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Complex Geodesics on Convex Domains

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Dedicated to M. Valdivia on the occasion of his 60$^{\mathrm{th}}$ birthday

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Abstract

Existence and uniqueness of complex geodesics joining two points of a convex bounded domain in a Banach space X are considered. Existence is proved for the unit ball of X under the assumption that X is 1-complemented in its double dual. Another existence result for taut domains is also proved. Uniqueness is proved for strictly convex bounded domains in spaces with the analytic Radon-Nikodym property. If the unit ball of X has a modulus of complex uniform convexity with power type decay at 0, then all complex geodesics in the unit ball satisfy a Lipschitz condition. The results are applied to classical Banach spaces and to give a formula describing all complex geodesics in the unit ball of the sequence spaces ℓ^p $(1 \leq p < \infty)$.

In this article, we discuss the existence, uniqueness and continuity of complex geodesics on a convex domain $\mathcal{D}$ in a complex Banach space X. The term 'complex geodesic' is due to Vesentini [33], although the concept was discussed by Carathéodory [5] and Reiffen [27] under the name 'metric plane'. Recent results on this topic are to be found in [11, 14, 15, 16, 34, 35, 36, 37]. Applications of complex geodesics to the study of biholomorphic automorphisms and to fixed point sets are to be found in [5, 33, 34, 35, 36, 37].

Our results on the existence problem depend on topological properties of the Banach space X, the results on uniqueness depend on the geometry of the boundary $\partial\mathcal{D}$ and on an analytic-geometric property of X (the analytic Radon-Nikodym property), while the continuity (*i.e.* continuous extensions to the boundary) is obtained using complex uniform convexity.

In section 1, we introduce complex geodesics and related concepts and prove some basic results. In section 2, we show that every pair of points in the unit ball B_X of X can be joined by a complex geodesic provided X is 1-complemented in its double dual X^{**}. In proving this we show that the canonical embedding of B_X in $B_{X^{**}}$ is an isometry for the Kobayashi metrics. We also obtain a new simple proof of a result of Davie and Gamelin [7] that bounded analytic functions on B_X extend to $B_{X^{**}}$. In section 3, we use the analytic Radon-Nikodym property and extreme points to obtain uniqueness results and in Section 4 we prove continuity properties of complex geodesics. In Section 5 we apply the results of the preceding three sections to ℓ^p and related spaces.

An examination of the methods of section 2 shows that topological considerations are only used to obtain 'Montel type' theorems for mappings defined on the unit disc in $\mathbb{C}$ with values in the domain $\mathcal{D} \subset X$ under consideration. This led us to consider (in an infinite-dimensional setting) the concepts of taut and complete hyperbolic domains (section 6). In section 6, we also apply our results to show that certain domains have constant negative curvature in the Kobayashi metric.

A number of the results in this paper were announced in [10]. We refer to [10] for background results and further details on complex geodesics.

1 COMPLEX GEODESICS

$\mathbb{D}$ is the open unit disc in $\mathbb{C}$ and ρ will denote the Poincaré distance on $\mathbb{D}$, i.e.

$$\rho(z,w) = \tanh^{-1}\left(\left|\frac{z-w}{1-\bar{w}z}\right|\right) \qquad (z,w \in \mathbb{D}).$$

The infinitesimal Poincaré distance α is defined by $\alpha(z,v) = |v|/(1-|z|^2)$ (for $v \in \mathbb{C}$ and $z \in \mathbb{D}$).

For $\mathcal{D}_1$ and $\mathcal{D}_2$ domains in complex Banach spaces, $H(\mathcal{D}_1, \mathcal{D}_2)$ denotes the set of all $\mathcal{D}_2$-valued holomorphic functions on $\mathcal{D}_1$. For a domain $\mathcal{D}$ in a complex Banach space X, $p, q \in \mathcal{D}$ and $v \in V$,

$$\begin{aligned}
C_{\mathcal{D}}(p,q) &= \sup\{\rho(f(p), f(q)) : f \in H(\mathcal{D}, \mathbb{D})\} \\
c_{\mathcal{D}}(p,v) &= \sup\{|f'(p)(v)|) : f \in H(\mathcal{D}, \mathbb{D})\} \\
\delta_{\mathcal{D}}(p,q) &= \inf\{\rho(u,v) : \exists f \in H(\mathbb{D}, \mathcal{D}), f(u) = p, f(v) = q\} \\
K_{\mathcal{D}}(p,q) &= \inf\left\{\sum_{i=1}^{n} \delta_{\mathcal{D}}(w_{i-1}, w_i) : n \geq 1, p = w_0, w_1, \ldots, w_n = q \in \mathcal{D}\right\} \\
k_{\mathcal{D}}(p,v) &= \inf\{\eta > 0 : \exists f \in H(\mathbb{D}, \mathcal{D}), f(0) = p, f'(0)\eta = v\}.
\end{aligned}$$

$C_{\mathcal{D}}$ is called the Carathéodory distance on $\mathcal{D}$, $K_{\mathcal{D}}$ the Kobayashi distance and $c_{\mathcal{D}}$ and $k_{\mathcal{D}}$ are the corresponding infinitesimal metrics.

For the unit disc, $C_{\mathbb{D}} = K_{\mathbb{D}} = \rho$ and $c_{\mathbb{D}} = k_{\mathbb{D}} = \alpha$. In general $C_{\mathcal{D}} \leq K_{\mathcal{D}} \leq \delta_{\mathcal{D}}$. Unlike $C_{\mathcal{D}}$ and $K_{\mathcal{D}}$, $\delta_{\mathcal{D}}$ does not obey the triangle inequality in general. $K_{\mathcal{D}}$ is the largest distance function on $\mathcal{D}$ smaller than $\delta_{\mathcal{D}}$. Holomorphic mappings are contractions relative to any one of the above distances or infinitesimal metrics.

Definition 1.1 *Let $\mathcal{D}$ be a domain in a complex Banach space X and let d be a distance on $\mathcal{D}$. A mapping $\phi \in H(\mathbb{D}, \mathcal{D})$ is called a* complex d-geodesic *if*

$$\rho(u, v) = d(\phi(u), \phi(v)) \text{ for } u, v \in \mathbb{D}.$$

If $z, w \in \phi(\mathbb{D})$ are distinct points, then we refer to ϕ as a complex d-geodesic joining z and w.

We will use the term complex geodesic *for 'complex $C_{\mathcal{D}}$-geodesic'.*

Proposition 1.2 *For a domain $\mathcal{D}$ in a Banach space and $\phi \in H(\mathbb{D}, \mathcal{D})$, the following are equivalent*

(a) ϕ is a complex geodesic.

(b) there exist distinct points $u, v \in \mathbb{D}$ such that $\rho(u, v) = C_{\mathcal{D}}(\phi(u), \phi(v))$.

(c) there exists a point $z \in \mathbb{D}$ such that $\alpha(z, 1) = c_{\mathcal{D}}(\phi(z), \phi'(z))$.

(d) ϕ is biholomorphic from $\mathbb{D}$ to an analytic set $\phi(\mathbb{D})$ and $\phi(\mathbb{D})$ is a holomorphic retract of $\mathcal{D}$ (i.e. there exists $f \in H(\mathcal{D}, \mathcal{D})$ such that $f \circ f = f$ and $f(\mathcal{D}) = \phi(\mathbb{D})$).

(e) ϕ is a complex $K_{\mathcal{D}}$-geodesic and

$$\delta_{\mathcal{D}}|\phi(\mathbb{D}) = K_{\mathcal{D}}|\phi(\mathbb{D}) = C_{\mathcal{D}}|\phi(\mathbb{D}).$$

PROOF: The equivalence of (a), (b) and (c) is due to Vesentini [34] and the fact that (d) is equivalent to (a) is due to Reiffen [27, p. 19] (see also Lempert [24]). (e) $\Rightarrow$ (a) and it remains to show that (a) $\Rightarrow$ (e).

Fix $u \neq v \in \mathbb{D}$ and write $p = \phi(u)$, $q = \phi(v)$. Since $\rho(u, v) = C_{\mathcal{D}}(p, q)$, Montel's theorem implies the existence of $f \in H(\mathcal{D}, \mathbb{D})$ such that $f(p) = u$ and $f(q) = v$. Now $f \circ \phi \in H(\mathbb{D}, \mathbb{D})$ and

$$\rho(f \circ \phi(u), f \circ \phi(v)) = C_{\mathcal{D}}(p, q) = \rho(u, v).$$

By the Schwarz-Pick lemma (see [10, p.5]), $f \circ \phi$ is a biholomorphic automorphism of $\mathbb{D}$ and $\rho(f \circ \phi(z), f \circ \phi(w)) = \rho(z, w)$ for all $z, w \in \mathbb{D}$. Since holomorphic mappings are contractions, we have

$$\begin{aligned} \rho(z, w) &= \rho(f \circ \phi(z), f \circ \phi(w)) \leq C_{\mathcal{D}}(\phi(z), \phi(w)) \\ &\leq K_{\mathcal{D}}(\phi(z), \phi(w)) \leq \delta_{\mathcal{D}}(\phi(z), \phi(w)) \\ &\leq \rho(z, w) \end{aligned} \tag{1.1}$$

and (1.1) consists entirely of equalities. Thus (a) implies (e). ∎

Definition 1.3 *We call a domain $\mathcal{D}$ in a complex Banach space* C-connected *if every pair of points of $\mathcal{D}$ can be joined by a complex geodesic.*

We remark that it follows from results of Vigué [37] (see also [10, proposition 11.15 and corollary 11.17]) that a subset of a bounded convex finite dimensional domain $\mathcal{D}$ is the range of a complex geodesic if and only if it is a connected one-dimensional analytic subset and a holomorphic retract of $\mathcal{D}$.

An immediate consequence of the equivalence of (a) and (e) in Proposition 1.2 is the following result.

Proposition 1.4 *If $\mathcal{D}$ is a C-connected domain in a Banach space, then*

$$C_{\mathcal{D}} = K_{\mathcal{D}} = \delta_{\mathcal{D}}.$$

This leads to many examples of domains which are *not* C-connected (see for instance [10, p.103]). In particular a proper domain in $\mathbb{C}$ is C-connected if and only if it is simply connected.

On the other hand Lempert [23, 24] (see also Royden-Wong [29]) has shown that $C_{\mathcal{D}} = K_{\mathcal{D}} = \delta_{\mathcal{D}}$ and that $c_{\mathcal{D}} = k_{\mathcal{D}}$ if $\mathcal{D}$ is a convex bounded domain in $\mathbb{C}^n$. This result was extended to convex domains in arbitrary Banach spaces in [11], where it was used to prove that the following are C-connected (see also [10, pp.90–91])

(a) bounded convex domains in reflexive Banach spaces

(b) the open unit ball B_X of a dual Banach space X.

These facts lead us to propose the following conjecture.

Conjecture 1.5 *If $\mathcal{D}$ is a domain in a complex Banach space X which is biholomorphically equivalent to a bounded domain, then the following are equivalent:*

(a) $C_{\mathcal{D}} = K_{\mathcal{D}} = \delta_{\mathcal{D}}$

(b) $\mathcal{D}$ is biholomorphically equivalent to a convex domain

(c) $\mathcal{D}$ is C-connected.

(Note that (c) implies (a) and (b) implies (a).)

2 EXTENSION AND EXISTENCE THEOREMS

In this section, we consider a complex Banach space X as a subspace of its double dual space X^{**} via the natural embedding. B_X denotes the open unit ball of X and $H^\infty(B_X)$ denotes the space of scalar-valued bounded holomorphic functions on B_X, with the supremum norm. $H^\infty(B_X)$ is a Banach algebra. We abbreviate K_{B_X} as K_X and k_{B_X} as k_X from now on.

If $\mathcal{U}$ is an ultrafilter on a set I, then $(X)_\mathcal{U}$ will denote the ultrapower of a Banach space X. More specifically, if $\ell^\infty(I, X)$ denotes the space of bounded X-valued functions on I (with the supremum norm) and

$$N_\mathcal{U} = \{(x_i)_{i\in I} \in \ell^\infty(I, X) : \lim_\mathcal{U} \|x_i\| = 0\},$$

then $(X)_\mathcal{U}$ is the quotient space $\ell^\infty(I, X)/N_\mathcal{U}$.

There is a canonical isometric embedding $j\colon X \to (X)_\mathcal{U}$ given by

$$j(x) = (x)_{i\in I} + N_\mathcal{U}$$

(where $(x)_{i\in I}$ denotes the constant function x).

The principle of local reflexivity in its ultrapower formulation (see [20]) asserts that given a Banach space X, there exists an ultrapower $(X)_\mathcal{U}$ such that

(i) there is an isometric embedding $J\colon X^{**} \to (X)_\mathcal{U}$ which extends the canonical embedding $j\colon X \to (X)_\mathcal{U}$;

(ii) the map

$$\begin{array}{rcl} Q\colon (X)_\mathcal{U} & \to & X^{**} \\ Q((x_i)_{i\in I} + N_\mathcal{U}) & = & w^*\text{-}\lim_\mathcal{U} x_i \end{array}$$

satisfies $QJ = \mathrm{id}_{X^{**}}$, $\|Q\| = 1$.

(Thus JQ is a contractive projection of $(X)_\mathcal{U}$ onto the isometric copy $J(X^{**})$ of X^{**}.)

The next lemma provides a crude version of the Schwarz inequality, which we prove for completeness.

Lemma 2.1 *Suppose X is a complex Banach space, $f \in H(B_X, \mathbb{D})$, $0 < \varepsilon < 1$ and $a, b \in (1-\varepsilon)B_X$. Then there is a constant A_ε depending only on ε such that*

$$\rho(f(a), f(b)) \le A_\varepsilon \|a - b\|.$$

PROOF: We can first dispose of the case where a and b are far apart, specifically the case $\|a-b\| > \varepsilon$. Define $\phi: \mathbb{D} \to B_X$ by

$$\phi(z) = \frac{z}{1-\varepsilon} a.$$

Then $f \circ \phi \in H(\mathbb{D}, \mathbb{D})$ and it follows from the classical Schwarz-Pick lemma that

$$\rho(f(a), f(0)) = \rho(f \circ \phi(1-\varepsilon), f \circ \phi(0)) \leq \rho(0, 1-\varepsilon).$$

Since the same estimate also applies to b

$$\rho(f(a), f(b)) \leq 2\rho(0, 1-\varepsilon) \leq A_\varepsilon \varepsilon \leq A_\varepsilon \|a-b\|$$

for A_ε chosen suitably large (depending only on ε).

For $\|a-b\| \leq \varepsilon$, we can define $\phi \in H(\mathbb{D}, B_X)$ by

$$\phi(z) = \frac{a+b}{2} + \frac{\varepsilon(a-b)}{\|a-b\|} z$$

and we will then have

$$\phi\left(\frac{\|a-b\|}{2\varepsilon}\right) = a, \quad \phi\left(-\frac{\|a-b\|}{2\varepsilon}\right) = b.$$

Applying the Schwarz lemma to $f \circ \phi$, we find that

$$\rho(f(a), f(b)) \leq \rho\left(\|a-b\|/2\varepsilon, -\|a-b\|/2\varepsilon\right) = 2\rho(\|a-b\|/2\varepsilon, 0) \leq A_\varepsilon \|a-b\|$$

since $\|a-b\|/2\varepsilon \leq 1/2$. ∎

Theorem 2.2 *Suppose $Y = (X)_\mathcal{U}$ is an ultrapower of a complex Banach space X, where $\mathcal{U}$ is an ultrafilter on the set I.*

(i) Suppose, for each $i \in I$, $f_i: B_X \to D$ is a holomorphic function from B_X to the unit disc D in the complex plane, and

$$\lim_{\mathcal{U}} |f_i(a_i)| < 1 \tag{2.1}$$

for one point $(a_i)_{i\in I} + N_\mathcal{U}$ of B_Y. Then the function

$$\begin{array}{rcl} F: B_Y & \to & D \\ F((x_i)_{i\in I} + N_\mathcal{U}) & = & \lim_{\mathcal{U}} f_i(x_i) \end{array}$$

is well-defined and holomorphic.

(ii) The Kobayashi distance K_Y on B_Y is given by

$$K_Y((x_i)_{i\in I} + N_{\mathcal{U}}, (y_i)_{i\in I} + N_{\mathcal{U}}) = \lim_{\mathcal{U}} K_X(x_i, y_i).$$

PROOF: (i) Note first of all that if $(x_i)_{i\in I} + N_{\mathcal{U}} \in B_Y$, then $\lim_{\mathcal{U}} \|x_i\| < 1$ and so there exists $U \in \mathcal{U}$ so that $\|x_i\| < 1$ for all $i \in U$. Thus $f_i(x_i)$ is defined for $i \in U$ and $\lim_{\mathcal{U}} f_i(x_i)$ makes sense. In fact, it is possible to change x_i for $i \notin U$ so as to ensure $\sup_{i\in I} \|x_i\| < 1$ without changing the coset $(x_i)_{i\in I} + N_{\mathcal{U}}$ or the value of the limit. We will make this change without comment from now on for all points in B_Y and, in particular we will assume that $S = \sup_i \|a_i\| < 1$.

By compactness, $\lim_{\mathcal{U}} f_i(x_i)$ certainly exists in the closed unit disc $\bar{\mathbb{D}}$. To show it is in the open disc fix $(x_i)_i$ and choose $0 < \varepsilon < 1 - S$ so that $\sup_i \|x_i\| < 1 - \varepsilon$. By Lemma 2.1,

$$\rho(f_i(x_i), f_i(a_i)) \le 2A_\varepsilon \quad (i \in I).$$

Choose now $U \in \mathcal{U}$ so that $T = \sup_{i\in U} |f_i(a_i)| < 1$. It follows by the triangle inequality that

$$\rho(f_i(x_i), 0) \le 2A_\varepsilon + \rho(0, f_i(a_i)) \le 2A_\varepsilon + \rho(0, T).$$

Hence $\sup_{i\in U} |f_i(x_i)| < 1$ and F does indeed map B_Y into $\mathbb{D}$.

Next we check that F is continuous. For this, fix $(x_i)_i + N_{\mathcal{U}} \in B_Y$ and choose $\varepsilon > 0$ with $x = \sup_i \|x_i\| < 1 - 2\varepsilon$. For $y = (y_i)_i + N_{\mathcal{U}} \in B_Y$ and $\|x - y\|_Y < \varepsilon$, we have $\|y_i\|_X < 1 - \varepsilon$ and so we can apply Lemma 2.1 to see that

$$\rho(f_i(x_i), f(y_i)) \le A_\varepsilon \|x_i - y_i\|$$

Taking limits along $\mathcal{U}$, we deduce that

$$\rho(F(x), F(y)) \le A_\varepsilon \|x - y\|_Y,$$

which is enough to show continuity of F at x.

Finally analyticity of F follows from continuity together with analyticity of the restricton of F on complex lines in Y. For $x = (x_i)_i + N_{\mathcal{U}} \in B_Y$ and $y = (y_i)_i + N_{\mathcal{U}} \in Y$, Montel's theorem shows that

$$F(x + zy) = \lim_{\mathcal{U}} f_i(x_i + zy_i)$$

is an analytic function of z on $\{z \in \mathbb{C} : \|x + zy\|_Y < 1\}$.

(ii) Fix $x = (x_i)_i + N_{\mathcal{U}}, y = (y_i)_i + N_{\mathcal{U}} \in B_Y$ and let $K_{\text{lim}} = \lim_{\mathcal{U}} K_X(x_i, y_i)$. Choose $\varepsilon > 0$ and put $r = \tanh(K_{\text{lim}} + \varepsilon)$. Then $\rho(0, r) = K_{\text{lim}} + \varepsilon$. There exists $U \in \mathcal{U}$ so that $K_X(x_i, y_i) < K_{\text{lim}} + \varepsilon$ for $i \in U$. Thus, since $\delta_X(x_i, y_i) = K_X(x_i, y_i)$, there exists $g_i \in H(\mathbb{D}, B_X)$ satisfying $g_i(0) = x_i$, $g_i(r) = y_i$ (for $i \in U$). For $i \in I \setminus U$, set $g_i = 0$.

Now define $g\colon \mathbb{D} \to B_Y$, by $g(z) = (g_i(z))_i + N_{\mathcal{U}}$. Of course we must check first that $\sup_i \|g_i(z)\|_X < 1$ for all $z \in \mathbb{D}$. To this end, observe that

$$\begin{aligned} K_X(g_i(z),0) &\leq K_X(g_i(z),g_i(0)) + K_X(g_i(0),0) \\ &\leq \rho(z,0) + K_X(x_i,0) \\ &= \rho(z,0) + \rho(\|x_i\|,0) \\ &\leq \rho(z,0) + \sup_{j\in I}\rho(\|x_j\|,0) \\ &< \infty \end{aligned}$$

Here we have used the fact that holomorphic mappings are contractions with respect to the Kobayashi distance and our standing assumption that the x_j are chosen so that $\sup_j \|x_j\| < 1$. The equality $K_X(x,0) = \rho(\|x\|,0)$ (for $x \in B_X$) is elementary. Hence we have

$$\sup_i \rho(\|g_i(z)\|,0) < \infty$$

which implies $\sup_i \|g_i(z)\| < 1$.

Cauchy's formula shows that the functions $\|g_i''(z)\|$ are uniformly bounded on compact subsets of $\mathbb{D}$. This will enable us to show by a direct argument that g is analytic. Fix $z \in \mathbb{D}$ and suppose $|w-z| < (1-|z|)/2$. Then

$$\begin{aligned} \left\|\frac{g_i(w)-g_i(z)}{w-z} - g_i'(z)\right\| &= \left\|\frac{1}{w-z}\int_z^w g_i'(\zeta) - g_i'(z)\,d\zeta\right\| \\ &= \left\|\frac{1}{w-z}\int_z^w \int_z^\zeta g_i''(\eta)\,d\eta\,d\zeta\right\| \\ &\leq |w-z|\sup\{|g_i''(\zeta)| : |\zeta| \leq (1+|z|)/2, i \in I\} \end{aligned}$$

Hence, if $\ell = (g_i'(z))_i + N_{\mathcal{U}}$, we have

$$\left\|\frac{g(w)-g(z)}{w-z} - \ell\right\|_Y \leq M|w-z|$$

for $|w-z| < (1-|z|)/2$ and M a constant depending on z. Taking the limit as $w \to z$, we see that $g'(z)$ exists and is ℓ.

Therefore $g \in H(\mathbb{D}, B_Y)$, $g(0) = x$ and $g(r) = y$. It follows that

$$K_Y(x,y) = K_Y(g(0),g(r)) \leq \rho(0,r) = K_{\lim} + \varepsilon.$$

Since this is true for all $\varepsilon > 0$, we have

$$K_Y(x,y) \leq K_{\lim} = \lim_{\mathcal{U}} K_X(x_i,y_i). \tag{2.2}$$

To establish the reverse inequality, we use the fact that $K_X = C_X$ to select functions $f_i \in H(B_X, \mathbb{D})$ satisfying $f_i(x_i) = 0$, $f_i(y_i) > 0$ and $K_X(x_i, y_i) = \rho(0, f_i(y_i))$. Applying part (i), we get a function $F \in H(B_Y, \mathbb{D})$ which satisfies $F(x) = 0$ and

$$\rho(0, F(y)) = \lim_{\mathcal{U}} \rho(0, f_i(y_i)) = \lim_{\mathcal{U}} K_X(x_i, y_i) = K_{\text{lim}}.$$

But now the distance decreasing property of Kobayashi distances under F allows us to conclude that

$$K_{\text{lim}} = \rho(0, F(y)) = \rho(F(x), F(y)) \leq K_Y(x, y).$$

Combining this with (2.2) completes the proof. ∎

Our next result is a new proof of a result of Davie and Gamelin [7]. It has come to our attention that M. Lindström and R. Ryan have independently obtained a proof of this result using ultrapower techniques.

Theorem 2.3 *For X a complex Banach space, there exists an algebra homomorphism of norm one,*

$$E : H^\infty(B_X) \to H^\infty(B_{X^{**}})$$

which satisfies

$$Ef|B_X = f$$

for all $f \in H^\infty(B_X)$.

PROOF: Given X and $f \in H^\infty(B_X)$, we choose an ultrafilter $\mathcal{U}$ according to the principle of local reflexivity. To apply Theorem 2.2(i), we take $f_i = f/\|f\|_\infty$ for all i. Unless f is constant, the hypothesis (2.1) is satisfied for $a_i = 0$. In any case, it follows that the function

$$\begin{array}{rcl} F \colon B_Y & \to & \mathbb{C} \\ F((x_i)_{i\in I} + N_{\mathcal{U}}) & = & \lim_{\mathcal{U}} f(x_i) \end{array}$$

is holomorphic and $\|F\|_\infty \leq \|f\|_\infty$. Put $Ef = F \circ J$, where J is as in the principle of local reflexivity. It is straightforward to check that E has the required linearity and multiplicative properties and that $\|E\| \leq 1$. Since J coincides with the canonical embedding $j \colon X \to (X)_{\mathcal{U}}$ on B_X, it is also easy to see that Ef coincides with f on B_X. ∎

Theorem 2.4 *For X a complex Banach space,*

$$K_{X^{**}}(x, y) = K_X(x, y)$$

for all $x, y \in B_X$.

PROOF: Since the canonical inclusion from B_X to $B_{X^{**}}$ is continuous and linear, it is holomorphic and the distance decreasing property of the Kobayashi metric implies

$$K_{X^{**}}(x, y) \leq K_X(x, y)$$

for $x, y \in B_X$.

Fix $x, y \in B_X$. By Montel's theorem, we can find $f \in H(B_X, \mathbb{D})$ so that $f(0) = 0$ and

$$K_X(x, y) = C_X(x, y) = \rho(f(x), f(y)).$$

By Propostion 2.3, we can find an extension $\tilde{f} \in H^\infty(B_{X^{**}})$ of f with $\|\tilde{f}\|_\infty \leq 1$. Since $\tilde{f}(x) = f(x) \in \mathbb{D}$, $\tilde{f}$ has all its values in $\mathbb{D}$. We conclude

$$K_X(x, y) = \rho(f(x), f(y)) = \rho(\tilde{f}(x), \tilde{f}(y)) \leq K_{X^{**}}(x, y). \quad \blacksquare$$

We are now in a position to extend Théorème 4.3 of [11].

Theorem 2.5 *If a complex Banach space X is 1-complemented in its second dual, then B_X is C-connected.*

PROOF: Let P denote a norm 1 projection from X^{**} onto X and let $p \neq q \in B_X$. Since $B_{X^{**}}$ is the unit ball of a dual Banach space, Théorème 4.3 of [11] implies that there exists $\phi \in H(\mathbb{D}, B_{X^{**}})$ and $u, v \in \mathbb{D}$ satisfying $\phi(u) = p$, $\phi(v) = q$ and $\rho(u, v) = K_{X^{**}}(p, q)$.

Now $P \circ \phi \in H(\mathbb{D}, B_X)$ and $P \circ \phi(u) = p$, $P \circ \phi(v) = q$. By Theorem 2.4 and [11, Théorème 2.5],

$$\rho(u, v) = K_{X^{**}}(p, q) = K_X(p, q) = C_X(p, q).$$

Hence $P \circ \phi$ is a complex geodesic in B_X. Since p and q are arbitrary, B_X is C-connected. $\blacksquare$

Remark 2.6 *Preduals of C*-algebras satisfy the hypotheses of Theorem 2.5 (see [32]) and these include examples which are not covered by the results in [11].*

On the other hand, c_0 is well known not to satisfy the hypotheses of Theorem 2.5, although B_{c_0} is C-connected (which follows from homogeneity — see Remarks 5.10).

3 UNIQUENESS RESULTS

If ϕ is a complex geodesic joining the points p and q of a domain $\mathcal{D}$ and f is a biholomorphic automorphism of $\mathbb{D}$, then $\phi \circ f$ is also a complex geodesic joining p and q (because f is a ρ-isometry). Thus there is never a unique complex geodesic joining p and q, because of this possibility of reparametrizing complex geodesics. However, Vesentini [33] has shown that if ϕ and ψ are complex geodesics then they have the same range $\phi(\mathbb{D}) = \psi(\mathbb{D})$ if and only if $\psi = \phi \circ f$ for some biholomorphic automorphism f of $\mathbb{D}$ (this can also be deduced from the global vector-valued subordination theorem of Finkelstein and Whitley [13]). We thus discuss uniqueness of complex geodesics up to reparametrization, by means of the following normalization.

We call a complex geodesic ϕ a *normalized geodesic joining* p *and* q if $\phi(0) = p$ and $\phi(s) = q$ for some positive real number s. The number s is uniquely determined by p and q — in fact $s = \tanh C_{\mathcal{D}}(p, q)$. By the homogeneity of the unit disc and the result of Vesentini cited above, it follows that there is a unique normalized complex geodesic joining two points $p, q \in \mathcal{D}$ if and only if all complex geodesics joining p and q have the same range.

The following are known results concerning uniqueness.

(a) If B_X is the unit ball of a Banach space X and $x \in B_X$, $x \neq 0$, then there is a unique normalized complex geodesic joining 0 and x if and only if $x/\|x\|$ is a complex extreme point of B_X. (Vesentini [33]).

(b) If $\mathcal{D}$ is a strictly convex domain (i.e. each point of the boundary $\partial\mathcal{D}$ is a (real) extreme point of $\mathcal{D}$) in a finite dimensional space, then there exist unique normalized complex geodesics joining all pairs of points in $\mathcal{D}$. (Lempert [23]).

In this section, we extend (b) to a class of Banach spaces which includes all reflexive Banach spaces and give a general criterion for uniqueness of complex geodesics which highlights the problem of interpolating between the results (a) and (b) above. A more detailed study of non-uniqueness of complex geodesics has been undertaken by Gentili [14, 15, 16] (see also Section 6).

Definition 3.1 *A complex Banach space X has the* analytic Radon-Nikodym property *(aRNP) if each $f \in H^\infty(\mathbb{D}, X)$ has radial limits almost everywhere on the unit circle.*

$H^\infty(\mathbb{D}, X)$ means those functions in $H(\mathbb{D}, X)$ which have bounded range. If X has aRNP and $f \in H^\infty(\mathbb{D}, X)$, we can extend f to almost all points $e^{i\theta} \in \partial\mathbb{D}$ (almost all with respect to Lebesgue measure on $\partial\mathbb{D}$) by

$$f(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta}).$$

Moreover f is uniquely determined by the boundary values $f(e^{i\theta})$. Reflexive Banach spaces and Banach spaces with the Radon-Nikodym property (e.g separable dual spaces) have aRNP. The Banach space c_0 does not have aRNP. For further details we refer to [10, Chapter 12] and [18].

Now suppose we have a convex bounded domain $\mathcal{D}$ in a Banach space X with aRNP. Let $p \neq q \in \mathcal{D}$ and let $G(p, q)$ denote the set of all normalized complex geodesics joining p and q. If $\phi, \psi \in G(p, q)$ and $0 < \lambda < 1$, and $s = \tanh C_{\mathcal{D}}(p, q)$, then $\lambda\phi + (1 - \lambda)\psi \in G(p, q)$.

This follows from convexity of $\mathcal{D}$ (which ensures that $\lambda\phi + (1-\lambda)\psi(\mathbb{D}) \subset \mathcal{D}$) together with the facts that

$$(\lambda\phi + (1-\lambda)\psi)(0) = \phi(0) = \psi(0) = p \text{ and } (\lambda\phi + (1-\lambda)\psi)(s) = \phi(s) = \psi(s) = q.$$

In other words, $G(p,q)$ is a convex subset of $H(\mathbb{D}, \mathcal{D})$

Next, notice that if $\phi \in G(p,q)$, then

$$\lim_{r\to 1^-} C_{\mathcal{D}}(\phi(re^{i\theta}), \phi(0)) = \lim_{r\to 1^-} \rho(re^{i\theta}, 0) = \infty.$$

It follows that at points $e^{i\theta}$ where $\phi(e^{i\theta})$ is defined (almost all points on $\partial\mathbb{D}$ by aRNP), $\phi(e^{i\theta}) \in \partial\mathcal{D}$. Now if ψ is another element of $G(p,q)$, then for almost all θ,

$$\phi(e^{i\theta}), \psi(e^{i\theta}) \text{ and } \left(\frac{\phi+\psi}{2}\right)(e^{i\theta}) = (1/2)(\phi(e^{i\theta}) + \psi(e^{i\theta}))$$

are all in $\partial\mathcal{D}$. If we now assume that $\mathcal{D}$ is strictly convex, then we must have $\phi(e^{i\theta}) = \psi(e^{i\theta})$ for almost all $e^{i\theta}$. This implies $\phi = \psi$. We have thus proved the following result.

Theorem 3.2 *If X is a complex Banach space with the analytic Radon-Nikodym property and $\mathcal{D} \subset X$ is a strictly convex bounded domain, then there exists at most one normalized complex geodesic joining p and q.*

We now restrict our attention to the case where $\mathcal{D}$ is the open unit ball B_X of X. Let $\phi, \psi \in G(p,q)$ for two points $p, q \in B_X$ and suppose again that X has aRNP. Let $g = \psi - \phi$ and $s = \tanh C_X(p,q)$. Since $\phi(0) = \psi(0) = p$ and $\phi(s) = \psi(s) = q$, $g(0) = g(s) = 0$ and we can therefore write $g(z) = z(z-s)h(z)$ for some $h \in H^\infty(\mathbb{D}, X)$. Using aRNP and the convexity of $G(p,q)$, we can see that the following result holds.

Proposition 3.3 *If X has aRNP and ϕ is a normalized complex geodesic joining two points $p, q \in B_X$, then ϕ is the unique such geodesic if and only if the zero function is the only element $h \in H^\infty(\mathbb{D}, X)$ satisfying*

$$\|\phi(e^{i\theta}) + \lambda e^{i\theta}(e^{i\theta} - s)h(e^{i\theta})\| = 1 \tag{3.1}$$

for almost all θ, all $\lambda \in [0,1]$ (where $s = \tanh C_X(p,q)$).

Examples 3.4 *(a) If X has aRNP and is 1-complemented in X^{**}, and if B_X is strictly convex, then Theorems 2.5 and 3.2 show that there exists a unique normalized complex geodesic joining each pair of points $p, q \in B_X$.*

(b) This applies in particular when X is the space ℓ^p of p-summable sequences ($1 < p < \infty$), because ℓ^p is reflexive and has a strictly convex unit ball.

4 CONTINUOUS COMPLEX GEODESICS

In this section we show that complex geodesics can be extended continuously to the boundary under a complex uniform convexity hypothesis.

Definition 4.1 *If $\mathcal{D} \subset X$ is a domain in a complex Banach space X, then we define*

$$\delta_{\mathcal{D}}(z, v) = \sup\{r > 0 : z + rv\mathbb{D} \subset \mathcal{D}\}$$

for $z \in \mathcal{D}$, $v \in X$, $\|v\| = 1$.

We define the modulus of complex convexity *of $\mathcal{D}$ to be*

$$\delta_{\mathcal{D}}(\varepsilon) = \sup\{\delta_{\mathcal{D}}(z, v) : z \in \mathcal{D}, d(z, \partial\mathcal{D}) \leq \varepsilon, \|v\| = 1\},$$

(where $d(z, \partial\mathcal{D})$ denotes the distance from a point z in $\mathcal{D}$ to the boundary).

The domain $\mathcal{D}$ is called complex uniformly convex *if $\delta_{\mathcal{D}}(\varepsilon) \to 0$ as $\varepsilon \to 0$.*

Remarks 4.2 *For the case where $\mathcal{D} = B_X$ is the unit ball of X, Globevnik [17] introduced*

$$\omega_c(\varepsilon) = \sup\{\|y\| : \|x + \zeta y\| \leq 1 + \varepsilon \textit{ for all } \|x\| = 1, \zeta \in \mathbb{D}\}$$

and this function is closely related to $\delta_{\mathcal{D}}(\varepsilon)$. In fact, it is easy to check that

$$\frac{\omega_c(\varepsilon)}{1+\varepsilon} = \sup\{\delta_{\mathcal{D}}(z, v) : \|v\| = 1, \|z\| = 1/(1+\varepsilon)\} = \delta_{\mathcal{D}}\left(\frac{\varepsilon}{1+\varepsilon}\right)$$

and thus that

$$\delta_{\mathcal{D}}(\varepsilon/2) \leq \omega_c(\varepsilon) \leq 2\delta_{\mathcal{D}}(\varepsilon) \quad (0 < \varepsilon \leq 1).$$

Functions which are inverse to $\delta_{\mathcal{D}}(\varepsilon)$ and $\omega_c(\varepsilon)$ were considered by Davis, Garling and Tomczak-Jaegermann [8] and called h_∞^X and H_∞^X (respectively). Dilworth [9, theorem 2.1] has shown that complex uniform convexity of B_X (or uniform H_∞-convexity of X in the notation of [8]) is equivalent to the notion of uniform PL-convexity which was studied intensively in [8].

A result similar to the following one can be obtained for the case where the domain is the unit ball using theorem 2 of [17].

Proposition 4.3 *If $\mathcal{D} \subset X$ is a convex domain, then*

$$c_{\mathcal{D}}(z, v) \geq \frac{\|v\|}{2\delta_{\mathcal{D}}(\varepsilon)}$$

holds for $v \in X$, $z \in \mathcal{D}$, $\varepsilon = d(z, \partial\mathcal{D})$.

PROOF: We will make use of the fact that $c_{\mathcal{D}} = k_{\mathcal{D}}$ for $\mathcal{D}$ convex (see [11]). Fix $z \in \mathcal{D}$, $v \in X$ and consider holomorphic mappings $f\colon \mathbb{D} \to \mathcal{D}$ with $f(0) = z$, $f'(0) = v/r$, $r > 0$.

Consider the function

$$g(\zeta) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}\zeta)(1 + \cos\theta)\, d\theta.$$

Since $(1 + \cos\theta)\, d\theta/(2\pi)$ is a probability measure on $[0, 2\pi]$, we may regard $g(\zeta)$ as a limit of convex combinations of points in $\mathcal{D}$. So g takes values in the closure $\bar{\mathcal{D}}$. Since $g(0) = f(0) \in \mathcal{D}$ and $\mathcal{D}$ is convex, it follows that g maps $\mathbb{D}$ into $\mathcal{D}$ [33, p. 376].

Working with the power series representation of f, we find that

$$g(\zeta) = f(0) + \frac{1}{2} f'(0)\zeta.$$

(In fact, $g(\zeta)$ is a Cesaro mean of the power series of f.). Thus

$$g(\zeta) = f(0) + \frac{v}{2r}\zeta = z + \frac{\|v\|}{2r} \frac{v}{\|v\|}\zeta$$

maps $\mathbb{D}$ into $\mathcal{D}$, which shows that

$$\frac{\|v\|}{2r} \le \delta_{\mathcal{D}}(z, v/\|v\|) \le \delta_{\mathcal{D}}(\varepsilon).$$

Rearranging this, we find $r \ge \|v\|/(2\delta_{\mathcal{D}}(\varepsilon))$. Since $k_{\mathcal{D}}(z, v) = c_{\mathcal{D}}(z, v)$ is the infimum of all possible values of r, the result follows. ∎

Theorem 4.4 *Let $\mathcal{D}$ be the unit ball B_X of a complex Banach space X. If $\mathcal{D}$ is complex uniformly convex and $\delta_{\mathcal{D}}(\varepsilon) \le A\varepsilon^s$ for some constants $A > 0$, $s > 0$, then all complex geodesics $\phi\colon \mathbb{D} \to \mathcal{D}$ extend to continuous functions $\phi\colon \bar{\mathbb{D}} \to \bar{\mathcal{D}}$.*

PROOF: By Proposition 4.3, we have (for $\zeta \in \mathbb{D}$)

$$c_{\mathcal{D}}(\phi(\zeta), \phi'(\zeta)) \ge \frac{\|\phi'(\zeta)\|}{2\delta_{\mathcal{D}}(1 - \|\phi(\zeta)\|)}.$$

Using the hypothesis and the fact that ϕ is a complex geodesic, we deduce

$$c_{\mathcal{D}}(\phi(\zeta), \phi'(\zeta)) = c_{\mathbb{D}}(\zeta, 1) = \frac{1}{1 - |\zeta|^2} \ge \frac{\|\phi'(\zeta)\|}{2A(1 - \|\phi(\zeta)\|)^s}$$

or

$$\|\phi'(\zeta)\| \le \frac{2A(1 - \|\phi(\zeta)\|)^s}{1 - |\zeta|^2}. \tag{4.1}$$

Next observe that

$$C_{\mathcal{D}}(\phi(0), \phi(\zeta)) = \rho(0, \zeta) = \tanh^{-1}(|\zeta|) \leq C_{\mathcal{D}}(0, \phi(\zeta)) + C_{\mathcal{D}}(0, \phi(0)).$$

Recall that $C_{\mathcal{D}}(0, z) = \tanh^{-1}\|z\|$ for $z \in \mathcal{D} = B_X$. Using elementary estimates, we conclude that

$$1 - \|\phi(\zeta)\| \leq A_\phi(1 - |\zeta|)$$

where A_ϕ is a constant depending on $\|\phi(0)\|$.

Combining this observation with (4.1), we see that

$$\|\phi'(\zeta)\| \leq A_1 \frac{(1-|\zeta|)^s}{1-|\zeta|^2} \leq A_1 \frac{1}{(1-|\zeta|)^{1-s}}$$

which implies (see for instance [12, theorem 5.5]) that ϕ satisfies a Lipschitz condition

$$\|\phi(\zeta) - \phi(\eta)\| \leq C|\zeta - \eta|^s.$$

Hence $\phi\colon \mathbb{D} \to X$ is uniformly continuous and extends continuously to a function $f\colon \bar{\mathbb{D}} \to X$. ∎

If a complex geodesic $\phi\colon \mathbb{D} \to \mathcal{D}$ extends to a continuous function $\phi\colon \bar{\mathbb{D}} \to \bar{\mathcal{D}}$, we call ϕ a *continuous complex geodesic*.

Examples 4.5 *(i) For $X = L^1$, $\mathcal{D} = B_X$, Globevnik [17] proved that $\delta_{\mathcal{D}} \leq A\sqrt{\varepsilon}$ (or rather, he proves the equivalent fact that $\omega_c(\varepsilon) \leq A\sqrt{\varepsilon}$). Thus all complex geodesics in B_X are continuous, by Theorem 4.4.*

(ii) More generally, if X is the predual of a C-algebra and $\mathcal{D} = B_X$, we deduce from a result due to Haagerup (see [8, theorem 4.3]) that $\delta_{\mathcal{D}}(\varepsilon) \leq A\sqrt{\varepsilon}$.*

Thus all complex geodesics in B_X are continuous by Theorem 4.4. Existence of complex geodesics in B_X is guaranteed by Theorem 2.5.

Remark 4.6 *From corollary 2.5 of [15], it follows that if all complex geodesics in the unit ball B_X are continuous, then all points of ∂B_X are complex extreme points. Theorem 4.4 falls short of being a converse to this.*

5 EXAMPLES IN CLASSICAL BANACH SPACES

We apply the results of the preceding sections to give a complete description of the complex geodesics in the unit ball of ℓ^p, $1 \leq p < \infty$. To obtain these examples, we require a Banach space version of a result of Lempert [23, proposition 1] and [29]. Various extensions are possible and we have chosen one which is suitable for the applications we have in mind.

Definition 5.1 *For X a complex Banach space with dual space X^*, we let $H^\infty_*(\mathbb{D}, X^*)$ denote the space of X^*-valued bounded analytic functions on $\mathbb{D}$ which have weak*-radial limits at almost all boundary points.*

In other words $f \in H^\infty_*(\mathbb{D}, X^*)$ means that $f \in H^\infty(\mathbb{D}, X^*)$ and there exists a function $\tilde{f}\colon \partial\mathbb{D} \to X^*$ so that

$$\lim_{r\to 1^-} \langle x, f(re^{i\theta})\rangle = \langle x, \tilde{f}(e^{i\theta})\rangle \qquad (\text{all } x \in X)$$

holds for almost all $\theta \in \mathbb{R}$.

If X^* has aRNP, then all functions in $H^\infty(\mathbb{D}, X^*)$ have norm radial limits at almost all points of $\partial\mathbb{D}$, and therefore $H^\infty_*(\mathbb{D}, X^*) = H^\infty(\mathbb{D}, X^*)$. By a result of Danilevich [6, theorem 1.4], this equality also holds if X is separable.

In general, the limit function $\tilde{f}(e^{i\theta})$ may only be weak*-measurable, a rather intractable condition. Moreover it is possible that the space $H^\infty_*(\mathbb{D}, X^*)$ depends on the choice of a predual X for X^*. However, the function $\tilde{f}(e^{i\theta})$ determines the holomorphic function $f(\zeta)$ uniquely as can be seen by applying standard results (see [12]) to the scalar-valued bounded analytic functions $\langle x, f(\zeta)\rangle$. We will therefore not cause confusion by using the notation $f(e^{i\theta})$ instead of $\tilde{f}(e^{i\theta})$ for the boundary function.

Lemma 5.2 *Let X be a complex Banach space, $f \in H^\infty_*(\mathbb{D}, X^*)$ and $h\colon \bar{\mathbb{D}} \to X$ a continuous function which is holomorphic on $\mathbb{D}$. Then*

$$\lim_{r\to r^-} \langle h(re^{i\theta}), f(re^{i\theta})\rangle = \langle h(e^{i\theta}), f(e^{i\theta})\rangle$$

for almost all θ and

$$\langle h(\zeta), f(\zeta)\rangle = \frac{1}{2\pi}\int_0^{2\pi} \langle h(e^{i\theta}), f(e^{i\theta})\rangle \frac{1-|\zeta|^2}{1+|\zeta|^2 - 2\mathrm{Re}\,(e^{-i\theta}\zeta)}\, d\theta$$

for all $\zeta \in \mathbb{D}$.

PROOF: The first assertion follows from the inequality

$$\begin{aligned} &|\langle h(re^{i\theta}), f(re^{i\theta})\rangle - \langle h(e^{i\theta}), f(e^{i\theta})\rangle| \\ \le\ & |\langle h(re^{i\theta}) - h(e^{i\theta}), f(re^{i\theta})\rangle| + |\langle h(e^{i\theta}), f(re^{i\theta}) - f(e^{i\theta})\rangle| \\ \le\ & \|h(re^{i\theta}) - h(e^{i\theta})\|\, \|f\|_\infty + |\langle h(e^{i\theta}), f(re^{i\theta}) - f(e^{i\theta})\rangle| \\ \to\ & 0 \end{aligned}$$

for almost all θ by continuity of h and the definition of $H^\infty_*(\mathbb{D}, X^*)$. The second assertion follows from the standard fact (see [12]) that scalar-valued bounded analytic functions like $\langle f(\zeta), h(\zeta)\rangle$ are the Poisson integrals of their (almost everywhere) boundary values. ∎

If X is a Banach space and $x \in \partial B_X$, then the Hahn-Banach theorem assures us of the existence of at least one supporting hyperplane for B_X at x. That is, there exists $N_x \in X^*$ such that $\langle x, N_x \rangle = 1$ and $\operatorname{Re} \langle p, N_x \rangle < 1$ for $p \in B_X$. We will use the notation N_x for a choice of one such functional, bearing in mind the possibility that it may not be unique.

Proposition 5.3 *Let X be a complex Banach space and $\phi: \bar{\mathbb{D}} \to \bar{B}_X$ a continuous map satisfying*

(i) $\phi \mid \mathbb{D}$ is holomorphic and $\phi(\mathbb{D}) \subset B_X$;

(ii) $\phi(\partial\mathbb{D}) \subset \partial B_X$; and

(iii) there exists a choice of $N_{\phi(\zeta)}$ for almost all $\zeta \in \partial\mathbb{D}$ and a measurable function $p: \partial\mathbb{D} \to \mathbb{R}^+$ such that the mapping

$$h(e^{i\theta}) = e^{i\theta} p(e^{i\theta}) N_{\phi(e^{i\theta})}$$

is almost everywhere the weak-radial limit of a function $h \in H^\infty_*(\mathbb{D}, X^*)$.*

Then ϕ is a complex geodesic.

PROOF: Let $g: \mathbb{D} \to B_X$ be a holomorphic mapping with $g(0) = \phi(0)$ and $g'(0) = \lambda\phi'(0)$, $\lambda \geq 0$. Let $g_r(\zeta) = g(r\zeta)$ for $0 < r < 1$ and $\zeta \in \mathbb{D}$. Then g_r is continuous on $\bar{\mathbb{D}}$ and holomorphic on $\mathbb{D}$, $g_r(0) = \phi(0)$ and $g_r'(0) = \lambda r \phi'(0)$. Moreover $g_r(\zeta) \in B_X$ for $\zeta \in \bar{\mathbb{D}}$. From the hypotheses, we see that

$$1 = \langle \phi(e^{i\theta}), N_{\phi(e^{i\theta})} \rangle > \operatorname{Re} \langle g_r(e^{i\theta}), N_{\phi(e^{i\theta})} \rangle$$

for almost all θ. Hence

$$\operatorname{Re} \left\langle \frac{\phi(e^{i\theta}) - g_r(e^{i\theta})}{e^{i\theta}}, e^{i\theta} p(e^{i\theta}) N_{\phi(e^{i\theta})} \right\rangle = p(e^{i\theta}) \operatorname{Re} \langle \phi(e^{i\theta}) - g_r(e^{i\theta}), N_{\phi(e^{i\theta})} \rangle > 0$$

for almost all θ.

Since $\dfrac{\phi(\zeta) - g_r(\zeta)}{\zeta}$ is holomorphic on $\mathbb{D}$ and continuous on $\bar{\mathbb{D}}$, the function

$$H(\zeta) = \left\langle \frac{\phi(\zeta) - g_r(\zeta)}{\zeta}, h(\zeta) \right\rangle$$

is the Poisson integral of its boundary values $H(e^{i\theta})$ by Lemma 5.2. By the above remark and Lemma 5.2, $\operatorname{Re} H(e^{i\theta}) > 0$ for almost all θ and it follows from the Poisson formula that $\operatorname{Re} H(0) > 0$, *i.e.*

$$\operatorname{Re} \langle \phi'(0) - g_r'(0), h(0) \rangle = (1 - \lambda r) \operatorname{Re} \langle \phi'(0), h(0) \rangle > 0.$$

Applying this to the special case where $g(\zeta) = \phi(0)$ is constant (and $\lambda = 0$) we see that $\operatorname{Re}\langle \phi'(0), h(0)\rangle > 0$. Thus, returning to the general case, we have $1 - r\lambda > 0$. Since this is true for all $0 < r < 1$, we deduce that $\lambda \leq 1$.

Since this is true for all g, we have established

$$k_X(\phi(0), \phi'(0)) = c_X(\phi(0), \phi'(0)) = 1$$

which shows, by Proposition 1.2, that ϕ is a complex geodesic. ∎

For μ a σ-finite measure on a σ-algebra Σ of subsets of a set Ω and $1 \leq p < \infty$, we use the standard notation $L^p(\mu)$ for the Banach space of (equivalence classes of) p-summable Σ-measurable ($\mathbb{C}$-valued) functions on Ω normed by $\|f\|_p = \left(\int_\Omega |f|^p \, d\mu\right)^{1/p}$. $L^\infty(\mu)$ denotes the essentially bounded Σ-measurable functions with the essential sup norm $\|f\|_\infty$. These include as special cases the sequence spaces ℓ^p (where μ is counting measure on the natural numbers) and the finite-dimensional spaces ℓ^p_n (which are $\mathbb{C}^n$ with the norm $\|(z_1, z_2, \ldots, z_n)\|_p = \left(\sum_{j=1}^n |z_j|^p\right)^{1/p}$ for $1 \leq p < \infty$).

In order to discuss complex geodesics in the unit ball B_p of $L^p(\mu)$ for $1 \leq p < \infty$ we will consider nonconstant mappings $\phi\colon \mathbb{D} \to B_p$ of the form

$$\phi(\zeta)(\omega) = c(\omega) \left(\frac{\zeta - \alpha(\omega)}{1 - \overline{\alpha(\omega)}\zeta}\right)^{\beta(\omega)} \left(\frac{1 - \overline{\alpha(\omega)}\zeta}{1 - \bar{\gamma}\zeta}\right)^{2/p} \tag{5.1}$$

($\zeta \in \mathbb{D}$, $\omega \in \Omega$) where the parameter γ and the measurable functions $\alpha(\omega)$, $\beta(\omega)$ and $c(\omega)$ satisfy

$$\left.\begin{array}{ll} (a) & \gamma \in \mathbb{D}, \quad \alpha \in L^\infty(\mu), \quad \|\alpha\|_\infty \leq 1, \\ & \beta \text{ takes only the values } 0 \text{ and } 1. \\ (b) & \int_\Omega |c(\omega)|^p (1 + |\alpha(\zeta)|^2) \, d\mu(\omega) = 1 + |\gamma|^2 \\ (c) & \int_\Omega |c(\omega)|^p \alpha(\omega) \, d\mu = \gamma \end{array}\right\} \tag{5.2}$$

Later, we will specialise to the case where $L^p(\mu) = \ell^p$ and then we will start to use subscript notation — $\phi_j(\zeta)$ rather than $\phi(\zeta)(j)$, α_j instead of $\alpha(j)$, *etc.* — and of course summation over j in place of integrals.

Proposition 5.4 *Let B_p denote the open unit ball of $L^p(\mu)$, $1 \leq p < \infty$. Then every non-constant mapping ϕ of the form (5.1) where γ, $\alpha(\omega)$, $\beta(\omega)$ and $c(\omega)$ satisfy the conditions (5.2) is a complex geodesic in B_p.*

Note that at points where $c(\omega) = 0$, the values of $\alpha(\omega)$ and $\beta(\omega)$ are immaterial. Thus we can suppose if we wish that

$$\text{support}\,(\beta) = \{\omega : \beta(\omega) = 1\} \subset \{\omega : c(\omega) \neq 0\} = \text{support}\,(c)$$

and that support $(\alpha) \subset$ support (c). Since $(\zeta - \alpha(\omega))/(1 - \overline{\alpha(\omega)}\zeta)$ is a constant function of ζ when $|\alpha(\omega)| = 1$ (the constant is of modulus 1), we can also assume support $(\beta) \subset \{\omega : |\alpha(\omega)| < 1\}$. (Then there is no problem defining $\phi(\zeta)$ for $|\zeta| = 1$.) With these assumptions, the condition for ϕ to be non-constant is

$$\mu(\{\omega : c(\omega) \neq 0\} \cap \{\omega : \beta(\omega) = 1 \text{ or } \alpha(\omega) \neq \gamma\}) > 0.$$

Note also that condition (5.2)(b) implies that $c \in L^p(\mu)$.

PROOF: Observe that for $\delta = \left(\dfrac{2}{1-|\gamma|}\right)^{2/p}$,

$$|\phi(\zeta)(\omega)| \leq \delta |c(\omega)| \qquad (\zeta \in \bar{\mathbb{D}}, \omega \in \Omega) \tag{5.3}$$

so that $\phi(\bar{\mathbb{D}}) \subset L^p(\mu)$. Next if (ζ_n) is a sequence in $\bar{\mathbb{D}}$ which converges to a point $\zeta \in \bar{\mathbb{D}}$, then $\phi(\zeta_n)(\omega) \to \phi(\zeta)(\omega)$ for all ω. Hence by the Lebesgue dominated convergence theorem and (5.3), $\|\phi(\zeta_n) - \phi(\zeta)\|_p \to 0$ as $n \to \infty$. This shows that $\phi\colon \bar{\mathbb{D}} \to L^p(\mu)$ is a continuous mapping.

For each fixed $\omega \in \Omega$ the map $\zeta \mapsto \phi(\zeta)(\omega)$ is analytic on $\mathbb{D}$ and from (5.3) we see that its Taylor series expansion

$$\phi(\zeta)(\omega) = \sum_n a_n(\omega)\zeta^n \tag{5.4}$$

with coefficients satisfying $|a_n(\omega)| \leq \delta|c(\omega)|$ (all n and all ω). Calculating the $a_n(\omega)$ using the binomial theorem and multiplication of power series, we can check that the functions $a_n(\omega)$ are measurable and then the estimate on the coefficients implies that $a_n \in L^p(\mu)$.

Now, for a fixed $\zeta \in \mathbb{D}$ the sequence $\sum_{j=0}^n a_j(\cdot)\zeta^j$ of measurable functions converges pointwise to $\phi(\zeta)(\cdot)$ as $n \to \infty$ by (5.4). Since

$$\left|\sum_{j=0}^n a_j(\omega)\zeta^j\right| \leq \delta|c(\omega)| \sum_{j=0}^\infty |\zeta|^j = \frac{\delta|c(\omega)|}{1-|\zeta|}$$

and $\delta c(\cdot)/(1-|\zeta|) \in L^p(\mu)$, the Lebesgue dominated convergence theorem shows that

$$\phi(\zeta)(\cdot) = \sum_{n=0}^\infty a_n(\cdot)\zeta^n$$

in $L^p(\mu)$ for each $\zeta \in \mathbb{D}$. Hence ϕ is holomorphic on $\mathbb{D}$.

For $\theta \in \mathbb{R}$, we have

$$\begin{aligned}
\|\phi(e^{i\theta})\|_p^p &= \int_\Omega |\phi(e^{i\theta})(\omega)|^p \, d\mu(\omega) \\
&= \int_\Omega |c(\omega)|^p \left|\frac{1 - \overline{\alpha(\omega)}e^{i\theta}}{1 - \bar{\gamma}e^{i\theta}}\right|^2 \, d\mu(\omega)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{1+|\gamma|^2 - 2\mathrm{Re}\,(\bar{\gamma}e^{i\theta})} \int_\Omega |c(\omega)|^p (1 + |\alpha(\omega)|^2 - 2\mathrm{Re}\,(\overline{\alpha(\omega)}e^{i\theta})\,)\,d\mu(\omega) \\
&= 1
\end{aligned}
$$

by conditions (5.2) (b) and (c). Hence $\phi(\partial\mathbb{D}) \subset \partial B_p$ and $\phi(\bar{\mathbb{D}}) \subset \bar{B}_p$. Since ϕ is non-constant and all unit vectors in $L^p(\mu)$ are complex extreme points of the unit ball the strong maximum modulus theorem of Thorp and Whitley (see for instance [10, proposition 6.19]) implies that $\phi(\mathbb{D}) \subset B_p$.

We consider the dual space $(L^p(\mu))^*$ of $L^p(\mu)$ to be identified with $L^q(\mu)$ $(1/p+1/q = 1$, $1 < q \leq \infty)$ in a complex linear fashion (rather than the conjugate linear identification frequently used) so that $g \in L^q(\mu)$ acts on $f \in L^p(\mu)$ via $\langle f, g\rangle = \int_\Omega f(\omega)g(\omega)\,d\mu(\omega)$. If $f \in \partial B_p$, then one choice of a supporting hyperplane $N_f \in L^q(\mu)$ is given by

$$
N_f(\omega) = |f(\omega)|^{p-2}\overline{f(\omega)}
$$

(where $0^{p-2} = 0$ for all p).

To complete the proof, we apply Proposition 5.3 with

$$
\begin{aligned}
p(e^{i\theta}) &= |1 - \bar{\gamma}e^{i\theta}|^2 \\
h(\zeta)(\omega) &= \tilde{c}(\omega)(1 - \overline{\alpha(\omega)}\zeta)^{2-2/p}(1 - \bar{\gamma}\zeta)^{2/p}\left(\frac{\zeta - \alpha(\omega)}{1 - \overline{\alpha(\omega)}\zeta}\right)^{1-\beta(\omega)}
\end{aligned}
$$

with $\tilde{c}(\omega) = |c(\omega)|^{p-2}\overline{c(\omega)}$. Note that $\tilde{c} \in L^q(\mu)$. If $1 < p < \infty$ an argument similar to the one given above to show that ϕ is holomorphic on $\mathbb{D}$ and continuous on $\bar{\mathbb{D}}$ shows that the same is true of h. This is more than enough to show that $h \in H^\infty_*(\mathbb{D}, (L^p(\mu))^*)$. Since $h(e^{i\theta}) = e^{i\theta}p(e^{i\theta})N_{\phi(e^{i\theta})}$, this shows that ϕ is a complex geodesic when $p > 1$.

When $p = 1$,

$$
h(\zeta)(\omega) = (1 - \bar{\gamma}\zeta)^2\tilde{c}(\omega)\left(\frac{\zeta - \alpha(\omega)}{1 - \overline{\alpha(\omega)}\zeta}\right)^{1-\beta(\omega)}
$$

and $\|\tilde{c}\|_\infty \leq 1$. It is quite easy to see that h is holomorphic on $\mathbb{D}$ and that $\|h(\zeta)\|_\infty \leq (1+|\gamma|)^2$ for all $\zeta \in \mathbb{D}$. For each $\omega \in \Omega$ and $\theta \in \mathbb{R}$, $\lim_{r\to 1^-} h(re^{i\theta})(\omega) = h(e^{i\theta})(\omega)$. Using the boundedness of h and the Lebesgue dominated convergence theorem it is then easy to see that h has weak* radial limits at all points $e^{i\theta} \in \partial\mathbb{D}$, *i.e.* that

$$
\lim_{r\to 1^-} \langle f, h(re^{i\theta})\rangle = \langle f, h(e^{i\theta})\rangle
$$

for all $f \in L^1(\mu)$. Thus $h \in H^\infty_*(\mathbb{D}, (L^1(\mu))^*)$ and ϕ is also a complex geodesic in the $p = 1$ case. ∎

Proposition 5.5 *Let B_p denote the open unit ball of ℓ^p, $1 \le p < \infty$. Then any two distinct points in B_p can be joined by a normalized complex geodesic ϕ of the form given in (5.1).*

That is, there exists a complex geodesic $\phi(\zeta) = (\phi_j(\zeta))_{j=1}^\infty$ joining the points where

$$\phi_j(\zeta) = c_j \left(\frac{\zeta - \alpha_j}{1 - \bar{\alpha}_j \zeta}\right)^{\beta_j} \left(\frac{1 - \bar{\alpha}_j \zeta}{1 - \bar{\gamma}\zeta}\right)^{2/p} \tag{5.5}$$

$\gamma \in \mathbb{D}$, $\alpha_j \in \bar{\mathbb{D}}$, $c_j \in \mathbb{C}$, $\beta_j = 0$ or 1, $\sum_j |c_j|^p(1 + |\alpha_j|^2) = 1 + |\gamma|^2$ and $\sum_j |c_j|^p \alpha_j = \gamma$.

PROOF: The existence of some complex geodesic joining the two points follows from Theorem 2.5 (or from [11]). We also know that all complex geodesics in B_p are continuous from Theorem 4.4. For the $p = 1$ case the uniform complex convexity hypothesis was established by Globevnik [17] (see Examples 4.5) and the $p > 1$ case is more straightforward because ℓ^p is uniformly convex in the real sense (see [8]).

Our proof that there exists geodesics of the required form relies heavily on finite-dimensional results (*i.e.* results for the case of ℓ^p_n) of Poletskiĭ [26] and Gentili [16]. For $1 < p < \infty$, Poletskiĭ [26] proved that all geodesics in the unit ball $B_{p,n}$ of ℓ^p_n are of the above form (except that he omitted the possibility that β_j could be 0). Gentili [16, Theorem 6] proved that all continuous complex geodesics in $B_{1,n}$ are of the above form. Now that we know all complex geodesics in $B_{1,n}$ are continuous, it follows that all complex geodesics in $B_{1,n}$ are of the above form. (We could actually circumvent Gentili's result. By taking a limiting argument based on Poletskiĭ's result and the fact that $\cap_{p>1} B_{p,n} = B_{1,n}$, we could establish the existence of complex geodesics of the required form joining any pair of points in $B_{1,n}$.)

Now let $x, y \in B_p$ be two distinct points.

We consider ℓ^p_n as being identical with the subspace $\{(z_1, z_2, \ldots, z_n, 0, 0, \ldots)\}$ of ℓ_p and we will use the notation $x^{(n)}$ for the natural projection $(x_1, x_2, \ldots, x_n, 0, \ldots)$ of $x = (x_1, x_2, \ldots)$ into ℓ^p_n. Similarly for $y^{(n)}$. Consideration of the inclusion map : $B_{\ell^p_n} \to B_p$ and the projection $z \mapsto z^{(n)}: B_p \to B_{\ell^p_n}$, which are both holomorphic and therefore contractions with respect to the Carathéodory distance, shows that

$$C_{\ell^p_n}(x^{(n)}, y^{(n)}) = C_{\ell^p}(x^{(n)}, y^{(n)}) \le C_{\ell^p}(x, y) \quad (x, y \in B_p).$$

Let $\phi^{(n)}$ denote a normalized complex geodesic in the unit ball of ℓ^p_n with $\phi^{(n)}(0) = x^{(n)}$ and $\phi^{(n)}(s_n) = y^{(n)}$, where $s_n = \tanh^{-1} C_{\ell^p}(x^{(n)}, y^{(n)}) > 0$. (Strictly speaking this may not make sense for small n when it may happen that $x^{(n)} = y^{(n)}$.) Let the parameters associated with $\phi^{(n)}$ be denoted $\alpha_j^{(n)}$, $\beta_j^{(n)}$, $\gamma^{(n)}$ and $c_j^{(n)}$ ($1 \le j \le n$).

There is a subsequence of n's along which we have

$$\alpha_j^{(n)} \to \alpha, \quad \beta_j^{(n)} \to \beta_j, \quad \gamma^{(n)} \to \gamma, \quad c_j^{(n)} \to c_j$$

for each j. Since $\sum_j |c_j^{(n)}|^p < 2$ for all n, we have $\sum_j |c_j|^p \le 2$. We claim that $|\gamma| < 1$. Observe that

$$\phi_j^{(n)}(\zeta) \to \phi_j(\zeta) = c_j \left(\frac{\zeta - \alpha_j}{1 - \bar{\alpha}_j \zeta}\right)^{\beta_j} \left(\frac{1 - \bar{\alpha}_j \zeta}{1 - \bar{\gamma}\zeta}\right)^{2/p}$$

uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$ along the subsequence. Hence $\phi_j(0) = x_j$ and $\phi_j(s) = y_j$ where

$$s = \lim_n s_n = \lim_n \tanh^{-1} C_{\ell^p}(x^{(n)}, y^{(n)}) = \tanh^{-1} C_{\ell^p}(x, y).$$

Thus we can pick a j with ϕ_j non-constant. If $|\gamma| = 1$, then ϕ_j would be unbounded on $\mathbb{D}$ unless $\alpha_j = \gamma$. ϕ_j unbounded leads to a contradiction since each $\phi_j^{(n)}$ is bounded by 1 and in the case $\alpha_j = \gamma$, ϕ_j would be constant. Hence $|\gamma| < 1$. Using $\sum_j |c_j|^p < \infty$ we can argue as in the proof of Proposition 5.4 to show that

$$\phi = (\phi_1, \phi_2, \ldots)\colon \mathbb{D} \to \ell^p$$

is holomorphic on $\mathbb{D}$ and continuous on $\bar{\mathbb{D}}$.

Since $\sum_j |\phi_j^{(n)}(\zeta)|^p \leq 1$ for all n, we have $\sum_j |\phi_j(\zeta)|^p \leq 1$ for $|\zeta| < 1$. Since $\phi(0) = x$ we have $\phi(\mathbb{D}) \subset B_p$ (see [33, p. 376]) and since $\phi(s) = y$, ϕ must be a complex geodesic. It follows that $\|\phi(\zeta)\|_p^p = 1$ for $|\zeta| = 1$, and expanding this out as in the proof of Proposition 5.4 we find

$$1 + |\gamma|^2 - 2\mathrm{Re}\,(\bar{\gamma}\zeta) = \sum_{j=1}^{\infty} |c_j|^p (1 + |\alpha_j|^2 - 2\mathrm{Re}\,(\bar{\alpha}\zeta)) \qquad (|\zeta| = 1).$$

As both sides are harmonic for $\zeta \in \mathbb{D}$ and continuous on $\bar{\mathbb{D}}$, the equality remains valid for $\zeta \in \mathbb{D}$. Putting $\zeta = 0$ gives $\sum_j |c_j|^p (1 + |\alpha_j|^2) = 1 + |\gamma|^2$ and then it follows that $\sum_j |c_j|^p \alpha_j = \gamma$. ∎

Lemma 5.6 *Let $f \in H^\infty(\mathbb{D})$, $\gamma \in \mathbb{D}$. Then*

$$\frac{f(e^{i\theta})}{\left(\dfrac{e^{i\theta} - \gamma}{1 - \bar{\gamma} e^{i\theta}}\right)}$$

is a non-negative real number for almost all $\theta \in \mathbb{R}$ if and only if

$$f(\zeta) = t \left(\frac{\zeta - \alpha}{1 - \bar{\alpha}\zeta}\right) \left(\frac{1 - \bar{\alpha}\zeta}{1 - \bar{\gamma}\zeta}\right)^2$$

for some $t \geq 0$, $|\alpha| \leq 1$.

PROOF: The case $\gamma = 0$ is due to Gentili [16] (see lemma 2 and the proof of theorem 6). The general case follows from Gentili's result by the change of variables $\eta = (\zeta - \gamma)/(1 - \bar{\gamma}\zeta)$. ∎

Lemma 5.7 *If $X = \ell^1_n$ or $X = \ell^1$ and $x \neq y \in B_X$, then there is a unique normalized complex geodesic in B_X joining x and y.*

PROOF: As already noted, we know that there exists a normalized complex geodesic ϕ in B_X joining x and y. In fact we have an explicit form (5.5) of one such $\phi = (\phi_j)_j$ by Proposition 5.5. , where

$$\phi_j(\zeta) = c_j \left(\frac{\zeta - \alpha_j}{1 - \bar{\alpha}_j\zeta\zeta}\right)^{\beta_j} \left(\frac{1 - \bar{\alpha}_j\zeta}{1 - \bar{\gamma}\zeta}\right)^2$$

for all j

Now suppose ψ is a second normalized complex geodesic joining x and y. Suppose $\phi(0) = \psi(0) = x$ and $\phi(s) = \psi(s) = y$ where $s > 0$. The argument given earlier in the proof of Theorem 3.2 shows that $f = \lambda\phi + (1 - \lambda)\psi$ is also a complex geodesic joining x and y. Thus $\|f(e^{i\theta})\| = 1$ for all $\theta \in \mathbb{R}$. It follows that we must have equality in the triangle inequality

$$1 = \|f(e^{i\theta})\| = \sum_j |\lambda\phi_j(e^{i\theta}) + (1 - \lambda)\psi_j(e^{i\theta})| \leq \sum_j \lambda|\phi_j(e^{i\theta})| + (1 - \lambda)|\psi_j(e^{i\theta})| = 1.$$

This forces

$$\frac{\psi_j(e^{i\theta})}{\phi_j(e^{i\theta})}$$

to be a non-negative real number for all θ except those for which the denominator is zero.

To prove that $\psi_j = \phi_j$, we consider the case $c_j = 0$ and $c_j \neq 0$ separately. In the first case $\phi_j \equiv 0$ and hence $x_j = y_j = 0$. Since linear isometries of X map complex geodesics in B_X to complex geodesics, ψ and

$$\tilde{\psi} = (\psi_1, \ldots, \psi_{j-1}, -\psi_j, \psi_{j+1}, \ldots)$$

are both normalized complex geodesics joining x and y. Therefore, so is $g = (\psi + \tilde{\psi})/2$. Since $\|g(e^{i\theta})\| = \|\psi(e^{i\theta})\| = 1$ for all θ, it easily follows that $\psi_j \equiv 0 \equiv \phi_j$.

If $c_j \neq 0$, Lemma 5.6 applied to the function

$$\frac{\phi_j(\zeta)}{c_j} \left(\frac{\zeta - \alpha_j}{1 - \bar{\alpha}_j\zeta}\right)^{1-\beta_j}$$

shows that

$$\frac{\phi_j(e^{i\theta})}{c_j} \left(\frac{e^{i\theta} - \alpha_j}{1 - \bar{\alpha}_j e^{i\theta}}\right)^{1-\beta_j} \Bigg/ \left(\frac{e^{i\theta} - \gamma}{1 - \bar{\gamma}e^{i\theta}}\right)$$

is non-negative for almost all θ. Therefore the same holds for $\psi_j(e^{i\theta})$ in place of $\phi_j(e^{i\theta})$ and Lemma 5.6 then shows that there exist $t > 0$ and $|\beta| \le 1$ so that

$$\psi_j(\zeta) = tc_j\left(\frac{\zeta - \alpha_j}{1 - \bar{\alpha}_j\zeta}\right)^{\beta_j - 1}\left(\frac{\zeta - \beta}{1 - \bar{\beta}\zeta}\right)\left(\frac{1 - \bar{\beta}\zeta}{1 - \bar{\gamma}\zeta}\right)^2.$$

If $\beta_j = 0$ and $|\alpha_j| < 1$, then analyticity of $\psi_j(\zeta)$ forces $\alpha_j = \beta_j$ and two terms cancel in the expression for $\psi_j(\zeta)$. Then from $\phi_j(0) = \psi_j(0) = x_j$, we conclude that $c_j = tc_j$ and $t = 1$. So $\phi_j = \psi_j$ in this situation.

In the remaining cases, $\phi_j(0) = \psi_j(0) = x_j$ yields

$$tc_j(-\alpha_j)^{\beta_j - 1}(-\beta) + c_j(-\alpha_j)^{\beta_j}$$

and hence $\alpha_j = t\beta$. Then $\psi_j(s) = \phi_j(s) = y_j$ and some cancellation of common terms shows that

$$(s - \alpha_j)(1 - \bar{\alpha}_j s) = t(s - \beta)(1 - \bar{\beta}s).$$

Combining this with $\alpha_j = t\beta$ yields

$$(1 - t)s(1 - t|\beta|^2) = 0.$$

Hence $t = 1$ or $t = |\beta|^{-2}$. In the second case,

$$\alpha_j = t\beta = 1/\bar{\beta}.$$

Since $|\alpha_j| \le 1$ and $|\beta| \le 1$, we must have $|\alpha_j| = |\beta| = 1$ so that $t = 1$. Hence $\alpha_j = \beta$ and $\psi_j(\zeta) = \phi_j(\zeta)$. ∎

We can summarise our results for ℓ^p as follows.

Corollary 5.8 *Let B_p denote the unit ball of ℓ^p, $1 \le p < \infty$. Then*

(i) Any two distinct points in B_p can be joined by a unique normalized complex geodesic.

(ii) All complex geodesics in B_p are continuous.

(iii) A map $\phi: \mathbb{D} \to B_p$ is a complex geodesic if and only if it is a non-constant map of the form given in Proposition 5.5

PROOF: For all $1 \le p < \infty$, existence follows from Theorem 2.5 (or from [11]). Uniqueness for $p > 1$ follows from Theorem 3.2 and uniqueness for $p = 1$ has just been established in Lemma 5.7.

(ii) follows from Theorem 4.4. For $p = 1$, this has already been noted in Examples 4.5. It is straightforward that Theorem 4.4 applies to ℓ^p for $1 < p < \infty$ because ℓ^p is uniformly convex in the real sense (see [8]).

(iii) follows from Proposition 5.5 and (i). ∎

We suspect that a more general version of this result holds for $L^p(\mu)$ in place of ℓ^p, but we have not managed to prove L^p-versions of Propositions 5.5 or 5.7.

Example 5.9 *Let* $X = \ell^{p_1} \oplus_r \ell^{p_2} = \{x = (y,z) : y \in \ell^{p_1}, z \in \ell^{p_2}\}$ *normed by* $\|x\| = \left(\|y\|_{p_1}^r + \|z\|_{p_2}^r\right)^{1/r}$.

One can check using Proposition 5.3 that for $1 \le p_i < \infty$, $1 \le r < \infty$ *all nonconstant maps* $\phi\colon \mathbb{D} \to B_X$ *of the following form are complex geodesics.*

$$\begin{aligned} \phi(\zeta) &= ((\phi_{1j})_{j=1}^\infty, (\phi_{2j})_{j=1}^\infty) \\ \phi_{ij}(\zeta) &= c_{ij}\left(\frac{\zeta - \alpha_{ij}}{1 - \bar\alpha_{ij}\zeta}\right)^{\beta_{ij}} \left(\frac{1 - \bar\alpha_{ij}\zeta}{1 - \bar\gamma_i\zeta}\right)^{2/p_i} \left(\frac{1 - \bar\gamma_i\zeta}{1 - \bar\gamma\zeta}\right)^{2/r} \end{aligned}$$

where $|\alpha_{ij}| \le 1$, $|\gamma_i| < 1$, $|\gamma| < 1$, β_{ij} *is* 0 *or* 1, *and the following relations hold*

$$\begin{aligned} \sum_j |c_{ij}|^{p_i}\alpha_{ij} &= \gamma c_i^{p_i} \qquad (i = 1,2) \\ c_1^r\gamma_1 + c_2^r\gamma_2 &= \gamma \\ c_1^r(1 + |\gamma_1|^2) + c_2^r(1 + |\gamma_2|^2) &= 1 + |\gamma|^2 \end{aligned}$$

where

$$c_i = \left(\frac{1}{1 + |\gamma_i|^2}\sum_j |c_{ij}|^{p_i}(1 + |\alpha_{ij}|^2)\right)^{1/p_i} \qquad (i = 1,2).$$

The proof of this involves observing first that for $x = (y,z) \in X$ *with* $\|x\| = 1$,

$$N_x = (\|y\|^{r-1}N_{y/\|y\|}, \|z\|^{r-1}N_{z/\|z\|})$$

with $N_{y/\|y\|}$ *and* $N_{z/\|z\|}$ *given as in the proof of Proposition 5.4 for* ℓ^p. *To apply Proposition 5.3, take* $p(\zeta) = |1 - \bar\gamma\zeta|^2$ *and*

$$h(\zeta) = \left(\tilde c_{ij}(1 - |\alpha_{ij}|^2)^{2-2/p_i}(1 - \bar\gamma_i\zeta)^{2/p_i - 2/r}(1 - \bar\gamma\zeta)^{2/r}\left(\frac{\zeta - \alpha_{ij}}{1 - \bar\alpha_{ij}\zeta}\right)^{1-\beta_{ij}}\right)_{ij}$$

where

$$\tilde c_{ij} = c_i^{r-p_i}|c_{ij}|^{p_i-2}\bar c_{ij} \quad (i = 1,2; 1 \le j < \infty).$$

We suspect that all complex geodesics in B_X *are of this form. Other examples of complex geodesics in spaces which are direct sums of more than two summands of* ℓ^p*-type can also be exhibited.*

Remark 5.10 *The case* $p = \infty$ *is excluded in all of the previous calculations because it is well known that almost everything is different for* ℓ^∞ *(and* L^∞*). Even for the unit ball* $B_{\infty,2} = \{(z_1,z_2) \in \mathbb{C}^2 : \max_i |z_i| < 1\}$ *(polydisc) of* ℓ_2^∞, *many of the differences are apparent. The only points of* $\partial B_{\infty,2}$ *that are complex extreme points are those where* $|z_1| = |z_2| = 1$ *and* *therefore the result of Vesentini [33] cited at the beginning of Section 3 shows that there are many complex geodesics joining* 0 *to* $z = (z_1, z_2)$ *if* $|z_1| \neq |z_2|$*. In fact, if* $|z_2| < |z_1|$*, the normalized complex geodesics joining* 0 *to* (z_1, z_2) *are*

$$\phi(\zeta) = \left(\zeta \frac{z_1}{|z_1|}, g(\zeta)\right)$$

where g *is any analytic function on* $\mathbb{D}$ *with* $g(0) = 0$*,* $g(|z_1|) = z_2$ *and* $\sup_{\zeta \in \mathbb{D}} |g(\zeta)| < 1$*. (This can easily be verified using the fact that the Kobayashi distance between* $x, y \in B_{\infty,2}$ *is given by* $\max_{i=1,2} \rho(x_i, y_i)$*.) Thus we see non-uniqueness and discontinuity of complex geodesics. Since* $B_{\infty,2}$ *is a homogeneous domain, for any pair of points* $x, y \in B_{\infty,2}$*, we can find a biholomorphic automorphism* F *with* $F(0) = x$ *(in fact* $F(z_1, z_2) = ((x_1 - z_1)/(1 - \bar{x}_1 z_1), (x_2 - z_2)/(1 - \bar{x}_2 z_2))$ *will do). Then for any normalized complex geodesic* ϕ *joining* 0 *and* $F^{-1}(y)$*,* $F \circ \phi$ *is a normalized complex geodesic joining* x *and* y*.*

The infinite dimensional case ℓ^∞ *is somewhat similar to the finite dimensional one. We state without proof the following result (*B_∞ *is the unit ball of* ℓ^∞*).*

> *A map* $\phi = (\phi_1, \phi_2, \ldots) \colon \mathbb{D} \to B_\infty$ *is holomorphic if and only if each coordinate function* $\phi_j(\zeta)$ *is analytic.*

One can check that ϕ *is a complex geodesic if and only if either there exists* j *for which* ϕ_j *is an automorphism of* $\mathbb{D}$ *or else there is a subsequence of* $(\phi_j)_j$ *which converges to an automorphism. This can be done directly or via a result of Gentili [15] who gives a necessary and sufficient conditions for a holomorphic map* $\phi \colon X \to B_X$ *to be a complex geodesic in the case when* $X = C(K)$ *is the space of continuous functions on a compact Hausdorff space* K*. The condition is that there exists* $k \in K$ *so that* $\zeta \mapsto \phi(\zeta)(k)$ *is an automorphism of the unit disc. Since* ℓ^∞ *is the same as the space of continuous functions on the Stone-Čech compactification of the integers, we can apply his result. There will be more than one complex geodesic joining* $x, y \in B_\infty$ *unless* $|(x_j - y_j)/(1 - \bar{x}_j y_j)|$ *is constant.*

Similar remarks apply to c_0*. In this case there are no complex extreme points on the unit sphere and there is more than one normalized complex geodesic joining every pair of points in the unit ball.*

6 TAUTNESS AND CURVATURE

In previous sections, we have used (for convex domains) convergence principles to establish the existence of holomorphic mappings with certain extremal properties. These convergence properties have been formalised for finite-dimensional domains and manifolds and certain relationships established. In this section, we show that these results do not extend to arbitrary Banach spaces, even for convex bounded domains and give a result on curvature of the Kobayashi infinitesimal metric.

We define hyperbolic, complete hyperbolic and taut domains in a Banach space. The first two definitions are standard, while various versions of the third are possible.

Definition 6.1 *A domain $\mathcal{D}$ in a Banach space X is called* hyperbolic *if $K_{\mathcal{D}}$ induces the original topology on $\mathcal{D}$. If, moreover, $(\mathcal{D}, K_{\mathcal{D}})$ is a complete metric space, we call $\mathcal{D}$ a* complete hyperbolic domain.

If $\mathcal{D}$ is hyperbolic, then $K_{\mathcal{D}}$ is a distance on $\mathcal{D}$ (*i.e.* $K_{\mathcal{D}}$ separates the points of $\mathcal{D}$) and the converse is true for finite dimensional domains (see [28, 19]). We do not know if the converse is true for infinite dimensional Banach spaces. Harris [19, theorem 24] has proved that a convex bounded domain in a Banach space is complete hyperbolic and Barth [3] has proved that a convex domain in $\mathbb{C}^n$ which contains no complex lines is hyperbolic (and indeed biholomorphically equivalent to a bounded domain). Again, we do not know if this result extends to infinite dimensional Banach spaces.

Definition 6.2 *A domain $\mathcal{D}$ in a Banach space X is* taut *if there exists a Hausdorff locally convex topology τ on X such that every net $(f_\alpha)_\alpha$ in $H(\mathbb{D}, \mathcal{D})$ contains either a compactly divergent subnet or a subnet which converges with respect to τ, uniformly on compact subsets of $\mathbb{D}$, to some $f \in H(\mathbb{D}, \mathcal{D})$.*

A net $(f_\alpha)_{\alpha\in\Gamma}$ is compactly divergent if, given K compact in $\mathbb{D}$ and L a norm-compact subset of $\mathcal{D}$, there exists $\alpha_0 \in \Gamma$ such that $f_\alpha(K) \cap L = \emptyset$ for all $\alpha \geq \alpha_0$.

For $\mathcal{D}$ finite dimensional, τ has to be the same as the norm topology and we have

$$\text{complete hyperbolic} \Rightarrow \text{taut} \Rightarrow \text{hyperbolic}$$

(and the converses are both false [22, 2]). Also, for finite dimensional domains the unit disc $\mathbb{D}$ can be replaced by any finite dimensional domain in the definition of tautness [1, 39].

Proposition 6.3 *The unit ball B_X of a Banach space X is taut if and only if X is isometrically isomorphic to a dual space.*

Moreover, if X contains any bounded convex domain $\mathcal{D}$ which is taut (for a locally convex topology τ) then X is isomorphic to a dual Banach space and the topology τ is weaker than the norm topology.

PROOF: We first suppose that B_X is taut and that τ is a locally convex Hausdorff topology associated with tautness.

Let $(x_\alpha)_\alpha$ be a net in $\bar{B}_X$ and for each α let $f_\alpha(\zeta) = \zeta x_\alpha$ for $\zeta \in \mathbb{D}$. Then f_α is a net in $H(\mathbb{D}, B_X)$ and, since $f_\alpha(0) = 0$ for all α, it contains no compactly divergent subnet. Therefore it must have a subnet $(f_\beta)_\beta$ which converges relative to τ, uniformly on compact subsets of $\mathbb{D}$, to some $f \in H(\mathbb{D}, B_X)$. Since τ is a locally convex topology, it is easily seen that $f(\zeta) = \zeta x$ for some $x \in X$. Because $\|f(\zeta)\| < 1$ for all $\zeta \in \mathbb{D}$, we must have $\|x\| \leq 1$

and moreover $x_\beta \to x$ (with respect to τ). This shows that $(\bar{B}_X, \tau)$ is compact. A result of Ng [25] now implies that X is isometrically isomorphic to a dual space.

Conversely, suppose now that X is isometrically isomorphic to the dual of a Banach space Y. Then $\bar{B}_X$ is compact in the weak*- or $\sigma(X,Y)$-topology. Let $(f_\alpha)_\alpha$ denote a net in $H(\mathbb{D}, B_X)$. Consider the Taylor series expansions

$$f_\alpha(\zeta) = \sum_n a_{\alpha,n}\zeta^n.$$

From the Cauchy formula, we have $\|a_{\alpha,n}\| \leq 1$ for all α and n. By compactness of the product of infinitely many copies of $\bar{B}_X$ (for the product $\sigma(X,Y)$-topology) we can find a subnet $(f_\beta)_\beta$ so that $a_{\beta,n} \to a_n \in \bar{B}_X$ for all n.

If $\|a_0\| = 1$, then for any $0 < r < 1$, $f_\beta(0) = a_{\beta,n} \notin r\bar{B}_X$ if β is sufficiently large. Thus $\|f_\beta(0)\| \to 1$. Now

$$\begin{aligned} \tanh^{-1}\|f_\beta(\zeta)\| &= K_X(f_\beta(\zeta), 0) \\ &\geq K_X(f_\beta(0), 0) - K_X(f_\beta(\zeta), f_\beta(0)) \\ &\geq \tanh^{-1}\|f_\beta(0)\| - \rho(\zeta, 0) \end{aligned}$$

(here we have used the distance decreasing property for the Kobayashi metric) shows that $\|f_\beta(\zeta)\| \to 1$ uniformly on compact subsets of $\mathbb{D}$. Thus f_β is compactly divergent.

For the case $\|a_0\| < 1$, let $f(\zeta) = \sum_n a_n\zeta^n$. Since $\|a_n\| \leq 1$ for all n, clearly $f \in H(\mathbb{D}, X)$. Fix $y \in Y$, $0 < r < 1$ and $\epsilon > 0$. Choose N so that $\sum_{n=N+1}^\infty r^n < \epsilon$. Then, by considering power series, we obtain

$$\begin{aligned} \sup_{|\zeta|\leq r} |\langle y, f_\beta(\zeta) - f(\zeta)\rangle| &\leq \sum_{n=0}^N |\langle y, a_{\beta,n} - a_n\rangle| + 2\sum_{n=N+1}^\infty r^n\|y\| \\ &\leq \sum_{n=0}^N |\langle y, a_{\beta,n} - a_n\rangle| + 2\epsilon\|y\| \end{aligned}$$

for all β. Hence $f_\beta \to f$ with respect to $\sigma(Y,X)$, uniformly on compact subsets of $\mathbb{D}$.

Since $f_\beta(\zeta) \in B_X$ for all ζ and β, it follows that f has values in $\bar{B}_X$. Since $f(0) = a_0 \in B_X$, $f \in H(\mathbb{D}, B_X)$. This completes the proof that B_X is taut when X is a dual space.

It remains to show the last part of the proposition. Suppose $\mathcal{D}$ is a taut bounded convex domain in X. We can assume that $0 \in \mathcal{D}$ by a translation. A simple argument based on compactness of the unit circle shows that $\mathcal{D}_0 = \cap_{\theta\in\mathbb{R}} e^{i\theta}\mathcal{D}$ is a bounded balanced convex domain. By replacing the norm on X with an equivalent one, we can assume that $\mathcal{D}_0 = B_X$. The same argument as used at the beginning of the proof, together with the observation that a map $f\colon \mathbb{D} \to X$ of the form $f(\zeta) = \zeta x$ has its values in $\mathcal{D}$ if and only if it has values in $\mathcal{D}_0 = B_X$, shows that X is a dual space. Moreover the proof of Ng [25] implies that τ is weaker than the norm topology. He shows that X is the dual of a certain subspace $Y \subset X^*$ and that the identity map id: $(\bar{B}_X, \tau) \to (\bar{B}_X, \sigma(X, Y))$ is a homeomorphism. It follows that id: $(\bar{B}_X, \|\cdot\|) \to (\bar{B}_X, \tau)$ is continuous, and hence that id: $(X, \|\cdot\|) \to (X, \tau)$ is continuous. ∎

We remark that we could have appealed to this proposition for the proof of Proposition 5.5 in place of the more direct argument we used.

Example 6.4 *For $X = c_0$, B_X is complete hyperbolic (by [19]) but not taut.*

The second part of the proof of Proposition 6.3 can easily be modified to show that the following are taut

(a) convex bounded domains in reflexive Banach spaces (for τ the weak topology);

(b) convex bounded domains in dual Banach spaces with the property that their norm closures are weak*-compact (τ the weak*-topology).

Proposition 6.5 *If $\mathcal{D}$ is a convex bounded taut domain in a Banach space X, then $\mathcal{D}$ is C-connected.*

PROOF: Given two points $x, y \in \mathcal{D}$, choose $(f_n)_n$ in $H(\mathbb{D}, \mathcal{D})$ so that $f_n(0) = x$, $f_n(s_n) = y$, $s_n \in (0, 1)$ for all n and $s_n \to s = \tanh K_{\mathcal{D}}(x, y)$ as $n \to \infty$.

Since $\mathcal{D}$ is taut and $f_n(0) = x$ for all n, it follows that $(f_n)_n$ has a τ-convergent subnet (τ being the locally convex topology related to tautness of $\mathcal{D}$). Let $f \in H(\mathbb{D}, \mathcal{D})$ denote the limit of one such subnet. Clearly $f(0) = x$ and we claim $f(s) = y$ (which will show that f is a complex geodesic by convexity and Proposition 1.2).

For $r = (1 + s)/2$, we have

$$f_n(s_n) - f_n(s) = \frac{s_n - s}{2\pi i} \int_{|z|=r} \frac{f_n(z)}{(z - s_n)(z - s)}\, dz.$$

Since $\mathcal{D}$ is bounded and $s_n \to s$, it follows that $f_n(s_n) - f_n(s) \to 0$ in norm (hence also in the topology τ by Proposition 6.3) as $n \to \infty$. Now if $(f_{n_\alpha})_{\alpha \in \Gamma}$ is a subnet of $(f_n)_n$ which converges to f uniformly with respect to τ on compact subsets of $\mathbb{D}$, then

$$f_{n_\alpha}(s_{n_\alpha}) - f(s) = f_{n_\alpha}(s_{n_\alpha}) - f_{n_\alpha}(s) + f_{n_\alpha}(s) - f(s) \to 0$$

in the topology τ. Since $f_{n_\alpha}(s_{n_\alpha}) = y$, it follows that $f(s) = y$. ∎

We observe that one can similarly prove an infinitesimal version of Proposition 6.5: *if $\mathcal{D}$ is a convex bounded taut domain in a Banach space X, $x \in \mathcal{D}$ and $0 \neq v \in X$, then there exists a complex geodesic $f \in H(\mathbb{D}, \mathcal{D})$ such that $f(0) = x$ and $f'(0) = v/k_{\mathcal{D}}(x, v)$.*

Definition 6.6 *If $\mathcal{D}$ is a bounded domain in a Banach space X, then the* holomorphic sectional curvature *of the Kobayashi infinitesimal metric $k_{\mathcal{D}}$ at a point $x \in \mathcal{D}$ in the (nonzero) direction $v \in X$ is*

$$\kappa_k(x,v) = \sup\left\{ \frac{\triangle \log k^2_{\mathcal{D}}(f(z), f'(z)) \mid_{z=0}}{-2k^2_{\mathcal{D}}(f(0), f'(0))} : f \in H(r\mathbb{D}, \mathcal{D}), r > 0, f(0) = x, f'(0) = v \right\}.$$

Since $k_{\mathcal{D}}$ may not even be continuous in general, $\triangle$ above denotes the generalised Laplacian. This is defined (motivated by [21], see [4]) for upper semicontinuous functions u with values in $[-\infty, \infty)$ (but not identically $-\infty$) at points z with $u(z) \neq -\infty$ by

$$\triangle u(z) = 4 \liminf_{r \to 0^+} \frac{1}{r^2} \left\{ \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta})\, d\theta - u(z) \right\}.$$

At a local maximum point z, $\triangle u(z) \leq 0$ and at a finite local minimum $\triangle u(z) \geq 0$.

The following result for finite-dimensional domains is due to Wong [38], Burbea [4] and Suzuki [30, 31].

Proposition 6.7 *If $\mathcal{D}$ is a taut bounded convex domain in a Banach space X, then*

$$\kappa_k(x,v) = -4 \qquad (x \in \mathcal{D}, 0 \neq v \in X).$$

PROOF: We will use the fact that this result is true for the unit disc $\mathbb{D}$ (see for instance [10, §3.4]) — in fact every suitable function f attains the supremum in Definition 6.6 when $\mathcal{D} = \mathbb{D}$ and the supremum is -4.

For general $\mathcal{D}$ and f as in the definition of $\kappa_k(x,v)$, let $\lambda_f(z) = c^2_{\mathcal{D}}(f(z), f'(z))$ (which coincides with $k^2_{\mathcal{D}}(f(z), f'(z))$ by convexity of $\mathcal{D}$). By Montel's theorem, we can find $g \in H(\mathcal{D}, \mathbb{D})$ such that $g(x) = 0$ and

$$c_{\mathcal{D}}(x,v) = |g'(x)(v)| = c_{\mathbb{D}}(g(x), g'(x)(v)) = \alpha(g(x), g'(x)(v)).$$

Let $\beta(z) = \alpha^2(g(f(z)), g'(f(z))(f'(z)))$ for $z \in r\mathbb{D}$. Observe that

$$\beta(z) \leq c^2_{\mathcal{D}}(f(z), f'(z)) = \lambda_f(z)$$

and $\beta(0) = \alpha^2(g(x), g'(x)(v)) = c^2_{\mathcal{D}}(x,v) = \lambda_f(0)$. Since $v \neq 0$ and $\mathcal{D}$ is bounded, there is a neighbourhood of 0 where $\beta(z)$ does not vanish. Hence $\log(\lambda_f/\beta)$ has a local minimum at the origin and therefore

$$\triangle \log \frac{\lambda_f}{\beta}\bigg|_{z=0} = \triangle \log \lambda_f(0) - \triangle \log \beta(0) \geq 0,$$

where we have used the fact that $\log \beta$ is twice continuously differentiable (and hence the $\liminf$ in the definition of $\triangle \log \beta$ is a limit). It follows that

$$\frac{\triangle \log \lambda_f(0)}{-2\lambda_f(0)} \leq \frac{\triangle \log \beta(0)}{-2\beta(0)} = -4$$

and hence that $\kappa_k(x,v) \le -4$.

Since $\mathcal{D}$ is taut and bounded, there exists a complex geodesic $f \in H(\mathbb{D},\mathcal{D})$ such that $f(0)=x$ and $f'(0)=v/k_{\mathcal{D}}(x,v)$ (by the infinitesimal version of Proposition 6.5). For this f we have

$$k^2_{\mathcal{D}}(f(z),f'(z)) = \alpha^2(z,1) = \frac{1}{1-|z|^2}$$

(see Proposition 1.2) and the one variable result shows that f attains the upper bound of -4. ∎

Corollary 6.8 *If X is a dual Banach space, then the Kobayashi infinitesimal metric k_X on B_X has constant holomorphic sectional curvature -4.*

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