{0\sigma,0} \sim \mathcal{O}\left(\frac{|\mathrm{x}|^2}{\mathcal{R}}\right) \quad \Rightarrow \quad \nabla_U \nabla_U \xi^\alpha = \ddot{\xi}^\alpha$$

Assuming $\xi^0 = 0 \quad \Rightarrow \quad \dfrac{d^2 \xi^j}{dt^2} = -R^j{}_{0i0}\,\xi^i$

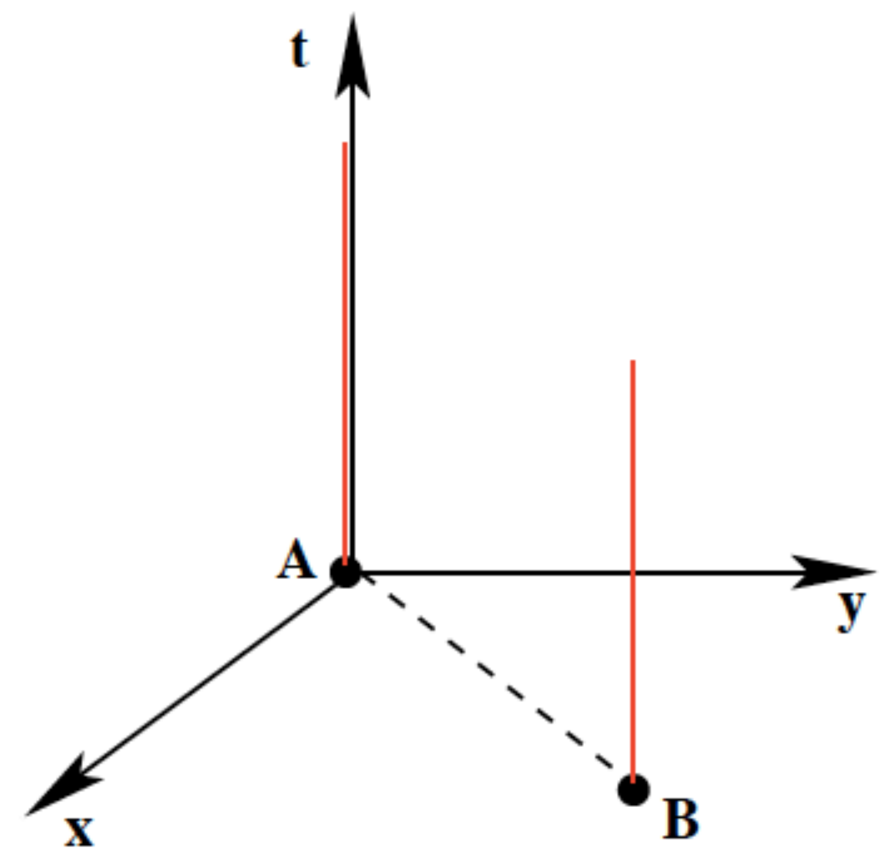
Curvature tensor in linearized gravity invariant. In TT gauge:

$$R^{\mathrm{TT}}_{j0i0} = -\frac{1}{2}\,\ddot{h}^{\mathrm{TT}}_{ji} \Rightarrow \frac{d^2\xi^j}{dt^2} = \frac{1}{2}\,\ddot{h}^{\mathrm{TT}}_{ji}\,\xi^i$$

Interaction of GWs with free-falling particles using TT gauge

- Two test particles A and B initially at rest one respect to the other in absence of GWs

- U^α being the 4-velocity of particle A



$$\frac{dU^\alpha}{d\tau} + \Gamma^\alpha_{\mu\nu} U^\mu U^\nu = 0$$

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\eta^{\alpha\beta} \left(h^{\rm TT}_{\beta\mu,\nu} + h^{\rm TT}_{\nu\beta,\mu} - h^{\rm TT}_{\mu\nu,\beta}\right)$$

- Initially $U^\alpha = \delta^\alpha_0 \quad \Rightarrow$

$$\frac{dU^\alpha}{d\tau} = -\Gamma^\alpha_{00} = -\frac{1}{2}(h^{\rm TT}_{0\beta,0} + h^{\rm TT}_{\beta 0,0} - h^{\rm TT}_{00,\beta}) = 0!$$

Coordinate position of test particles does not vary. Proper distance varies.

On geodesic deviation equation in LLF

If $\bar{x}^{\mu}$ and $\bar{g}^{\mu\nu}$ refer to TT gauge ($h_{\times}^{\rm TT} = 0, h_{+}^{\rm TT} \neq 0$):

$$\bar{g}^{\mu\nu} = \eta_{\mu\nu} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h^{\rm TT} & 0 & 0 \\ 0 & 0 & -h^{\rm TT} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The local Lorentz metric $g_{\mu\nu}$ should reduce to Minkowski metric at the origin and all its first derivatives must vanish

$$\bar{t} = t - \dot{h}^{\rm TT}(x^2 - y^2)/4$$
$$\bar{x} = x - h^{\rm TT}\, x/2$$
$$\bar{y} = y + h^{\rm TT}\, y/2$$
$$\bar{z} = z + \dot{h}^{\rm TT}(x^2 - y^2)/4$$

$$g^{\mu\nu} = \eta_{\mu\nu} - 2\begin{pmatrix} \Phi(t) & 0 & 0 & \Phi(t) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Phi(t) & 0 & 0 & \Phi(t) \end{pmatrix}$$

$$\Phi(t) = -\frac{1}{4}\ddot{h}^{\rm TT}(x^2 - y^2) \Rightarrow \frac{d^2\xi^j}{dt^2} = -\frac{\partial^2\Phi}{\partial x^j \partial x^k}\xi^k$$

On geodesic deviation equation in LLF (contd.)

$$R^i_{0j0} = \frac{\partial x^i}{\partial \bar{x}^\mu}\frac{\partial \bar{x}^\nu}{\partial x^0}\frac{\partial \bar{x}^\mu}{\partial x^j}\frac{\partial \bar{x}^\rho}{\partial x^0} R^{\mathrm{TT}\,\mu}{}_{\nu\lambda\rho} \simeq R^{\mathrm{TT}\,\mathrm{i}}{}_{0j0} \quad \Rightarrow \quad R^i_{0j0} = \frac{1}{2}\ddot{h}^{\mathrm{TT}}_{ij}\text{(-)}$$

$$\frac{d^2\xi^j}{dt^2} = \text{-}R^j{}_{0i0}\,\xi^i = \frac{1}{2}\ddot{h}^{\mathrm{TT}}_{ij}\,\xi^j(0) \quad \Rightarrow \quad \delta\xi^i = \frac{1}{2}h^{\mathrm{TT}}_{ij}\,\xi^j(0)$$

- The acceleration of particle B in the LLF of particle A is: $a^j = \frac{F^j}{m_B} = \frac{1}{2}\ddot{h}^{\mathrm{TT}}_{ij}\,\xi^i(0)$

- The observer in LLF of particle A concludes that particle B is subjected to the force $\mathbf{F}$, whereas for an observer in LLF of particle B, particle B is just free falling

In GW interferometers: <u>A</u> $\rightarrow$ mirror at beam splitter; <u>B</u> $\rightarrow$ mirror at end of arm cavity

$$\frac{\delta\xi}{\xi(0)} = \frac{\delta\xi}{L} = h \qquad \text{If } L \sim 3\text{ km},\ h \sim 10^{-21} \quad \Rightarrow \quad \delta\xi \sim 10^{-16}\,\mathrm{cm!}$$

On geodesic deviation equation in LLF (contd.)

The LLF is useful to do calculations as long as we can use the metric in the form $g_{\mu\nu} = \eta_{\mu\nu} + \mathcal{O}\left(\frac{\mathrm{x}^2}{\mathcal{R}^2}\right)$, i.e., as long as we can disregard $\mathbf{x}^2$ corrections

$$\mathcal{R}^{-2} = |R_{\mu\nu\rho\sigma}| \sim |R_{i0j0}| \sim \ddot{h} \sim \frac{h}{\lambda_{\mathrm{GW}}^2}$$

$$\frac{\mathrm{x}^2}{\mathcal{R}^2} \sim \frac{L^2\, h}{\lambda_{\mathrm{GW}}^2} \ll 1 \quad \text{if} \quad L \ll \lambda_{\mathrm{GW}}$$

- Ground-based detectors (LIGO, VIRGO, KAGRA, GEO): $L \ll \lambda_{\mathrm{GW}} \sim 10^3$ km
- Space-based detectors (LISA): $L \sim \lambda_{\mathrm{GW}} \sim 5 \times 10^6$ km

Equivalence between TT frame and local Lorentz frame

Proper distance in the two frames (assume that A and B are along x-axis and *only* $h_+ \neq 0$)

- LLF:

$$(\Delta s)^2 = g_{xx}\,(\Delta x)^2$$

$$g_{xx} = 1 \quad \text{but} \quad (\Delta x)^2 = (L + \tfrac{1}{2}h_+\,L)^2$$

$$\Rightarrow \Delta s = L\,\left(1 + \tfrac{1}{2}h_+\right)$$

- TTF:

$$(\Delta s)^2 = g_{xx}\,(\Delta x)^2$$

$$g_{xx} = 1 + h_+ \quad \text{but} \quad (\Delta x)^2 = L^2$$

$$\Rightarrow \Delta s = L\,\left(1 + \tfrac{1}{2}h_+\right)$$

Proper distances are the same!

GWs and ring of free-falling particles

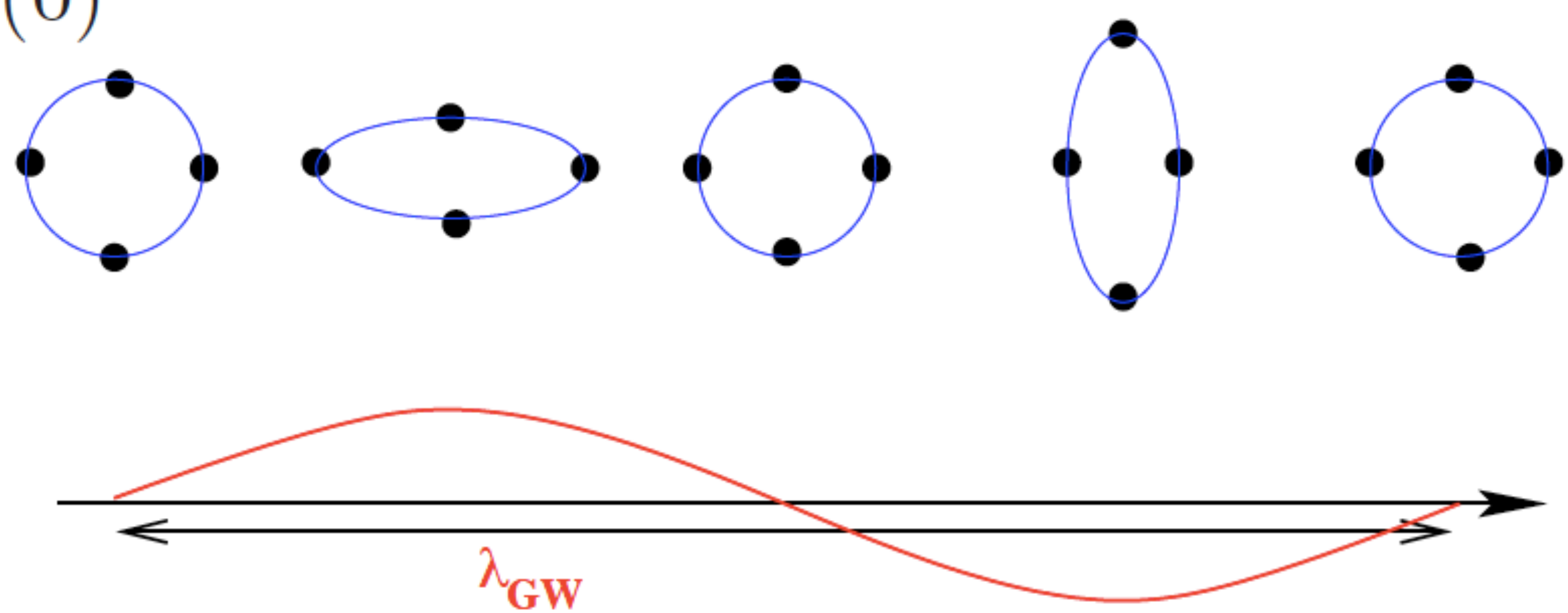
Interaction between GW and ring of free-falling particles: $h_{+}^{\rm TT}$

GW propagating along z-axis

Case: $h_{xx}^{\rm TT} = -h_{yy}^{\rm TT} \equiv h_{+}^{\rm TT} \neq 0 \qquad h_{xy}^{\rm TT} = h_{yx}^{\rm TT} \equiv h_{\times}^{\rm TT} = 0$

$$\delta\xi_x = +\frac{1}{2} h_{xx}^{\rm TT}\,\xi_x(0)$$

$$\delta\xi_y = -\frac{1}{2} h_{yy}^{\rm TT}\,\xi_y(0)$$



GWs and ring of free-falling particles (contd.)

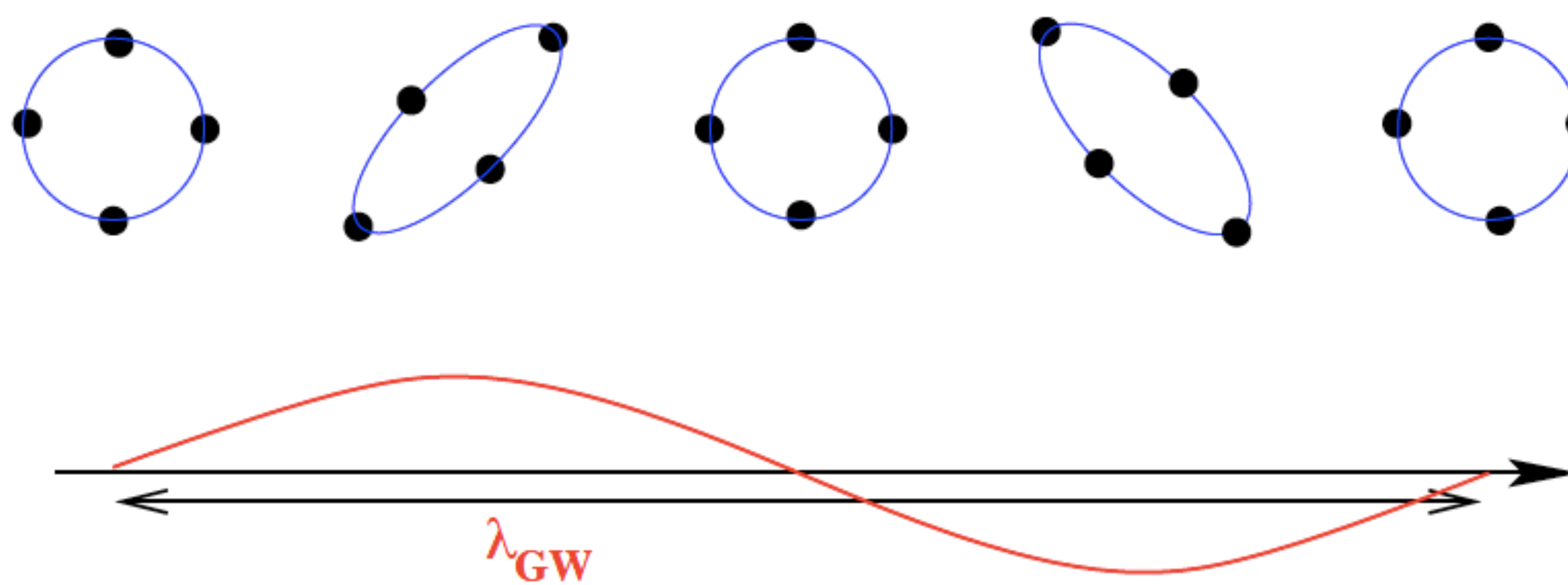
Interaction between GW and ring of free-falling particles: $h_{\times}^{\rm TT}$

GW propagating along z-axis

Case: $h_{xx}^{\rm TT} = -h_{yy}^{\rm TT} \equiv h_{+}^{\rm TT} = 0 \qquad h_{xy}^{\rm TT} = h_{yx}^{\rm TT} \equiv h_{\times}^{\rm TT} \neq 0$

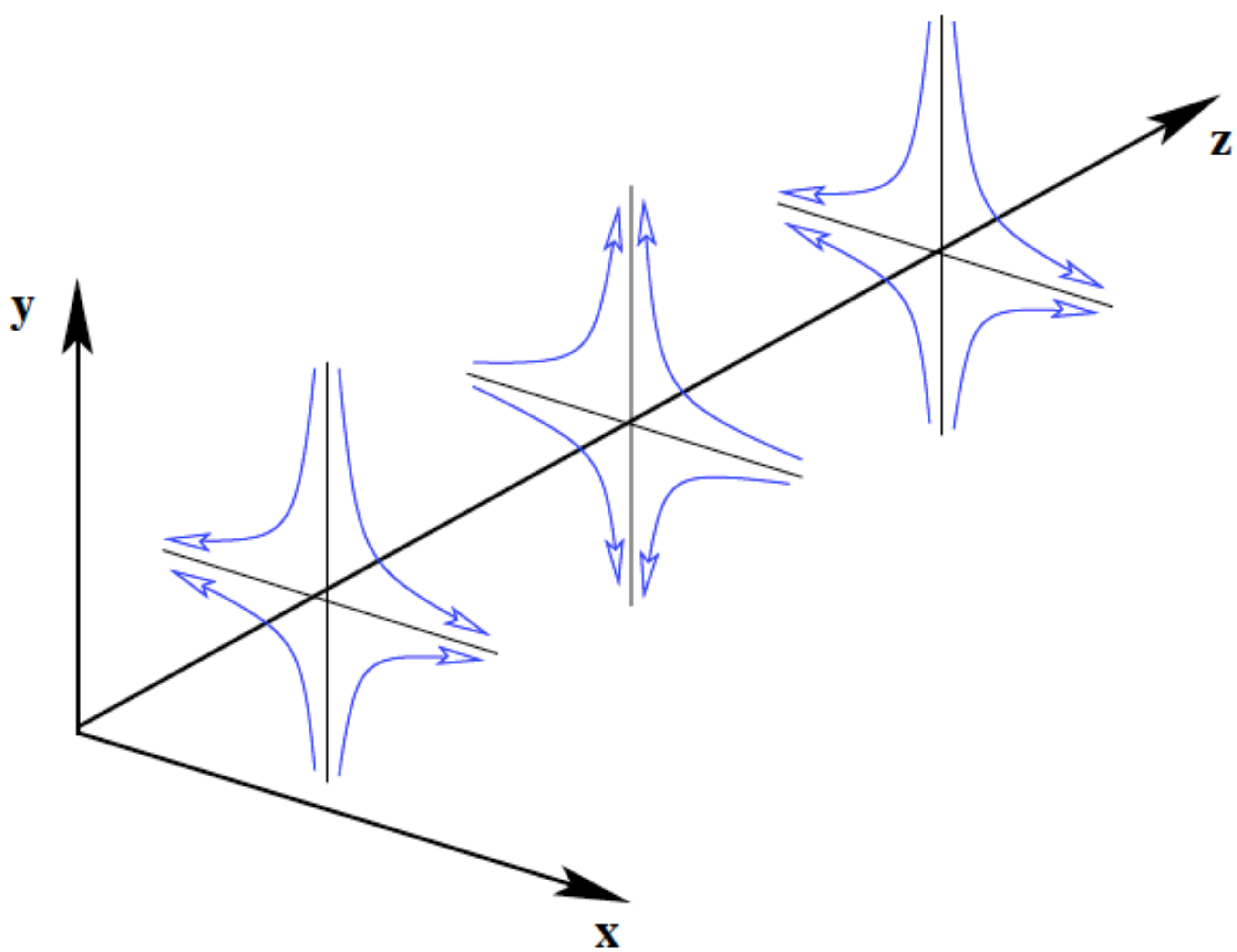
$\delta\xi_x = +\frac{1}{2} h_{xy}^{\rm TT}\, \xi_y(0)$

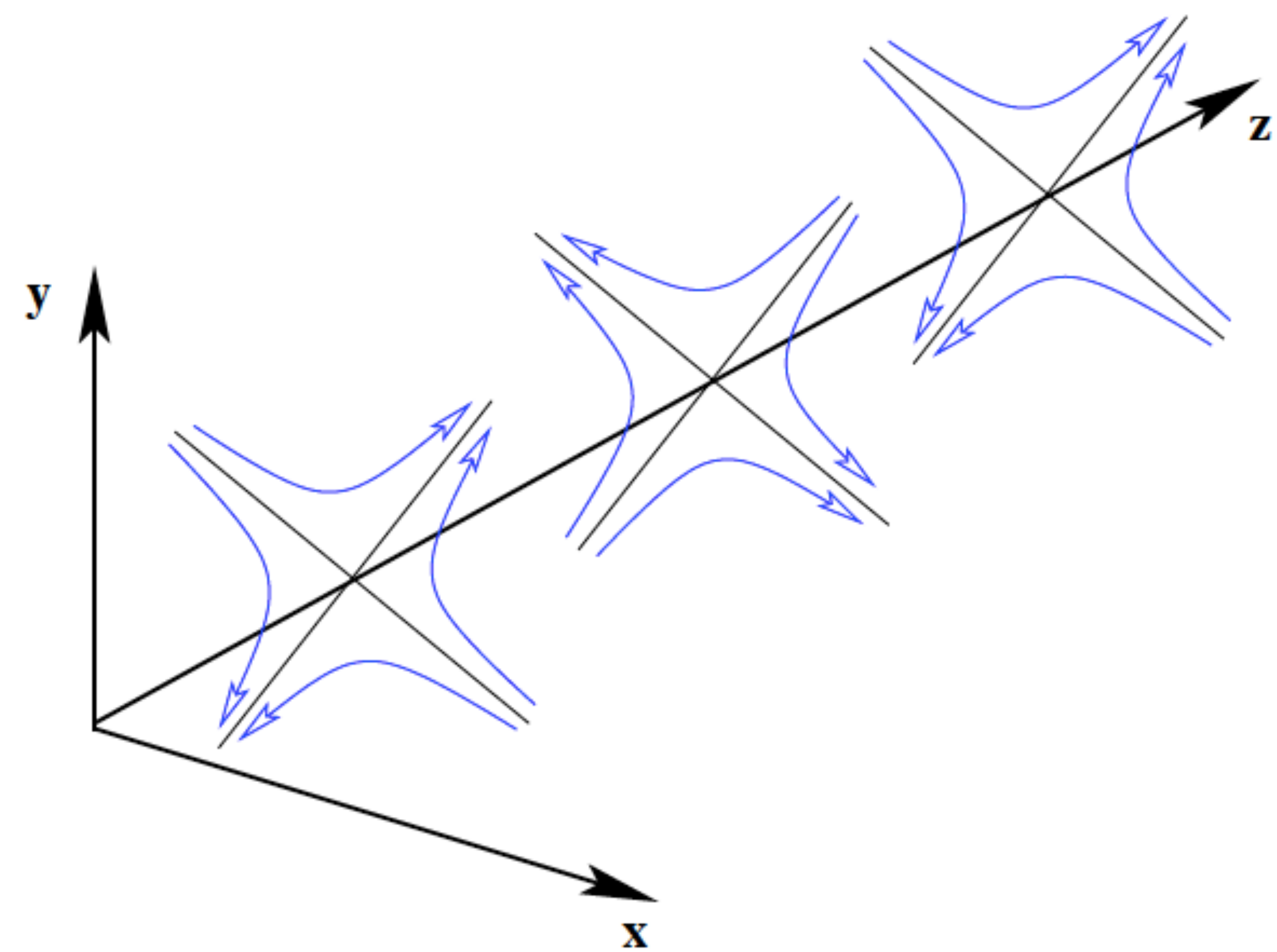
$\delta\xi_y = +\frac{1}{2} h_{xy}^{\rm TT}\, \xi_x(0)$



GWs and lines of force

Lines of force for h_+^{TT} and $h_\times^{\mathrm{TT}}$



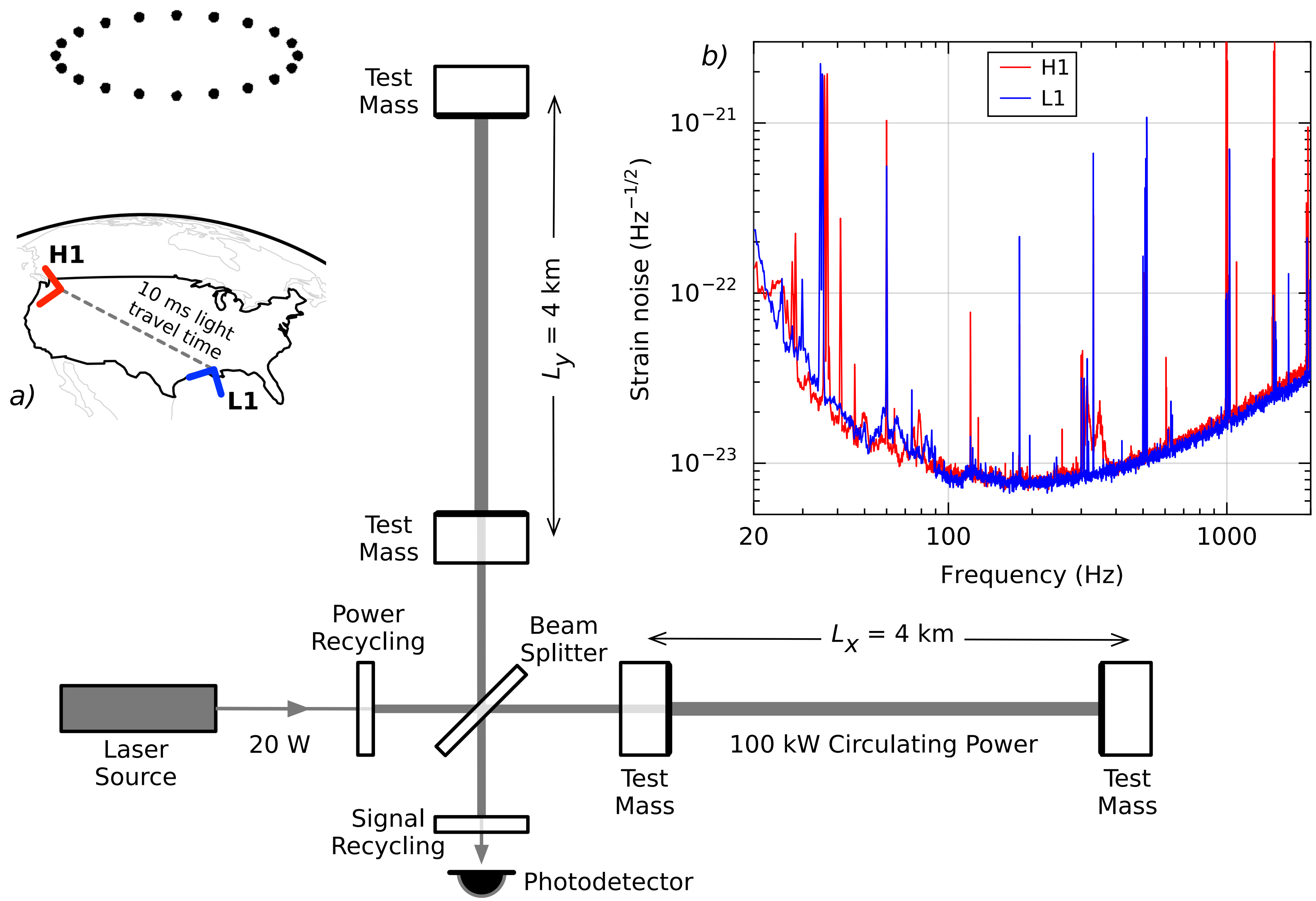


The two LIGO detectors & the Virgo detector

LIGO in Hanford, WA

Virgo in Pisa, Italy

(Abbott et al. PRL 116 (2016) 061102)

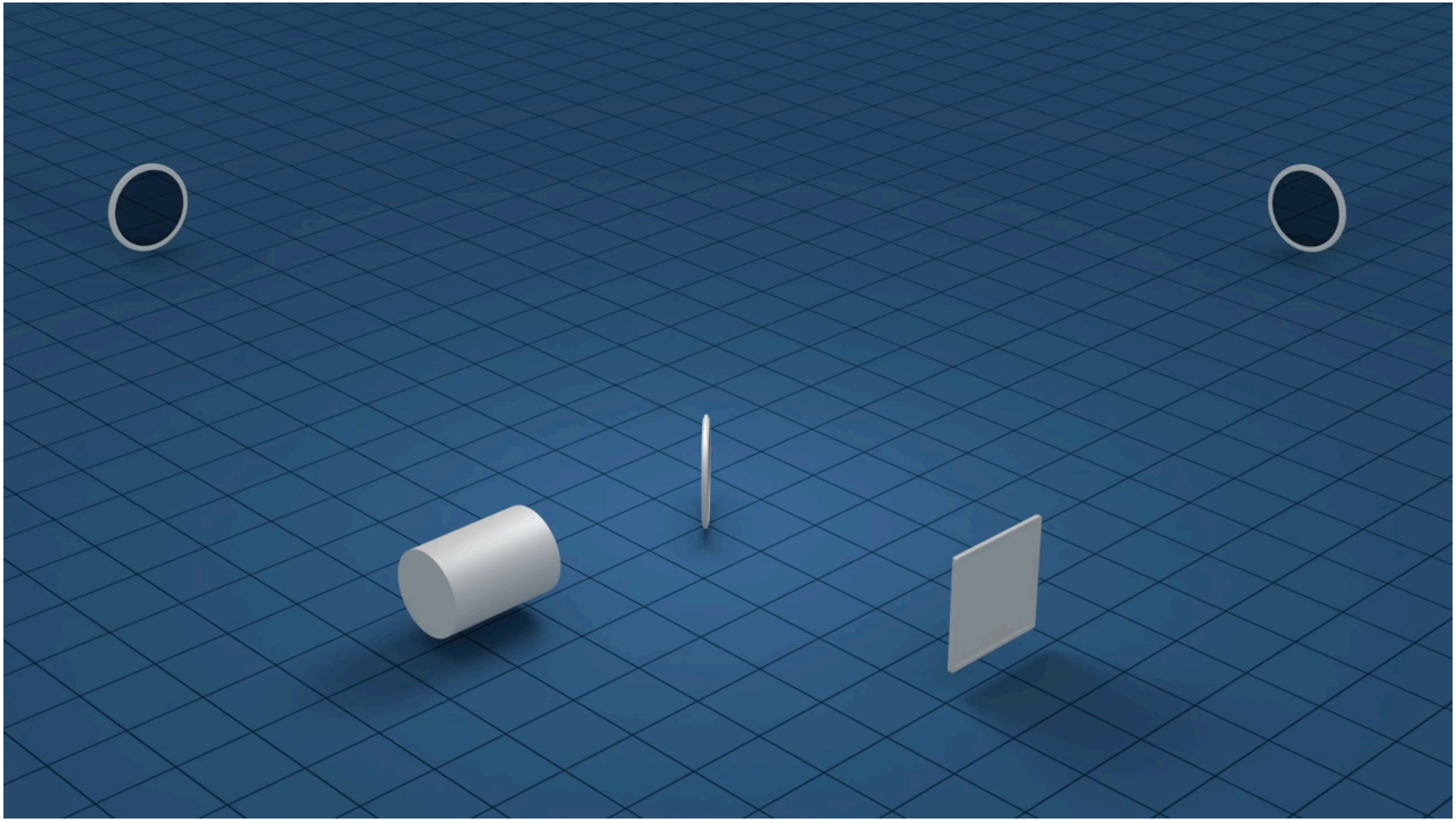


$$\Delta L = L\, h \sim 10^{-16}\,\mathrm{cm}$$

$$L = 4\,\mathrm{km} \Rightarrow h \sim 10^{-21}$$

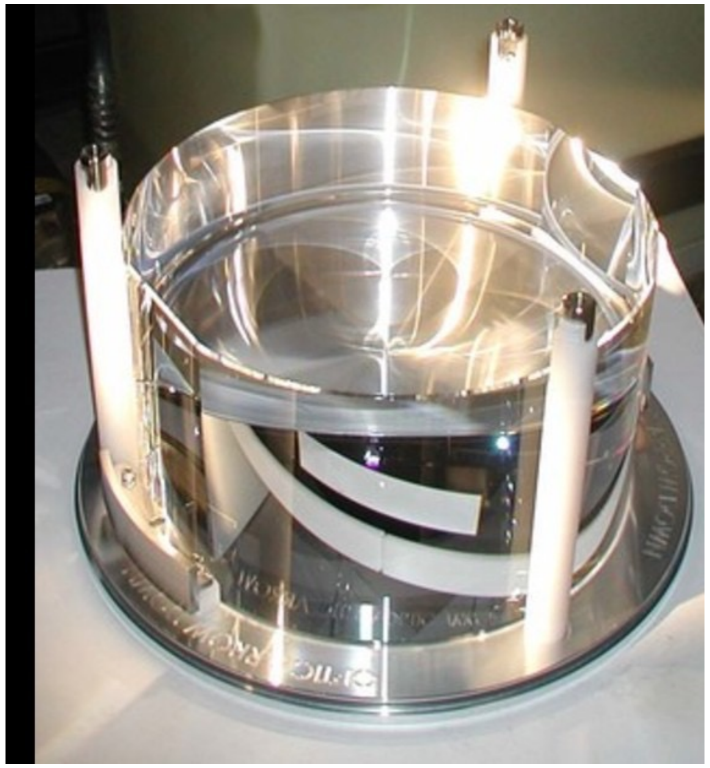
LIGO/Virgo measure (tiny) relative changes in separation of mirrors **(phase shifts of light at beamsplitter of 10^{-9} rad!)**

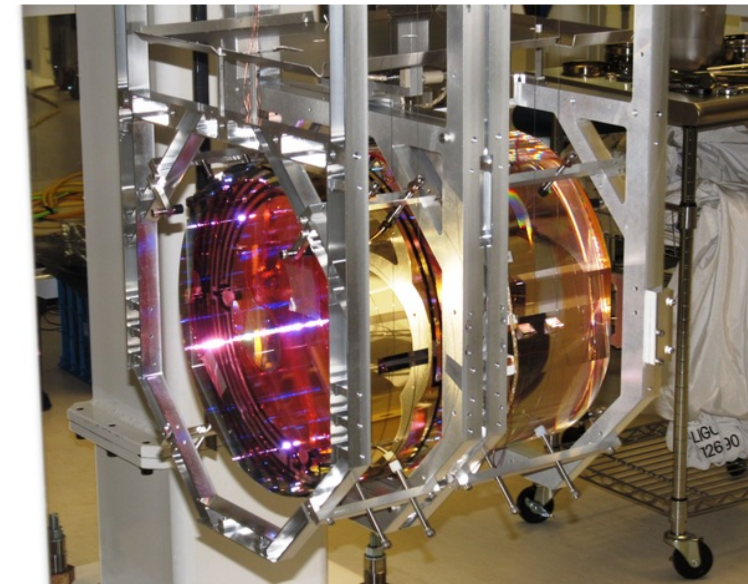
How LIGO/Virgo work



LIGO Scientific Collaboration

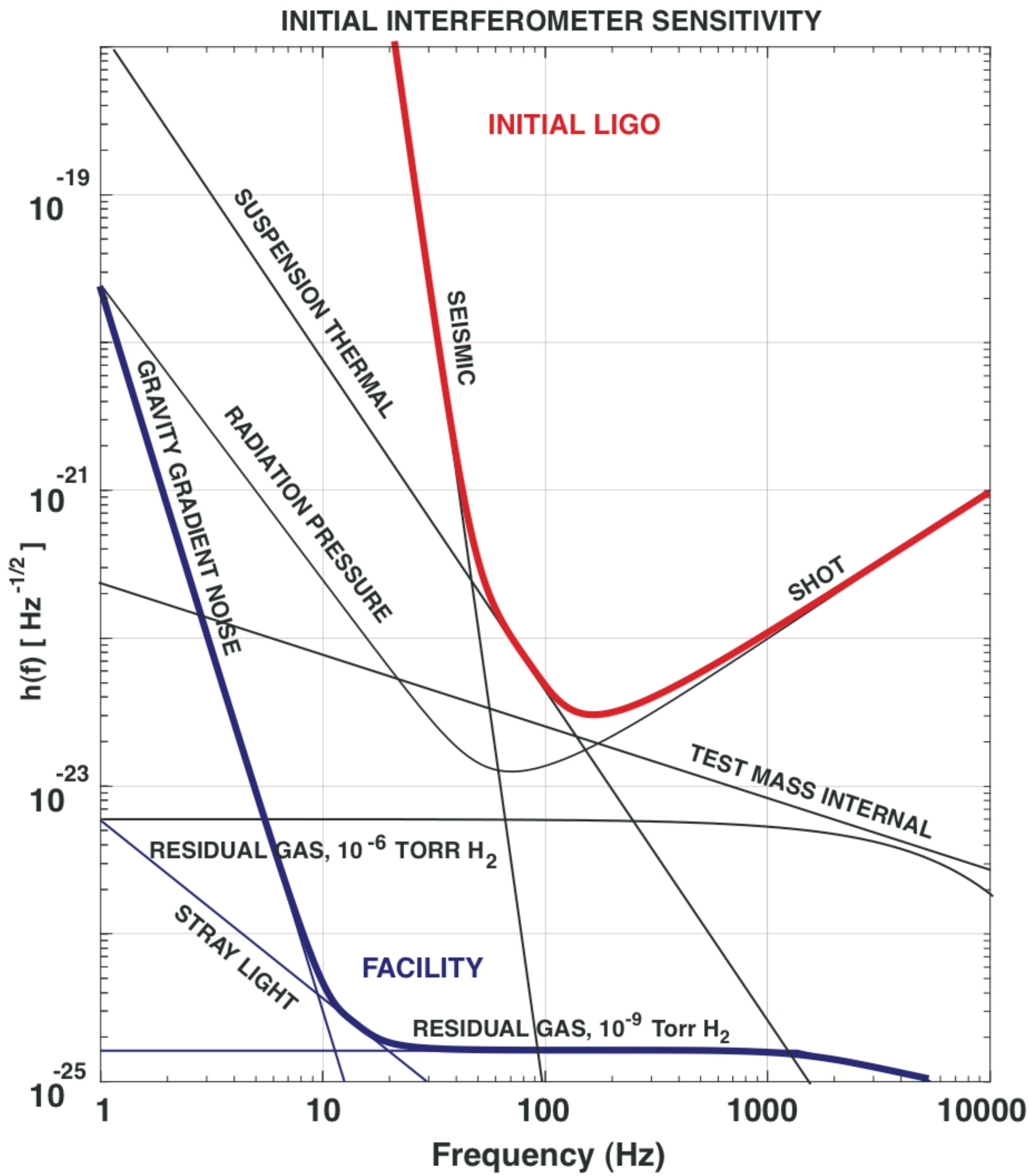
A glimpse inside the LIGO facility



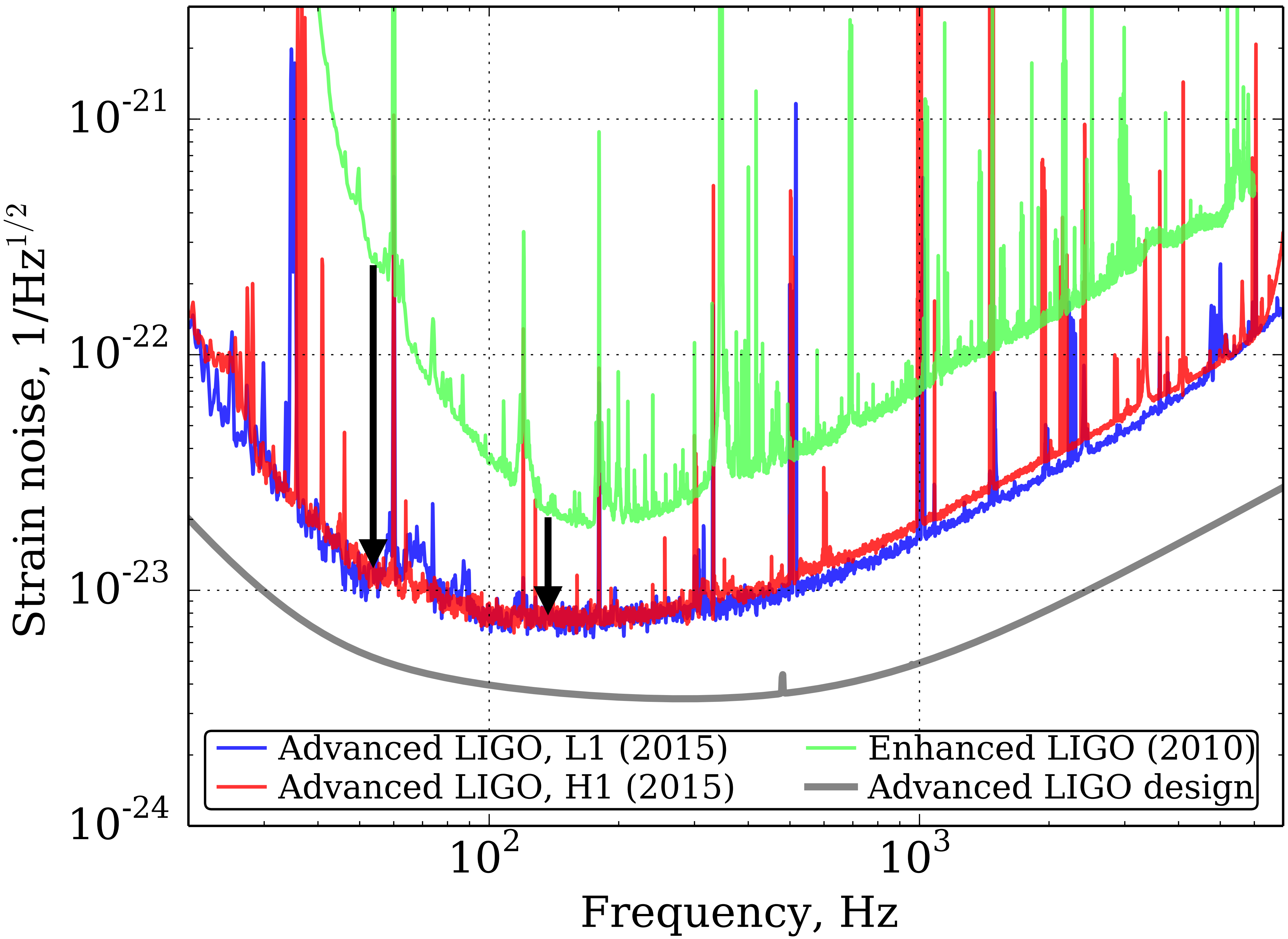


LIGO Scientific Collaboration

Typical noises in ground-based gravitational-wave detectors



Evolution of sensitivity from Enhanced to Advanced LIGO (O1)



(Martynov et al. arXiv:1604.00439)

Advanced Virgo joined Advanced LIGOs on August 1, 2017

(Abbott et al. arXiv:1709.09660)