Astrophysics and Data Analysis

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Lecture Outlines

- Principles of signal analysis. Understanding GW sensitivity curves. (Lecture 1, part I)
- Estimating detectability of sources. (Lecture 1, part II)
- Sources of gravitational waves and their properties. (Lecture 2)
- Data analysis techniques: searches. (Lecture 3, part I)
- Data analysis techniques: source characterisation/ parameter estimation. (Lecture 3, part II)

Principles of Signal Analysis

Principles of Signal Analysis

- Gravitational wave detectors are intrinsically noisy. The output $s(t)$ will consist of a (possible) signal $h(t)$ plus noise fluctuations $n(t)$.

$$s(t) = h(t) + n(t)$$

- The noise is a random process.
    - ➡ Future values are not uniquely determined by initial data, but evolves according to some probabilistic model.
    - ➡ We suppose the random process is drawn from an *ensemble of random processes characterised by probability distributions*

$$p_N(n_N, t_N; \ldots; n_2, t_2; n_1; t_1)\mathrm{d}n_N \ldots \mathrm{d}n_2 \mathrm{d}n_1$$

Principles of Signal Analysis

- We typically make various useful assumptions about the properties of a random process
    - *Stationarity*: A stationary process is one for which the probability distributions depend only on time *differences*, not absolute time.

$$p_N(n_N, t_N + \tau; \ldots; n_2, t_2 + \tau; n_1; t_1 + \tau) = p_N(n_N, t_N; \ldots; n_2, t_2; n_1; t_1) \;\forall\, \tau$$

    - *Gaussianity*: A process is Gaussian if and only if *all* of its (absolute) probability distributions are Gaussian.

$$p_N(n_N, t_N; \ldots; n_2, t_2; n_1; t_1) = A \exp\left[-\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} \alpha_{jk}(y_j - \bar{y})(y_k - \bar{y})\right]$$

    - *Ergodicity*: An ensemble of stationary random processes is ergodic if for any process *n(t)* drawn from the ensemble, the new ensemble {*n(t+KT)*: *K* an integer} has the same probability distributions.

Principles of Signal Analysis

- We are interested in how large the random fluctuations are about the mean value. We'll assume this is zero here, which can be arranged by a subtracting a constant.

- The fluctuations can be characterised by the power in a certain time interval $-T/2 < t < T/2$

$$\int_{-T/2}^{T/2} |n(t)|^2 \mathrm{d}t$$

- For stationary random processes this increases linearly with time. So, we instead use the mean power (or mean square fluctuations)

$$P_n = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |n(t)|^2 \mathrm{d}t$$

Principles of Signal Analysis

- Defining $n_T(t) = n(t)\mathbb{I}[|t| < T/2]$ and using Parseval's theorem we have

$$\int_{-T/2}^{T/2} [n(t)]^2 \mathrm{d}t = \int_{-\infty}^{\infty} [n_T(t)]^2 = \int_{-\infty}^{\infty} |\tilde{n}_T(f)|^2 \mathrm{d}f = 2\int_0^{\infty} |\tilde{n}_T(f)|^2 \mathrm{d}f$$

$$P_n = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} [n(t)]^2 = \lim_{T\to\infty} \frac{2}{T} \int_0^{\infty} |\tilde{n}_T(f)|^2 \mathrm{d}f$$

- This motivates defining *the spectral density*, $S_n(f)$, via

$$S_n(f) = \lim_{T\to\infty} \frac{2}{T} \left| \int_{-T/2}^{T/2} n(t) \exp(2\pi i f t) \mathrm{d}t \right|^2$$

- This is the *one-sided* spectral density which assumes the time series is real and we only consider positive frequencies. The *two-sided* spectral density is half this.

Principles of Signal Analysis

- The spectral density represents the power in the process at a particular frequency

$$P_n = \int_0^\infty S_n(f)\mathrm{d}f$$

- If we consider the evolution of the process over a time interval Δt, with corresponding *bandwidth* $\Delta f = 1/\Delta t$, the mean square fluctuations in *n* at that frequency are

$$[\Delta n(\Delta t, f)]^2 \equiv \lim_{N\to\infty} \frac{2}{N} \sum_{n=-N/2}^{N/2} \left| \frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} n(t)\exp(2\pi i f t)\mathrm{d}t \right|^2 = \frac{S_n(f)}{\Delta t} = S_n(f)\Delta f$$

- The *root mean square fluctuations at frequency* f *and measured over a time* Δt are just $\Delta n(\Delta t, f)_{\mathrm{rms}} = \sqrt{S_n(f)\Delta f}$

Principles of Signal Analysis

- The auto-correlation function of a (zero mean) time series is defined by

$$C(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} n(t)n(t+\tau)\mathrm{d}t$$

- For an ergodic (and hence stationary) random process this is equivalent to the expectation value over the ensemble

$$C(\tau) =< n(t)n(t+\tau) >$$

- The auto-correlation function is the Fourier transform of the spectral density (the Wiener-Khintchine theorem).

Principles of Signal Analysis

- For stationary processes a consequence of the Wiener-Khintchine theorem is that

$$\langle \tilde{n}^*(f)\tilde{n}(f')\rangle = S_n(f)\delta(f-f')$$

- where ~ denotes the Fourier transform, and * denotes complex conjugation.

- Examples of spectral densities include -

white noise spectrum	$S_n(f) = \mathrm{const.}$
flicker noise spectrum	$S_n(f) \propto 1/f$
random walk spectrum	$S_n(f) \propto 1/f^2$

Principles of Signal Analysis

- Can also define a *cross-spectral density* between two separate random process *n(t)* and *m(t)*

$$S_{nm}(f) = \lim_{T\to\infty} \frac{2}{T} \left[\int_{-T/2}^{T/2} n(t) \exp(-2\pi i f t) \mathrm{d}t \right] \left[\int_{-T/2}^{T/2} m(t) \exp(2\pi i f t') \mathrm{d}t' \right]$$

- Similarly we can define the cross-correlation between two time series

$$C_{nm}(\tau) = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} n(t) m(t+\tau) \mathrm{d}t$$

- As in the case of a single process, these are related to each other via a Fourier transform.

Principles of Signal Analysis

- For a Gaussian, stationary random process the spectral density conveys all the information about the statistical properties of the process.
- For gravitational wave detectors, it is natural therefore to plot the spectral density to characterise the detector sensitivity. But - how then do we represent sources on the same diagram?
- There is no unique way to do this. Different types of source are best represented in different ways.

Signal Sensitivity: Bursts

- A transient *burst* of gravitational waves can be characterised by its frequency, *f*, its duration, Δt, its *bandwidth*, Δf and its mean square amplitude, a proxy for signal power

$$\bar{P}_h = \frac{1}{\Delta t} \int_0^{\Delta t} |h(t)|^2 \mathrm{d}t = h_c^2$$

- The square root of this defines the *characteristic amplitude* of the burst, h_c.

- The power in the noise in the same bandwidth is

$$\Delta f S_n(f)$$

Signal Sensitivity: Bursts

- The square root of the ratio of the signal power to the noise power is the *signal-to-noise ratio.*

$$\left(\frac{\mathrm{S}}{\mathrm{N}}\right)^2 = \frac{\bar{P}_h}{\Delta f S_h(f)} = \frac{h_c^2}{\Delta f S_h(f)}$$

- This is a measure of detectability. If we window and bandpass the time series, this is the ratio of the root-mean-square signal contribution to the root-mean-square noise contribution.

- For a broad-band burst with $\Delta f \sim f$, the signal-to-noise ratio is approximately

$$\left(\frac{\mathrm{S}}{\mathrm{N}}\right)^2 = \frac{h_c^2}{f S_h(f)}$$

- This motivates plotting $f\, S_h(f)$ instead of PSD. Height above this curve is a measure of burst detectability.

Signal Sensitivity: Continuous Waves

- Consider now a monochromatic GW source

$$h(t) = h_0 \exp(2\pi i f_0 t)$$

- The signal power is constant over time and given by

$$P_h = \lim_{T \to \infty} \int_{-T/2}^{T/2} |h(t)|^2 \mathrm{d}t = \frac{1}{2} h_0^2$$

- However, this power is concentrated at f_0. With finite time series of length T we can resolve frequency to a precision

$$\Delta f \sim 1/T$$

- Noise power in this bandwidth is $S_n(f)/T$.

Signal Sensitivity: Continuous Waves

- This motivates representing sensitivity by plotting

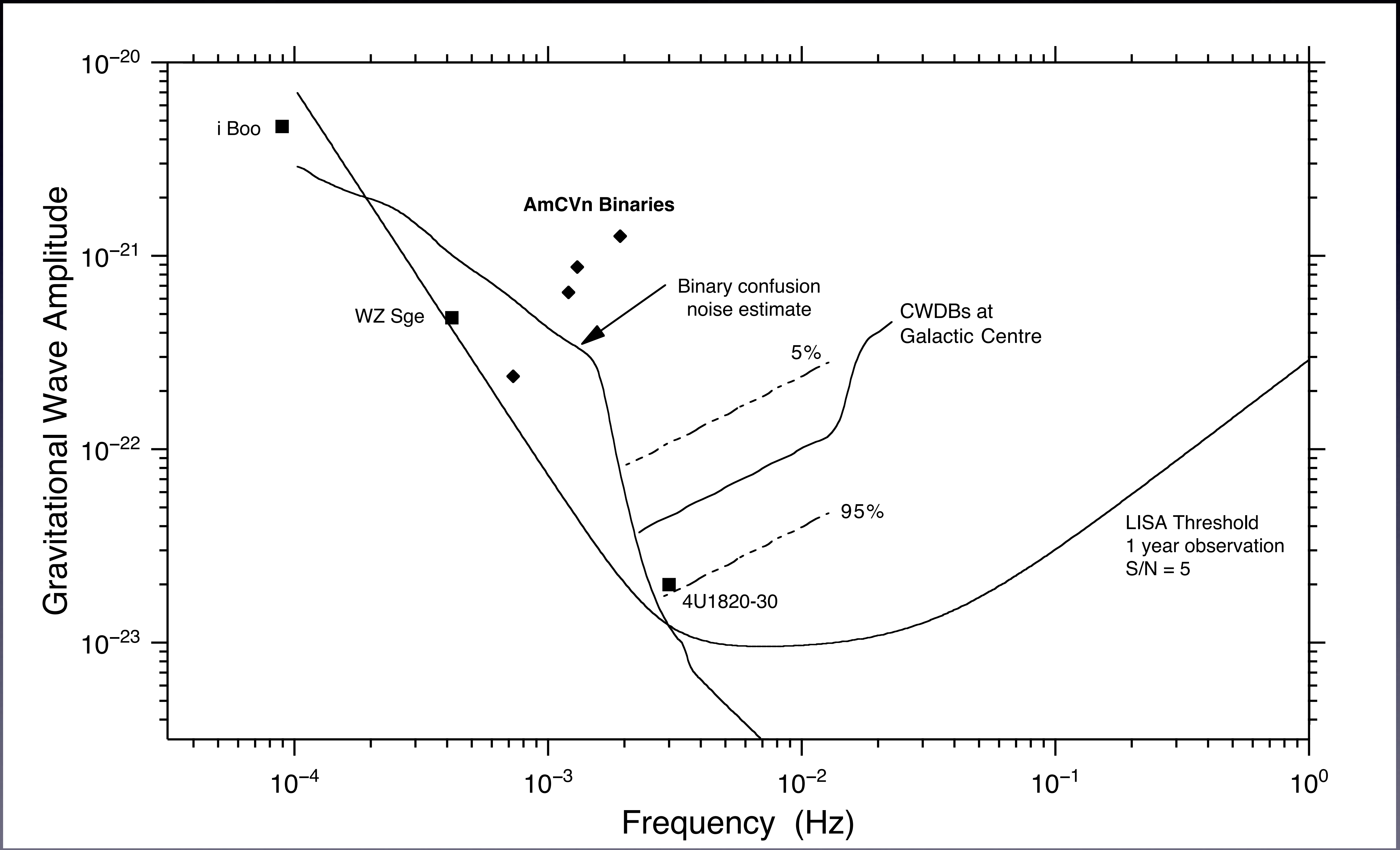
$$\sqrt{S_n(f)/T} \quad \text{or} \quad \rho_{\text{thresh}}\sqrt{S_n(f)/T}$$

- where ρ_{thresh} is the estimated threshold S/N needed for detection. This is the *strain spectral density*.

- Advantage: for a monochromatic source, height above curve gives expected S/N or, with specified threshold, an easy assessment of whether source is detectable or not.

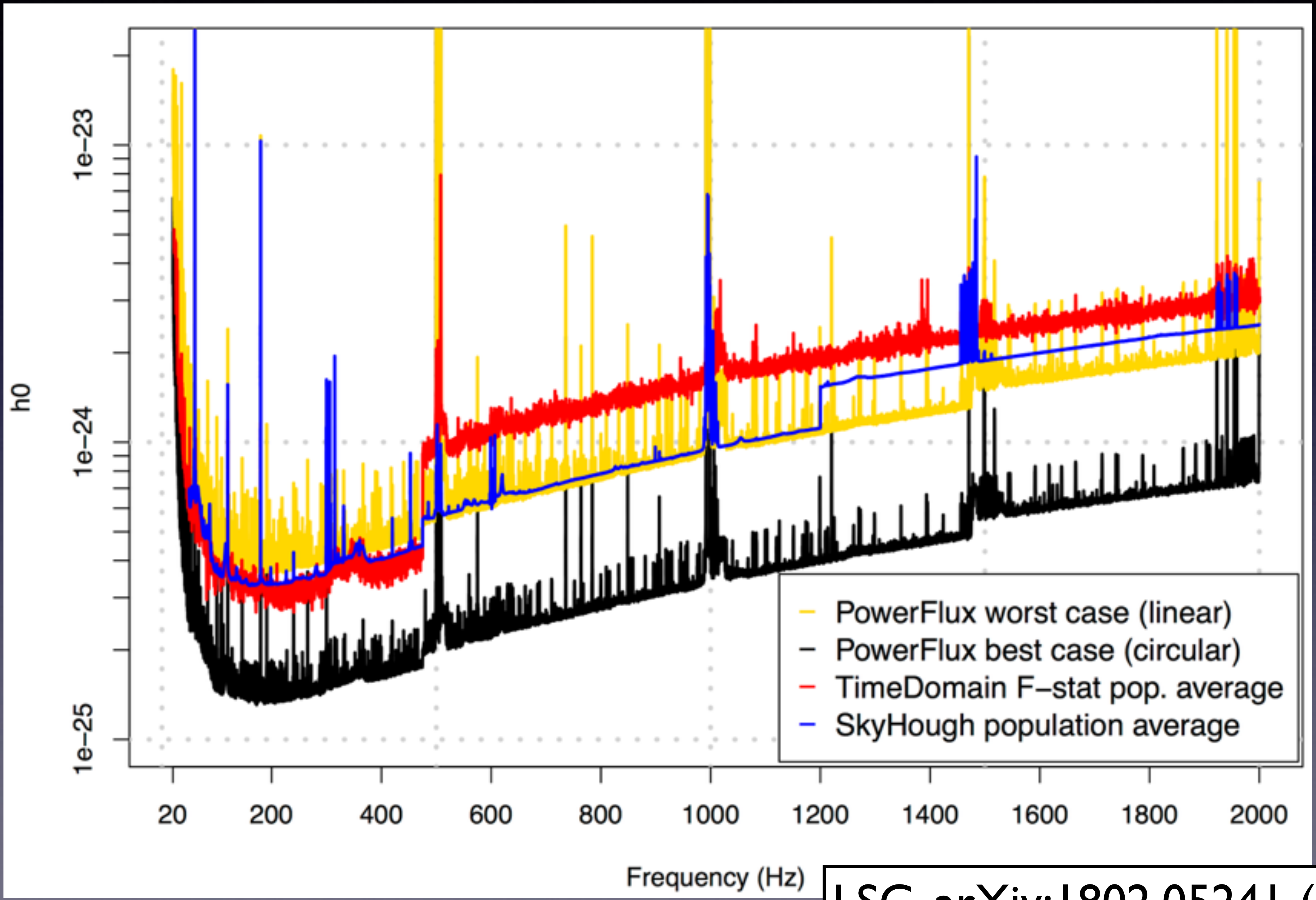
- Disadvantage: must specify length of observation. Not appropriate for ongoing experiments, e.g., LIGO. But - can produce this after each observing run.

Signal Sensitivity: Continuous Waves



LISA Pre-Phase A report (1998)

Signal Sensitivity: Continuous Waves



LSC, arXiv:1802.05241 (2018)

Signal Sensitivity: Continuous Waves

- SNRs also depend on the *sky position* and *orientation* of a source. This can be folded into the spectral density be using a sky and orientation averaged sensitivity, and using the strain of an optimally positioned and oriented source.

$$\langle S_h(f)\rangle_{\rm SA}^{LIGO} \approx 5 S_h(f) \qquad\qquad \langle S_h(f)\rangle_{\rm SA}^{LISA} \approx 20/3 S_h(f)$$

Signal Sensitivity: Inspiraling Sources

- For an inspiraling source, the total energy emitted in each frequency band is finite and so is the Fourier transform.
- Hence

$$\frac{1}{\sqrt{T}}\tilde{h}(f) \Rightarrow 0 \quad \text{as} \quad T \rightarrow \infty$$

- and so the spectral density is zero (over all time).
- Band passing and windowing can recover some of the power, but can we do better than this?
- Yes - using *filtering*.

Filtering

- A filtered time series is defined using a *kernel* $K(t-t')$.

$$w(t) = \int_{-\infty}^{\infty} K(t-t')s(t')\mathrm{d}t'$$

- We now apply a slightly modified definition of S/N. We compare the *amplitude* output of the filter due to the signal to the rms output of the filter due to the noise.

$$\left(\frac{\mathrm{S}}{\mathrm{N}}\right)(t) = \frac{\int_{-\infty}^{\infty} K(t-t')h(t')\mathrm{d}t'}{\sqrt{\left\langle \left|\int_{-\infty}^{\infty} K(t-t')n(t')\mathrm{d}t'\right|^2 \right\rangle}}$$

- The rms output of the filter $S+N$ is the signal amplitude to within an rms fractional error N/S, which is the reciprocal of the signal to noise ratio.

Optimal Filter

- We can ask what choice of filter maximises the value of S/N at *zero-lag*, i.e., t=0.

- From the convolution theorem for Fourier transforms we have

$$\tilde{w}(f) = \tilde{K}(f)\tilde{h}(f)$$

- The expression for S/N can thus be written

$$\frac{\mathrm{S}}{\mathrm{N}} = \frac{\int \tilde{K}(f)\tilde{h}(f)\mathrm{d}f}{\sqrt{\int |\tilde{K}(f')|^2 S_h(f')\mathrm{d}f'}}$$

Optimal Filter

- This motivates a natural inner product, $(\mathbf{h_1}|\mathbf{h_2})$, on the space of signals of the form

$$(\mathbf{h_1}|\mathbf{h_2}) = 2\int_0^\infty \frac{\tilde{\mathbf{h_1}}^*(f)\tilde{\mathbf{h_2}}(f) + \tilde{\mathbf{h_1}}(f)\tilde{\mathbf{h_2}}^*(f)}{S_h(f)}\mathrm{d}f$$

- in terms of which we have $\dfrac{\mathrm{S}}{\mathrm{N}} = \dfrac{(S_h K|h)}{\sqrt{(S_h K|S_h K)}}$
- which is maximised by the choice

$$\tilde{K}(f) \propto \frac{\tilde{h}(f)}{S_h(f)}$$

- This is the *Weiner optimal filter*. In the frequency domain the optimal kernel is equal to the signal weighted by the spectral density of the noise.

Optimal Filter

- A search using the optimal filter then amounts to taking the inner product $(\mathbf{s}|h)$ of the data stream, $\mathbf{s}$, with a *template* of the signal $\mathbf{h}$. This is *matched filtering*.

- The signal to noise ratio of a matched filtering search is

$$\frac{S}{N}[\mathbf{h}] = \frac{(\mathbf{h}|\mathbf{h})}{\sqrt{\langle(\mathbf{h}|\mathbf{n})(\mathbf{h}|\mathbf{n})\rangle}} = (\mathbf{h}|\mathbf{h})^{1/2}$$

- which follows from the fact that $\langle(\mathbf{h_1}|\mathbf{n})(\mathbf{h_2}|\mathbf{n})\rangle = (\mathbf{h_1}|\mathbf{h_2})$

- For a monochromatic source, the matched filter is just a Fourier transform, so this agrees with the previous result. In that case, the signal to noise ratio increases like the square root of the observation time.

Optimal Filter

- The matched filtering S/N squared is

$$\left(\frac{\mathrm{S}}{\mathrm{N}}\right)^2 = 4\int_0^\infty \frac{|\tilde{h}(f)|^2}{S_h(f)}\mathrm{d}f$$

- which can also be written as

$$\left(\frac{\mathrm{S}}{\mathrm{N}}\right)^2 = 4\int_0^\infty \frac{f|\tilde{h}(f)|^2}{S_h(f)}\mathrm{d}\ln f = 4\int_0^\infty \frac{f^2|\tilde{h}(f)|^2}{fS_h(f)}\mathrm{d}\ln f$$

- These expressions aid "integration by eye" in a logarithmic plot.

- For a source which has amplitude h_0 at frequency f and corresponding frequency derivative $\dot{f}$, we have

$$\tilde{h}(f) \sim \frac{h_0}{\sqrt{\dot{f}}}$$

Characteristic Strain

- The analogy with a broad-band burst therefore motivates the definition of a *characteristic strain,* h_c, for inspiraling sources (e.g., Finn and Thorne 2000).

$$h_c = h_o \sqrt{\frac{2f^2}{\mathrm{d}f/\mathrm{d}t}}$$

- The characteristic strain is a measure of the SNR accumulated while the frequency sweeps through a bandwidth equal to frequency. If we also plot the rms noise in a bandwidth equal to frequency, $h_n(f)$

$$h_n(f) \equiv \sqrt{f\langle S_h(f)\rangle_{\mathrm{SA}}} \qquad \left(\frac{S}{N}\right)^2_{f\to 2f} = \left[\frac{h_c(f)}{h_n(f)}\right]^2$$

- Plots of $h_c(f)$ and $h_n(f)$ allow us to see directly how the SNR of an evolving source builds up over the evolution.

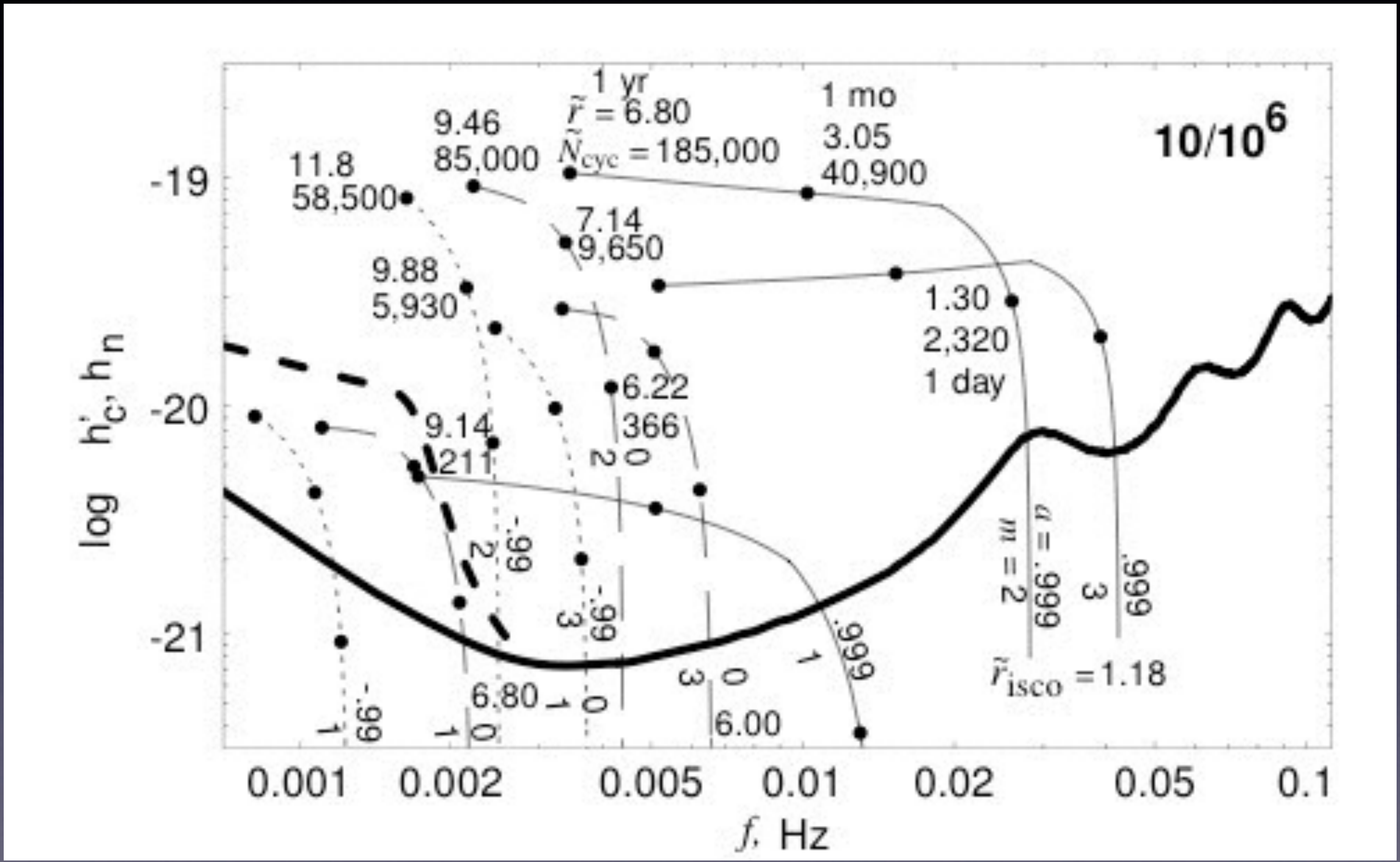
Characteristic Strain

- In the definition of *characteristic strain*

$$h_c = h_o\sqrt{\frac{2f^2}{\mathrm{d}f/\mathrm{d}t}}$$

- the term inside the square root is equal to the number of cycles the inspiral spends in the vicinity of the frequency *f*.
- You will read papers in which people talk about S/N being enhanced by the number of cycles spent int he vicinity of a certain frequency. This is what they are referring to.
- **Note**: plotting characteristic strain only makes sense if you are *also* plotting $f\,S_h(f)$. If you are plotting $S_h(f)$ directly your strain should be a factor of $\sqrt{f}$ lower.

Characteristic Strain



Build up of SNR for EMRIs observed by LISA
(Finn & Thorne 2000)

Representing Stochastic Backgrounds

- *Stochastic backgrounds* are characterised by a spectral density, so it is natural to compute the power spectral density and plot it on the same axes as the detector PSD.

- There are two caveats.

- Firstly, the "power" we have been talking about so far has not been a power in a physical sense since we have not specified any unites for the time series (and indeed for GW strain this is dimensionless). Better to use something that represents a physical energy density if possible.

- Plotting two PSDs does not convey any information about their distinguishability. Can we represent backgrounds in a way that allows the reader to assess detectability at a glance?

Representing Stochastic Backgrounds

- The energy density carried by a gravitational wave is

$$\frac{\mathrm{d}E}{\mathrm{d}t\mathrm{d}A} \propto \dot{h}_+^2 + \dot{h}_\times^2$$

- Therefore, we should consider the time derivative of the strain series to get a physical energy.

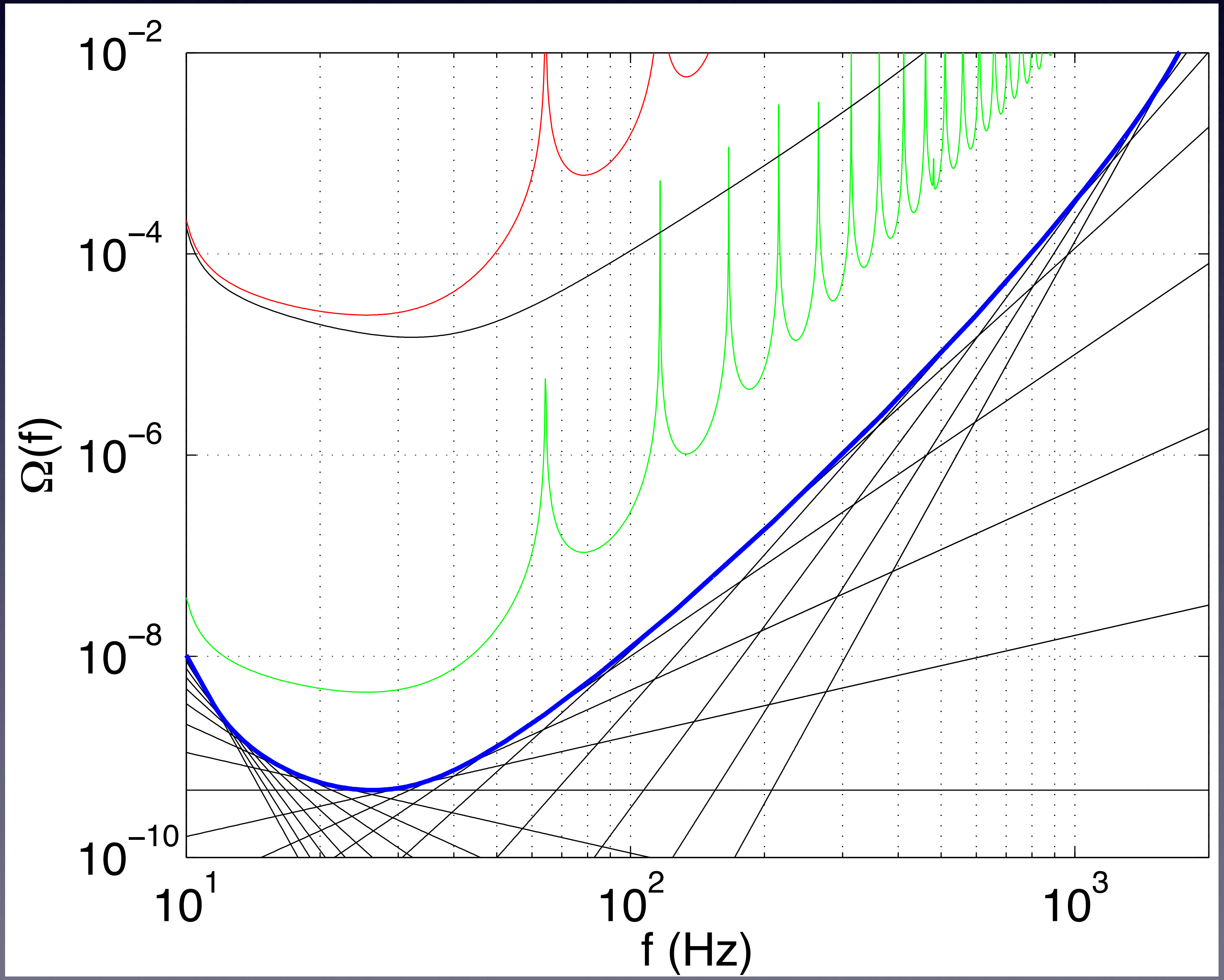
- The corresponding spectral density is $f^2 S_h(f)$ and fluctuations in a bandwidth equal to frequency are $f^3 S_h(f)$.

- Energy densities in astrophysical and cosmological backgrounds are often expressed as a fraction of the *closure density* of the Universe

$$\Omega_{\mathrm{GW}} = \frac{8\pi G}{3H_0^2}\frac{\mathrm{d}E_{\mathrm{GW}}}{\mathrm{d}\ln f} \propto f^2 h_c^2(f)$$

Representing Stochastic Backgrounds

- Quick assessment of background detectability can be derived from *power-law sensitivity curves* (Thrane & Romano 2013). Requires assumptions about data analysis procedures.



Sensitivity Curves Summary

- To summarise, there are four different types of sensitivity curve you might see in figures.

- *Power Spectral Density* - summarises statistical properties of noise

$$S_n(f)$$

- *Strain spectral density*

$S_n(f)/T$ - for monochromatic sources

$fS_n(f)$ - for inspirals and bursts

- *Energy spectral density* - for backgrounds

$$f^3 S_n(f)$$

Estimating Detectability of Sources

Electromagnetic Radiation and GWs “on the back of an envelope”

Recap of Electromagnetic Radiation

- Maxwell's equations describe electromagnetism in a fully covariant way

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \wedge \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

- Can introduce a scalar and vector potential to describe the fields

$$\mathbf{E} = -\nabla \phi + \dot{\mathbf{A}} \qquad \mathbf{B} = \nabla \wedge \mathbf{A}$$

- The vector potential is not unique (gauge freedom)

$$\begin{aligned} \mathbf{A}' &= \mathbf{A} + \nabla \chi \\ \phi' &= \phi + \dot{\chi} \end{aligned} \quad \Rightarrow \quad \begin{aligned} \mathbf{E}' &= \mathbf{E} \\ \mathbf{B}' &= \mathbf{B} \end{aligned}$$

Recap of Electromagnetic Radiation

- The theory can be rewritten in covariant form by defining

$$A_\mu = (\phi/c^2, -\mathbf{A}) \qquad\qquad J^\mu = (\rho, \mathbf{J})$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & -E_x/c^2 & -E_y/c^2 & -E_z/c^2 \\ E_x/c^2 & 0 & -B_z & B_y \\ E_y/c^2 & B_z & 0 & -B_x \\ E_z/c^2 & -B_y & B_x & 0 \end{pmatrix}$$

- Maxwell's equations become

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu \qquad \Rightarrow \qquad \partial_\mu \partial^\mu A^\nu - \partial_\nu \partial_\mu A^\mu = \mu_0 J^\nu$$

- Gauge freedom allows simplification to Lorenz gauge

$$\begin{aligned} A'_\mu &= A_\mu + \partial_\mu \chi \\ \partial_\mu \partial^\mu \chi &= -\partial_\mu A^\mu \end{aligned} \qquad \Rightarrow \qquad \begin{aligned} \partial_\mu A^{\mu\prime} &= 0 \\ \partial_\mu \partial^\mu A^\nu \equiv \Box A^\nu &= \mu_0 J^\nu \end{aligned}$$

Recap of Electromagnetic Radiation

- The general solution of the flat space wave equation may be found from a Green's function to be

$$A^\nu(\mathbf{x}, t) = -\frac{\mu_0}{4\pi}\int_V \frac{J^\nu\left(\mathbf{x}', t - |\mathbf{x}-\mathbf{x}'|/c\right)}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}V'$$

- We now consider a region far from all sources such that the source position $|\mathbf{x}| \gg R$, the size of the source, and can expand the factor $|\mathbf{x}-\mathbf{x}'|$

$$|\mathbf{x}-\mathbf{x}'|^{-1} = |\mathbf{x}|^{-1} + 2\mathbf{x}'\cdot\mathbf{n}|\mathbf{x}|^{-2} + \cdots$$

- In which $\mathbf{n} = \mathbf{x}/|\mathbf{x}|$. Similarly, the retarded time factor is nearly constant and equal to $t_0 \equiv t - |\mathbf{x}|/c$, and we can expand the difference

$$t - t_0 = (|\mathbf{x}| - |\,\mathbf{x}-\mathbf{x}'|)/c = -\mathbf{n}\cdot\mathbf{x}'/c + O(|\mathbf{x}|^{-1})$$

Recap of Electromagnetic Radiation

- The dominant $|\mathbf{x}|^{-1}$ field at large distances may thus be written as an expansion in time derivatives of the source distribution

$$A^\nu = -\frac{\mu_0}{4\pi}|\mathbf{x}|^{-1}\int_V \left\{J^\nu(t_0) - \partial_t J^\nu(t_0)\mathbf{n}\cdot\mathbf{x}'/c + \partial_{tt}J^\nu(t_0)\left(\mathbf{n}\cdot\mathbf{x}'\right)^2/(2c^2) + \cdots\right\}\mathrm{d}V'$$

- The first term is just the total charge/current, a constant

$$J^\nu = \int_V J^\nu(t_0)\mathrm{d}V$$

- This is the static monopole potential and is not radiative. The second term is the time derivative of the dipole moment

$$\int_V \partial_t J^\nu(t_0)\mathbf{n}\cdot\mathbf{x}'/c\mathrm{d}V = \mathbf{n}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_V J^\nu\mathbf{x}'\mathrm{d}V'\right)$$

- This is the dominant piece of the e.m. radiation field.

Gravitational Waves
"on the back of an envelope"

- B. Schutz, *American Journal of Physics* **52**, 412 (1984).

- In Newtonian theory, the gravitational potential satisfies the Poisson equation $\nabla^2\phi = 4\pi G\rho$ with solution

$$\phi(\mathbf{x},t) = -G\int_V \frac{\rho(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}V'$$

- Natural way to include special relativity, i.e., finite propagation speed, is to replace time by retarded time $t \to t - |\mathbf{x}-\mathbf{x}'|/c$

$$\phi(\mathbf{x},t) = -G\int_V \frac{\rho\left(\mathbf{x}',t-|\mathbf{x}-\mathbf{x}'|/c\right)}{|\mathbf{x}-\mathbf{x}'|}\mathrm{d}V'$$

- This is just the solution of the wave equation $\Box\phi = 4\pi G\rho$, and so includes gravitational waves. Can expand the solution far from any sources as in the electromagnetic case.

Gravitational Waves
"on the back of an envelope"

- As before, leading order term is just the monopole potential, in this case the total mass of the system. The dipole term can be simplified using the continuity equation.

$$\frac{\partial}{\partial t}\left(\rho(t,\mathbf{x}')\right) + \frac{\partial}{\partial \mathbf{x}'_\mathbf{j}}\left(\rho v'_j\right) = 0$$

$$\Rightarrow \int_V \dot{\rho}\mathbf{x}'_\mathbf{i}\mathrm{d}V' = -\int_V \mathbf{x}'_\mathbf{i}\frac{\partial}{\partial \mathbf{x}'_\mathbf{j}}\left(\rho v'_j\right) = \int_V \rho v_i \mathrm{d}V' = P_i$$

- This is just the momentum, which is also conserved. The radiative field appears first due to a time-varying quadrupole moment

$$\phi \approx -\frac{GM}{|\mathbf{x}|} + \frac{G\mathbf{n}\cdot\mathbf{P}}{c|\mathbf{x}|} - \frac{G\ddot{I}_{ij}n_i n_j}{2c^2|\mathbf{x}|} \qquad I_{ij} = \int \rho \mathbf{x}'_\mathbf{i}\mathbf{x}'_\mathbf{j}\mathrm{d}V'$$

- Characterize field by dimensionless $h \approx -(G/2c^4)\ddot{I}_{ij}n_i n_j/|\mathbf{x}|$

Gravitational Waves
"on the back of an envelope"

- In full general relativity, the propagating field is the spacetime metric, a tensor. It is the *transverse and traceless* components that propagate, and correct formula is

$$h_{ij} \approx -(G/2c^4)\ddot{\mathcal{I}}_{ij}/|\mathbf{x}| \qquad \mathcal{I}_{ij} = I_{ij} - \frac{1}{3}\delta_{ij}I_{kk}$$

- How much energy does the field carry? The toy scalar gravitational field has an energy flux associated with it of the form $F = -(1/4\pi G)(\mathbf{n}\cdot\nabla\phi)\dot{\phi}$

- In the wave zone, $\mathbf{n}\cdot\nabla\phi \approx -\dot{\phi}$, so the flux is $(c^3/4\pi G)\dot{h}^2$ and we deduce the source luminosity $L = 4\pi|\mathbf{x}|^2 F$ is approximately $L \sim (G/4c^2)\,\dddot{I}^2$ which is a good approximation to the full GR result

$$L = (G/5c^2)\dddot{\mathcal{I}}_{jk}\dddot{\mathcal{I}}_{jk}$$

Estimating GW Amplitudes for Astrophysical Sources

Typical strains of astrophysical sources

- Order of magnitude of quadrupole moment can be estimated from virial theorem

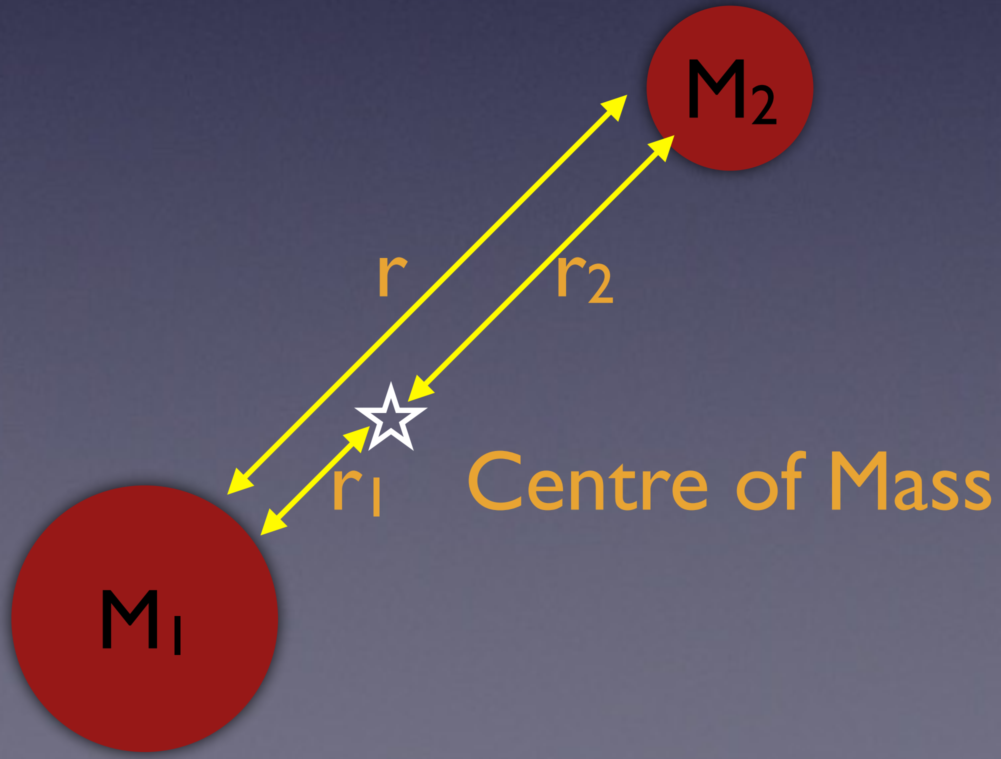
$$\ddot{I}_{ij} \sim 2 \int \rho v_i v_j \mathrm{d}V \Rightarrow |\ddot{I}_{ij} n_i n_j| \lesssim 2M\phi_{\mathrm{int}} \Rightarrow |h| \lesssim \phi_N \phi_{\mathrm{int}}/c^4$$

- Newtonian circular binary system with masses M_1, M_2 and separation *r* is equivalent to object of mass μ orbiting in a fixed potential of mass *M* at distance *r*.

$$\mu = \frac{M_1 M_2}{M_1 + M2}$$

$$M = M_1 + M_2$$

$$r_1 M_1 = r_2 M_2 = \mu r$$



Typical strains of astrophysical sources

- From general properties of Newtonian orbits

$$\omega^2 = \left(\frac{2\pi}{T}\right)^2 = (2\pi f)^2 = \frac{M}{r^3} \qquad E = -\frac{M\mu}{2r}$$

- we deduce

$$I \sim \mu r^2 \cos 2\omega t \sim \frac{M_1 M_2}{(M_1 + M_2)^{\frac{1}{3}}} \omega^{-\frac{4}{3}}$$

- And hence

$$h \sim \frac{1}{D} \frac{M_1 M_2}{(M_1 + M_2)^{\frac{1}{3}}} \omega^{\frac{2}{3}}$$

- with emission at $f_{\mathrm{GW}} = \omega/\pi$ Hz.

Typical strains of astrophysical sources

- Similarly we find

$$\dot{E} \sim \mu^2 M^{\frac{4}{3}} \omega^{\frac{10}{3}}$$

$$\dot{\omega} \sim \frac{M_1 M_2}{(M_1 + M_2)^{\frac{1}{3}}} \omega^{\frac{11}{3}} = M_c^{\frac{5}{3}} \omega^{\frac{11}{3}}$$

- where we have introduced the chirp mass

$$M_c = \frac{M_1^{\frac{3}{5}} M_2^{\frac{3}{5}}}{(M_1 + M_2)^{\frac{1}{5}}}$$

Typical strains of astrophysical sources

- Finally, we deduce expressions for the Fourier domain amplitude of the signal and the characteristic strain

$$\tilde{h}(f) \sim \frac{1}{D} M_c^{\frac{5}{6}} f^{-\frac{7}{6}} \qquad h_c(f) \sim \frac{1}{D} M_c^{\frac{5}{6}} f^{-\frac{1}{6}}$$

- Reintroducing dimensional and numerical factors the expressions for strain and luminosity are

$$h \sim \frac{G^{\frac{5}{3}}}{2c^4} \frac{1}{|\mathbf{x}|} \frac{M_1 M_2}{(M_1 + M_2)^{\frac{1}{3}}} \omega^{\frac{2}{3}} \qquad L \sim \frac{G^{\frac{7}{3}}}{5c^5} \frac{M_1^2 M_2^2}{(M_1 + M_2)^{\frac{2}{3}}} \omega^{\frac{10}{3}}$$

Typical strains of astrophysical sources

- Putting in physical units we have

$$h \sim 3 \times 10^{-16} \left(\frac{|\mathbf{x}|}{1\mathrm{pc}}\right)^{-1} \left(\frac{M_1}{M_\odot}\right) \left(\frac{M_2}{M_\odot}\right) \left(\frac{M_\odot}{M_1 + M_2}\right)^{\frac{1}{3}} \left(\frac{f}{100\mathrm{Hz}}\right)^{\frac{2}{3}}$$

$$L \sim 5 \times 10^{43} \mathrm{Js}^{-1} \left(\frac{M_1}{M_\odot}\right)^2 \left(\frac{M_2}{M_\odot}\right)^2 \left(\frac{M_\odot}{M_1 + M_2}\right)^{\frac{2}{3}} \left(\frac{f}{100\mathrm{Hz}}\right)^{\frac{10}{3}}$$

- For the Earth/Sun system, viewed at 1 parsec

$$f \sim 3 \times 10^{-8}\mathrm{Hz} \qquad h \sim 5 \times 10^{-28} \qquad L \sim 10\mathrm{Js}^{-1}$$

- For a $1M_\odot + 1M_\odot$ binary at merger

$$f = \frac{1}{2\pi}\sqrt{\frac{GM}{(6GM/c^2)^3}} = \frac{c^3}{12\sqrt{6}\pi GM} \sim 1000\,\mathrm{Hz} \qquad L \sim 10^{47}\,\mathrm{J\,s}^{-1}$$

Typical strains of astrophysical sources

- At a distance of 1 Mpc, we find $h \sim 10^{-21}$ at merger. This scales like $M^{2/3}$ so a $10^6 M_\odot + 10^6 M_\odot$ SMBH binary at 10 Gpc generates the same strain in the Solar System, with merger frequency of ~1mHz.

- A $1 M_\odot + 1 M_\odot$ millihertz compact binary at 1 kpc has

$$f \sim 3 \times 10^{-4} \mathrm{Hz} \qquad h \sim 5 \times 10^{-23} \qquad L \sim 10^{25} \mathrm{J\,s^{-1}}$$

- These are all sources we expect to be able to detect with current and future detectors. LIGO operates in the 10Hz to 10kHz range, while LISA will operate in the 0.1 mHz to 0.1Hz range.

Eccentric binaries

- Eccentric orbits have emission at all frequencies, with relative contribution

$$\dot{E}_n = \frac{32}{5}\mu^2 M^{\frac{4}{3}}(2\pi f)^{\frac{10}{3}} g(n,e)$$

$$g(n,e) = \frac{n^4}{32}\left\{\left[J_{n-2}(ne) - 2eJ_{n-1}(ne) + \frac{2}{n}J_n(ne) + 2eJ_{n+1}(ne) - J_{n+2}(ne)\right]^2 + (1-e^2)\left[J_{n-2}(ne) - 2J_n(ne) + J_{n+2}(ne)\right]^2 + \frac{4}{3n^2}\left[J_n(ne)\right]^2\right\}$$

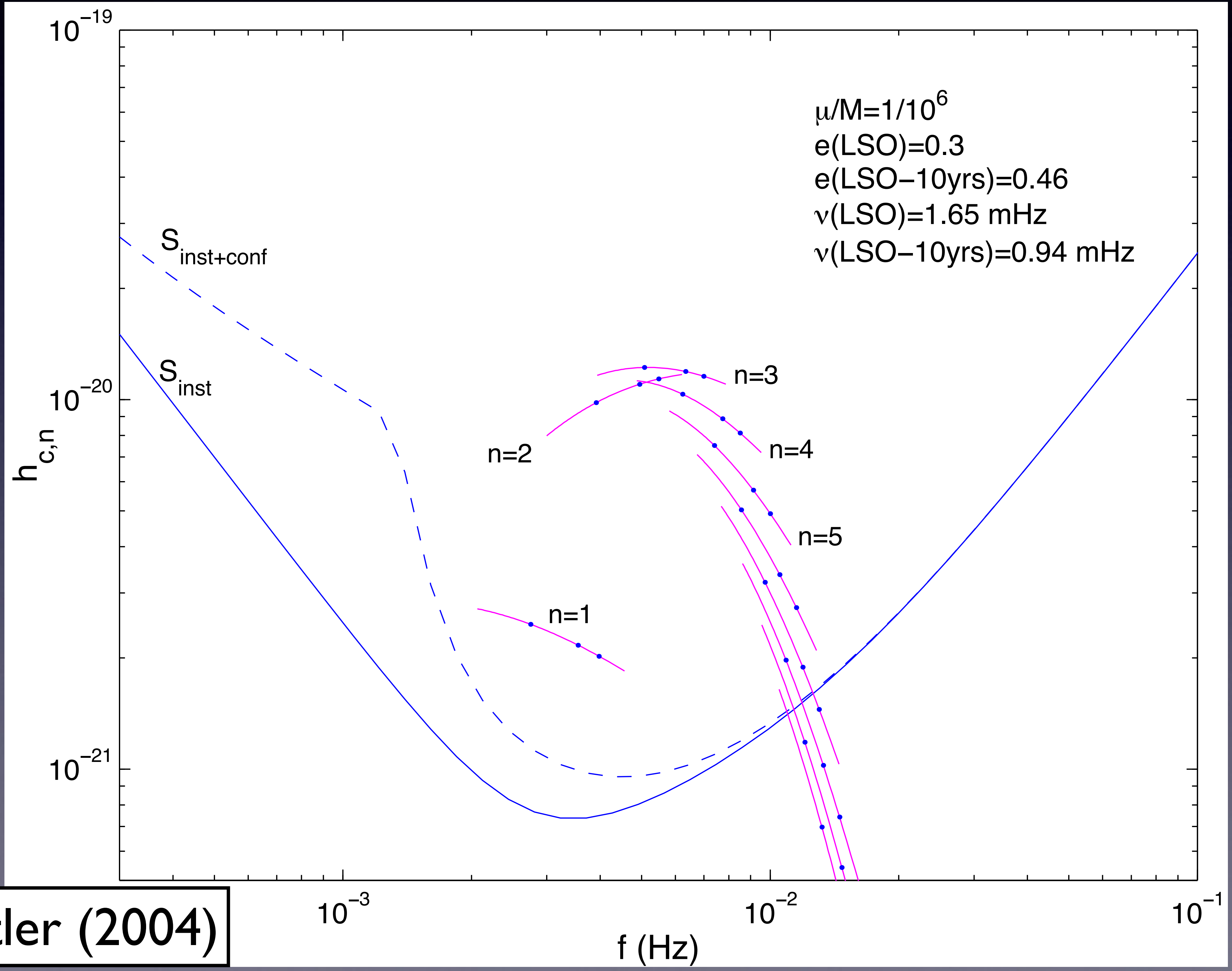
- Can compute a characteristic strain for a particular mode

$$h_{c,n}(f) = \frac{1}{\pi D}\sqrt{\frac{2\dot{E}_n(f/n)}{n\dot{f}(f/n)}} \sim M_c^{\frac{5}{6}} f^{-\frac{7}{6}} n^{\frac{2}{3}}\sqrt{g(n,e)}$$

- This is the characteristic strain of the n'th harmonic at frequency *f*. The argument *(f/n)* indicates the functions are to be evaluated when the orbital frequency is equal to *f/n*.

Eccentric binaries

- Represent these on a single "waterfall plot".



Barack & Cutler (2004)

Stochastic Backgrounds

- Recall definition

$$\Omega_{\rm GW} = \frac{8\pi G}{3H_0^2}\frac{{\rm d}E_{\rm GW}}{{\rm d}\ln f} \propto f^2 h_c^2(f)$$

- Suppose background is generated by a astrophysical population of sources with coming volume density *N(z)*. Then, total energy density in background today is

$$\mathcal{E}_{\rm GW} = \int_0^\infty \rho_c c^2 \Omega_{\rm GW} {\rm d}\ln f = \int_0^\infty \int_0^\infty N(z) \frac{1}{(1+z)} \frac{{\rm d}E}{{\rm d}f} f \frac{{\rm d}f}{f} {\rm d}z$$

- We deduce (Phinney 2001, astro-ph/0108028)

$$\rho_c c^2 \Omega_{\rm GW} = \frac{\pi}{4}\frac{c^2}{G} f^2 h_c^2(f) = \int_0^\infty \frac{N(z)}{1+z} \left(f_r \frac{{\rm d}E}{{\rm d}f_r} \right)_{|f_r = f(1+z)} {\rm d}z$$

Stochastic Backgrounds

- For a population of inspiraling binaries we have

$$f\frac{\mathrm{d}E}{\mathrm{d}f} \sim M_c^{\frac{5}{3}} f^{\frac{2}{3}}$$

- From which we deduce the scaling of the energy density

$$\Omega_{\mathrm{GW}}(f) \sim M_c^{\frac{5}{3}} f^{\frac{2}{3}} \int_0^\infty \frac{N(z)}{(1+z)^{\frac{1}{3}}} \mathrm{d}z$$

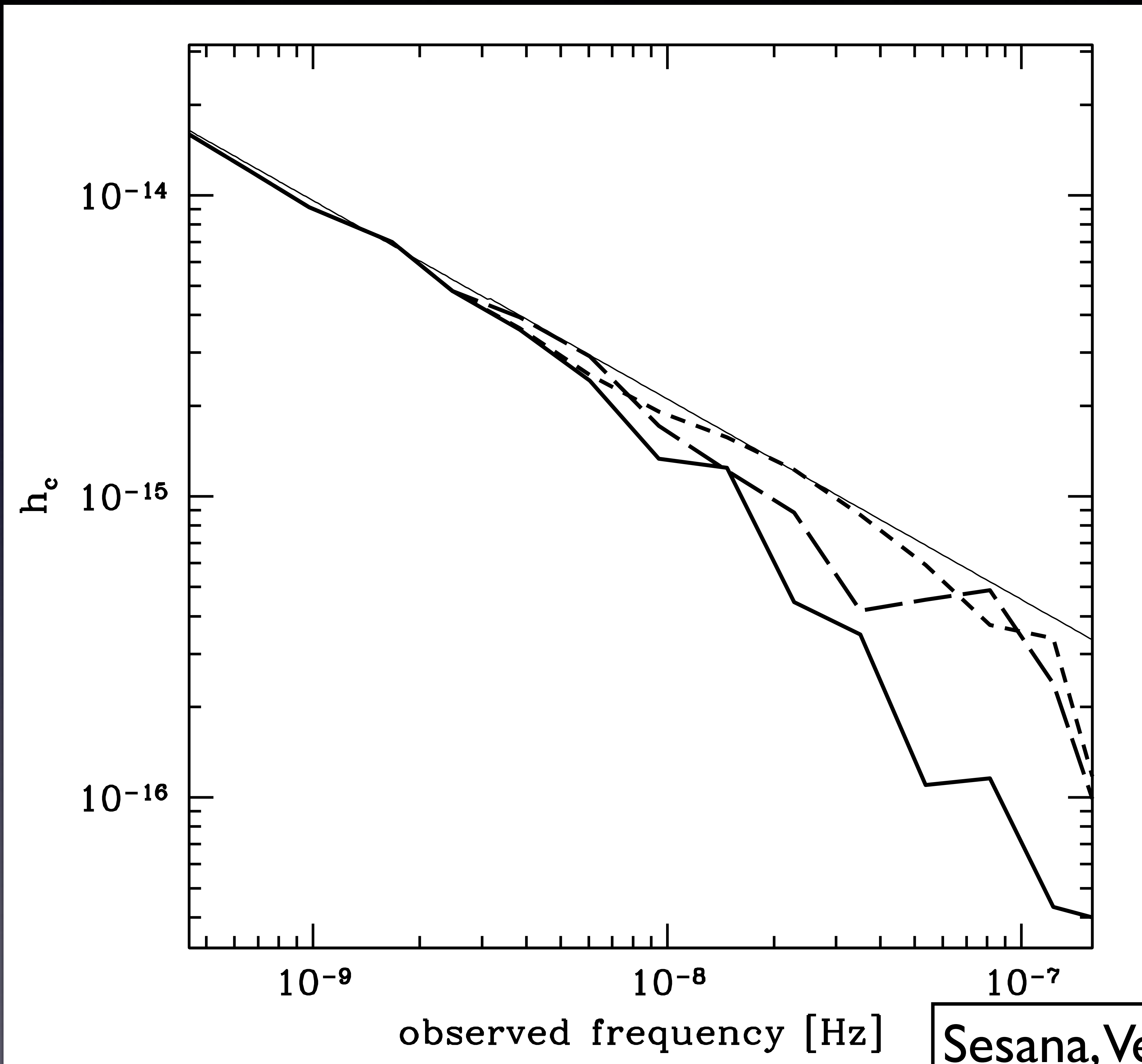
- and of the characteristic strain and spectral density

$$h_c(f) \sim \sqrt{\Omega_{\mathrm{GW}}(f)}/f \sim M_c^{\frac{5}{6}} f^{-\frac{2}{3}}$$

$$S_h(f) \sim \Omega_{\mathrm{GW}}(f)/f^3 \sim M_c^{\frac{5}{3}} f^{-\frac{7}{3}}$$

- Note the difference to the inspiraling source case. The latter is enhanced by matched filtering.

Stochastic Backgrounds



Sesana, Vecchio & Colacino (2008)

Typical strains of astrophysical sources

- Supernovae

➡ Collapse of the core of a massive star. If collapse is asymmetric, expect gravitational waves to be produced.

➡ Internal potential of system is potential of pre-collapse core, which can be modelled as a Neutron star, for which $\phi_{\mathrm{int}}/c^2 \sim 0.2$.

➡ At a distance of 10 Mpc, the Newtonian potential is

$$|\phi_{\mathrm{N}}/c^2| = \frac{GM_\odot/c^2}{d_{10} \times 10Mpc} \sim 5 \times 10^{-21} d_{10}^{-1}$$

➡ And we estimate a strain of $|h_{\mathrm{SN}}| \lesssim 10^{-21} d_{10}^{-1}$.

➡ 10 Mpc is the distance to the Virgo Cluster, in which volume the supernovae rate is ~one event per month.

Typical strains of astrophysical sources

- Rotating Neutron Stars

➡ Pulsars are rapidly rotating Neutron Stars.

➡ If the Neutron Star is deformed, i.e., it has a "mountain" on the surface, then this rotation leads to a time varying quadrupole moment and hence gravitational radiation.

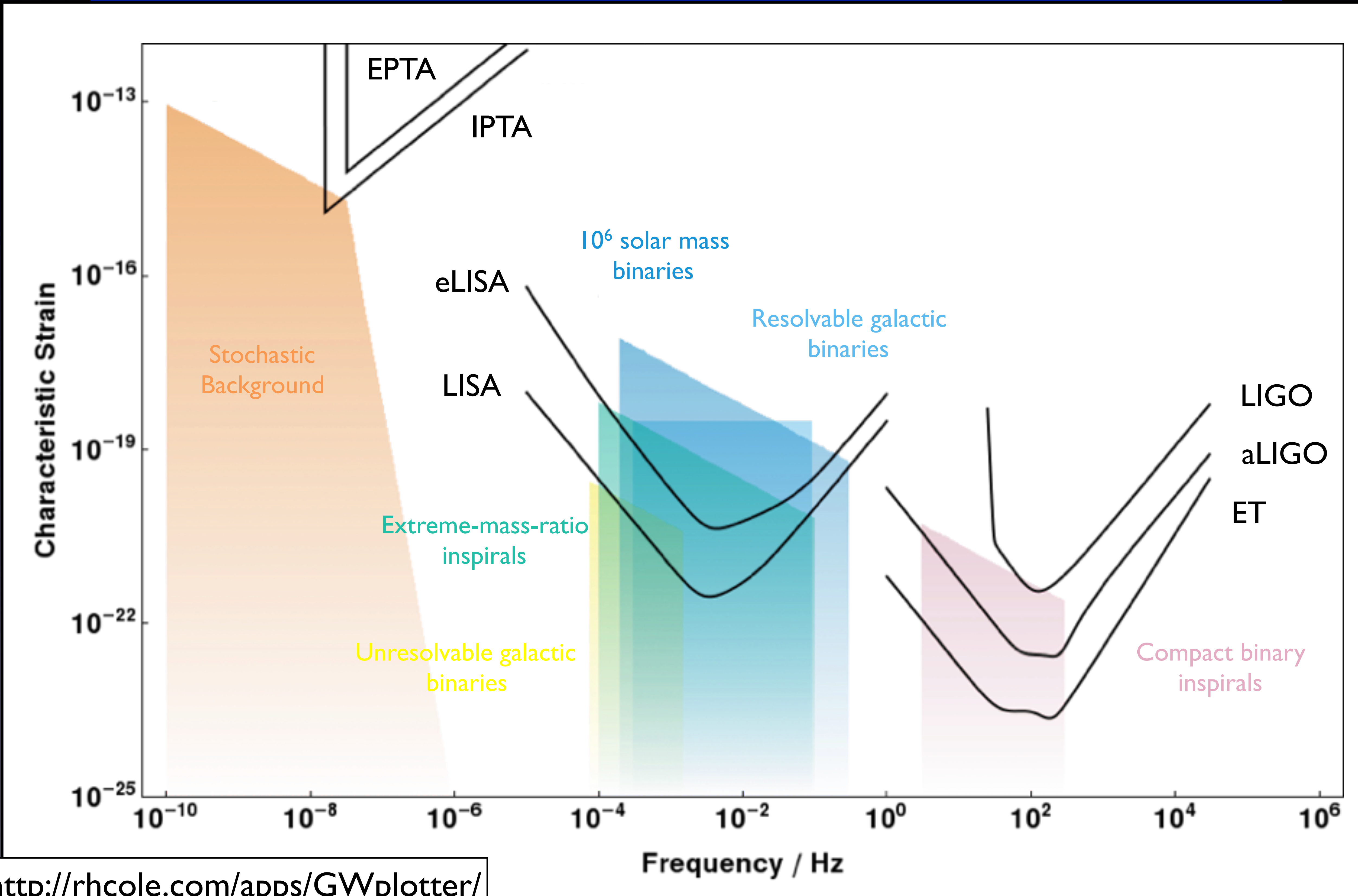
➡ The quadrupole moment of a "mountain" of mass δM on the surface of the star can be approximated as

$$I_{ij} \sim \delta M R_{\rm NS}^2 = \epsilon M_{\rm NS} R_{\rm NS}^2$$

➡ Hence, for a $1 M_\odot$ neutron star with $R_{\rm NS} = 10 \rm km$, rotating at a rate of $1 \rm kHz$, the luminosity and strain are

$$h \sim 10^{-20} \epsilon (d/1 \rm kpc)^{-1} \qquad L \sim 10^{41} \epsilon^2 \rm J\, s^{-1}$$

The Gravitational Wave Spectrum



http://rhcole.com/apps/GWplotter/