DYNAMICS OF BIHOLOMORPHIC SELF-MAPS ON BOUNDED SYMMETRIC DOMAINS

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Abstract

Let g be a fixed-point free biholomorphic self-map of a bounded symmetric domain B. It is known that the sequence of iterates (g^n) may not always converge locally uniformly on B even, for example, if B is an infinite dimensional Hilbert ball. However, $g = g_a \circ T$, for a linear isometry T, $a = g(0)$ and a transvection g_a, and we show that it is possible to determine the dynamics of g_a. We prove that the sequence of iterates (g_a^n) converges locally uniformly on B if, and only if, a is regular, in which case, the limit is a holomorphic map of B onto a boundary component (surprisingly though, generally not the boundary component of $\frac{a}{\|a\|}$). We prove (g_a^n) converges to a constant for all non-zero a if, and only if, B is a complex Hilbert ball. The results are new even in finite dimensions where every element is regular.

Introduction

In 1926 Wolff [25] and Denjoy [5] proved that if g is a holomorphic fixed-point free self-map of Δ, then its iterates (g^n) converge locally uniformly on Δ to a unimodular constant. This was first generalised to the finite dimensional Hilbert ball by Hervé [7] in 1963, and then again, by others, two decades later [18], [20]. Shortly afterwards, the result was shown to fail for the infinite dimensional Hilbert ball [24] even for biholomorphic fixed-point free self-maps. It is also easy to see that it fails for other bounded symmetric domains, as shown for the bidisc $\Delta \times \Delta$ in Example 1 of [4]. It may therefore appear hopeless to consider the iterates of such maps on arbitrary bounded symmetric domains, which is, nonetheless, the purpose of this paper.

Let Z be a JB^*-triple with open unit ball B. As is long known [11] B is a bounded symmetric domain and every bounded symmetric domain can be realised in this way. Let g be a biholomorphic self-map of B which has no fixed point in B. Then g has a unique decomposition into linear and non-linear parts, and the non-linear part is tractable, namely, we can trace its iterates. We recall that the group, G, of biholomorphic automorphisms of B, has the form [11] $G = PK = KP$, where K is all linear elements of G (equivalently, all linear isometries of Z [16]), and $P = \exp(\mathfrak{p})$, where $\mathfrak{p}$ is the Lie algebra generated by the quadratic vector fields, $X_\alpha(z) = (\alpha - \{z, \alpha, z\})\frac{\partial}{\partial z}$ on Z. In other words, each $g \in G$ may be written $g = g_a \circ T$, where $a = g(0), T \in K$, and $g_a \in P$ is called a transvection.

We show that the local uniform convergence properties of the sequence of iterates (g_a^n), unlike those of (g^n), are good, and it is our aim here to establish exactly the dynamics of (g_a^n). We use results on the boundary properties of bounded symmetric domains [17] to reduce the local uniform convergence properties of (g_a^n) to the norm convergence properties of the sequence $(g_a^n(0))$ in Z and we can thereby locate all accumulation points of (g_a^n) (with respect to the topology of local uniform convergence on B) as holomorphic maps of B onto certain boundary components. We present our main result, noting that if Z is finite rank, in particular if it is finite dimensional, then every element is (what is known as) regular, cf. section 3 of this paper, giving a much simpler statement than below.

Theorem 0.1. *Let Z be a JB^*-triple with open unit ball B and $a \in B$.*

The sequence of iterates (g_a^n) has an accumulation point, with respect to the topology of local uniform convergence on B if, and only if, a is regular. Moreover, if a is regular, then the iterates (g_a^n) converge locally uniformly on B to a holomorphic map $g_e : B \to K_e$, where K_e is the (holomorphic) boundary component of e and e is the support tripotent of a.

We note that the limit point g_e is not in general, even in finite dimensions, a constant map, and more crucially, its image, the boundary component K_e, may also not contain the point $\frac{a}{\|a\|}$, for $a \neq 0$. In fact, the following result shows that while, as one might expect, the above simplifies greatly in the case of the Hilbert ball, such simplification actually characterises the Hilbert ball within the class of all bounded symmetric domains.

Theorem 0.2. *Let Z be a JB^*-triple with open unit ball B. The following are equivalent.*

(i) (g_a^n) converges locally uniformly on B to a constant map, for all non-zero $a \in B$.

(ii) Z is (isometrically J^*-isomorphic to) a complex Hilbert space.

We note that the results are new even in finite dimensions. For a survey of the classical case $B = \Delta$ we refer to [3].

1. Preliminaries

Throughout $\Delta = \{z \in \mathsf{C} : |z| < 1\}$, H will denote a complex Hilbert space, $L(X, Y)$ the space of all continuous linear maps from a complex Banach space X to a complex Banach space Y, $L(X) = L(X, X)$ and $GL(X)$ is all invertible elements in $L(X)$.

1.1. *JB^*-triples*

Definition 1.1. A JB^*-triple is a complex Banach space Z with real tri-linear mapping $\{\cdot, \cdot, \cdot\} : Z \times Z \times Z \to Z$ satisfying

(i) $\{x, y, z\}$ is complex linear and symmetric in x and z, and is complex anti-linear in y;

(ii) the map $z \mapsto \{x, x, z\}$, denoted $x\Box x$, is Hermitian, $\sigma(x\Box x) \geq 0$ and $\|x\Box x\| = \|x\|^2$ for all $x \in Z$, where σ denotes the spectrum;

(iii) for all $a, b, x, y, x \in Z$ the Jordan triple identity holds, namely,

$$\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\}.$$

The triple product is continuous [6], namely, $\|\{x, y, z\}\| \leq \|x\|\|y\|\|z\|$. JB^*-triples that are also Banach dual spaces are known as JBW^*-triples, and have been much studied. It is known [11] that every bounded symmetric domain is biholomorphically equivalent to the open ball of a JB^*-triple and vice versa.

Example 1.2. (i) H is a JB^*-triple for product

$$\{x, y, z\} = \frac{\langle x, y\rangle z + \langle z, y\rangle x}{2}$$

(ii) If X is a locally compact Hausdorff space, then $C_0(X)$, the space of all continuous C-valued functions which vanish at infinity, is a JB^*-triple for $\{x, y, z\} = x\overline{y}z$.

All C^*-algebras, JB^*-algebras and J^*-algebras are JB^*-triples, so the class of triples is large and interesting. Since the triple product encodes the holomorphic structure of B (for example, the Kobayashi metric on B [21] and [22]), it is a key tool in the study of holomorphic maps on all of these spaces.

Let Z be a JB^*-triple with open unit ball B. The most important linear maps on Z are the Bergman operators $B(x,y) = I - 2x\Box y + Q(x)Q(y) \in L(Z)$ as they play a central role in the geometry of B. Here $x\Box y \in L(Z)$ is the map $z \mapsto \{x,y,z\}$ and $Q(x)$ maps $z \mapsto \{x,z,x\}$ so that $Q(x)Q(y) \in L(Z)$. We note that for all $x \in B$, $\sigma(B(x,x)) > 0$ and $B_x := B(x,x)^{\frac{1}{2}}$ exists in the sense of the holomorphic functional calculus on $L(Z)$ [13].

1.2. *Tripotents and Peirce decompositions*

A concept of orthogonality exists in Z and we say $x, y \in Z$ are orthogonal, $x \perp y$, if $x\Box y = 0$ (or equivalently [19] if $y\Box x = 0$). Analogues of idempotents for an algebra also exist in the form of tripotents, where $e \in Z$ is a tripotent if $\{e,e,e\} = e$. Every tripotent e induces a Peirce splitting $Z = Z_1(e) \oplus Z_{\frac{1}{2}}(e) \oplus Z_0(e)$ where $Z_k(e)$ is the k eigenspace of $e\Box e$. A tripotent e is said to be maximal if $Z_0(e) = 0$ and said to be minimal if $Z_1(e) = \mathsf{C}e$. It is known that for JB^*-triples real and complex extreme points of the closed unit ball coincide and are precisely the set of all maximal tripotents.

Z is said to have rank r if the cardinality of every set of non-zero pairwise orthogonal tripotents is $\leq r$ and there is at least one set of cardinality r. We say Z is finite rank if $r < \infty$. Of course, if Z is finite dimensional then it is finite rank. We say $a \in Z$ is algebraic if there exists a finite family of pairwise orthogonal tripotents $e_1, ..., e_r$ and $\lambda_1, \lambda_2, \cdots, \lambda_r \in \mathsf{C}$ such that $a = \lambda_1 e_1 + \cdots + \lambda_r e_r$. For $a \neq 0$ algebraic, the decomposition can be chosen so that each e_k is non-zero and $\lambda_1 = \|a\| > \lambda_2 > \cdots > \lambda_r > 0$. We refer to $e = e_1 + \cdots + e_r$ as the support tripotent of A and write $e = \mathrm{supp}(a)$ ($\mathrm{supp}(0) = 0$). If Z is finite rank then every $a \in Z$ is algebraic. We refer to [1] and [15] for additional details.

1.3. *Spectral Theory*

A well developed spectral theory exists for JB^*-triples, as follows. Given $a \in Z$, let Z_a denote the smallest closed subtriple of Z containing a. Then there is a unique compact subset $S = -S$ of R such that (i) 0 is not isolated in S and (ii) there is a unique triple isomorphism from Z_a onto

$C^-(S) := \{f \in C(S) : f(-s) = -f(s)\ \forall s \in S\}$ such that a becomes the function $a(s) \equiv s$ on S. Letting $S^+ = \{s \in S : s > 0\}$, the following are triple isomorphisms $Z_a \cong C^-(S) \cong C_0(S^+)$, where we identify a with the map $a(s) = s\ \forall s \in S$ and elements of $C_0(S^+)$ are identified with maps in $C^-(S)$ by extension in the obvious way. Proposition 3.5 of [15] gives all necessary details. The set S, denoted $\mathrm{Sp}(a)$, is called the (triple) spectrum of a ($\mathrm{Sp}(0) = \emptyset$) and we write $\mathrm{rank}(a) :=\dim(Z_a)$. Using this an odd functional calculus exists on Z [15]. We note that $a \perp b$ gives $Z_a \perp Z_b$ (this follows from the Jordan triple identity and double induction on odd powers of a and b). Moreover, as the direct product $Z_a \times Z_b$ with component-wise triple product and maximum norm is a JB*-triple, then $a \perp b$ implies that the map $Z_a \times Z_b \to Z : (x, y) \mapsto x + y$ is an injective triple homomorphism and is hence an isometry [16] giving $\|a + b\| = \max\{\|a\|, \|b\|\}$. In particular, $a \perp b$ for $a, b \in B$ also means $a + b \in B$.

1.4. *Automorphism Group*

The structure of the group, G, of all biholomorphic automorphisms of B is long known. We refer to section 3 of [14] for details. G is a (Banach) Lie group whose Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ consists of all complete holomorphic vector fields on B, with $\mathfrak{k} = \mathrm{aut}(Z)$ being all triple derivations of Z and

$$\mathfrak{p} = \{X_\alpha : \alpha \in Z\}, \text{ where } X_\alpha(z) = (\alpha - \{z, \alpha, z\})\frac{\partial}{\partial z}.$$

In particular, for each $X \in \mathfrak{g}$, the map $t \mapsto \exp(tX)$ is a 1-parameter subgroup of G. At the group level we have the decomposition $G = KP = PK$, where $K = \mathrm{Aut}(Z)$ is the subgroup of all surjective linear isometries (or equivalently, all triple isomorphisms) of Z and $P = \exp(\mathfrak{p})$ is a real submanifold, though not a subgroup, of G. Each $g \in G$ therefore has a unique representation $g = g_a \circ T$, where $T \in K, a = g(0)$, and $g_a \in P$, called a transvection, is given by

$$g_a(z) = a + B_a(I + z\Box a)^{-1}z, \quad z \in B$$

where $B_a := B(a, a)^{\frac{1}{2}} \in \mathrm{GL}(Z)$. Clearly $g_0 = I$.

Moreover, $g_a = \exp(X_\alpha)$, where $\alpha = \tanh^{-1}(a)$ is defined in terms of the odd functional calculus on Z (and α and a generate the same subtriple of Z).

Example. (i) If $B = \Delta$ and $a \in \Delta$ then $g_a(z) = \frac{z+a}{1+\bar{a}z}$, $z \in \Delta$, is the classical Möbius map. To distinguish the case of Δ throughout, we will write t_a for g_a whenever $a \in \Delta$. It is well known, cf. [3], that for $a \in \Delta \backslash \{0\}$, t_a^n converges locally uniformly on Δ to the constant map $\frac{a}{|a|}$ and hence, in particular, $\lim_n t_a^n(0) = \frac{a}{|a|}$. We make repeated use of the fact that $\lim_n t_a^n(0) = 1$ for all $a \in (0,1)$.

(ii) If $B = B_H$ is a complex Hilbert ball then for $a \neq 0$ in B

$$g_a(z) = \left(P_a + \sqrt{1 - \|a\|^2} Q_a\right) \left(\frac{z+a}{1 + \langle z, a\rangle}\right), \ z \in B$$

where P_a is the orthogonal projection onto the subspace $\mathsf{C}a$ and $Q_a = I - P_a$.

1.5. *Boundary Components*

The boundary components of a bounded symmetric domain B are classified [17] in terms of holomorphic maps called boundary transvections.

We recall that $A \subset \overline{B}$, $A \neq \emptyset$ is a (holomorphic) boundary component of B if A is minimal with respect to the fact that either $f(\Delta) \subset A$ or $f(\Delta) \subset \overline{B} \setminus A$, $\forall f : \Delta \to Z$ holomorphic with $f(\Delta) \subset \overline{B}$. We denote the boundary component of B containing a as K_a.

For $c \in \partial B$, the local uniform limit of g_a as $a \in B$ approaches c, namely $\lim_{a \to c} g_a$, exists as a holomorphic map : $B \to Z$, is denoted g_c and called a boundary transvection. Such maps classify the boundary components of B containing tripotents, namely, if $e \in Z$ is a tripotent then

$$K_e = g_e(B) = e + B_0(e), \text{ where } B_0(e) = B \cap Z_0(e)$$

and also $K_e = g_a(B)$, for all $a \in K_e$. Of course, if $e = 0$ then $K_0 = B$ and this is the unique open boundary component of B. We note that for $c \in \partial B$, the boundary transvection g_c, unlike g_a for $a \in B$, is neither biholomorphic nor injective in general. The map $(z, a) \mapsto g_a(z)$ is, however, a continuous map on $\overline{B} \times \overline{B} \setminus (\partial B \times \partial B)$. We refer to [17], in particular Theorem 2.1 and Proposition 4.3, for proofs and details of all results in this subsection.

2. Results: Algebraic Elements

Let Z be an arbitrary JB^*-triple with open unit ball B. For holomorphic functions on B, convergence is understood throughout to mean local uniform convergence on B. We note that on G, the automorphism group of B, this topology coincides with uniform convergence on subsets lying strictly inside B, cf. Theorem 3.1 of [10].

Let $a \in B$. We begin by examining the iterates, g_a^n, of g_a. Fix $n \in N$.

As $g_a = \exp(X_\alpha)$, where $\alpha = \tanh^{-1}(a)$ and since the map $t \mapsto \exp(tX_\alpha)$ is a 1-parameter subgroup of G then (recall that P is not a subgroup of G)

$$g_a^n = (\exp(X_\alpha))^n = \exp(nX_\alpha) = \exp(X_{n\alpha}) \in P$$

so that $g_a^n = g_{c_n}$, for $c_n \in B$, and evaluating at 0 gives $g_a^n(0) = c_n$.

In other words, for all $a \in B$ and $n \in \mathsf{N}$

$$g_a^n = g_{g_a^n(0)}. \tag{2.1}$$

This simple identity is crucial, since in light of section 1.5 above, it immediately simplifies the process of finding accumulation points of (g_a^n) with respect to the topology of local uniform convergence on B, by allowing us instead to focus on finding accumulation points of the sequence $(g_a^n(0))$ in Z with respect to the norm topology. To this end, it is important to notice that the sequence $(g_a^n(0))$ lies entirely in the JB^*-subtriple, Z_a, generated by a. If now Z_a is just $\mathsf{C}a$, then we are already almost done. Although this is generally not the case, it is true for Hilbert spaces, where $Z_a = \mathsf{C}a$, for all a in H. For Hilbert space enthusiasts therefore, who may wish to forgo Jordan theory, we present this separately.

THEOREM 2.1. *Let H be a complex Hilbert space with open unit ball $B, a \in B\backslash\{0\}$. The sequence of iterates (g_a^n) converges locally uniformly on B to the constant map $\frac{a}{\|a\|}$.*

PROOF. For $e \in \partial B$ and $\lambda, \mu \in \Delta$ then $g_{\lambda e}(\mu e) = \left(\frac{\lambda+\mu}{1+\bar{\lambda}\mu}\right) e = t_\lambda(\mu)e$, where t_λ is the Möbius transformation on the disc. By induction $g_{\lambda e}^n(0) = t_\lambda^n(0)e$, for $n \in \mathsf{N}$. Fix $a \in B\backslash\{0\}$. Then $g_a^n(0) = g_{\|a\|e}^n(0) = t_{\|a\|}^n(0)e$, for $e = \frac{a}{\|a\|} \in \partial B$. Since $(t_{\|a\|}^n(0))$ converges to 1, $(g_a^n(0))$ converges in norm to $\frac{a}{\|a\|}$ and hence, as in section

1.5, $(g_{g_a^n(0)})$ converges locally uniformly on B to the boundary transvection $g_{\frac{a}{\|a\|}}$. As $g_{\frac{a}{\|a\|}}(B) = K_{\frac{a}{\|a\|}}$ and, since every point in ∂B is complex extreme, $K_{\frac{a}{\|a\|}} = \{\frac{a}{\|a\|}\}$, so $g_{\frac{a}{\|a\|}}$ is the constant map $\frac{a}{\|a\|}$. The result follows from (2.1).

Comment. The following properties of Hilbert spaces are key to the above proof.

1. $g_a^n(0) \in \mathsf{C}a$, for all $n \in \mathsf{N}$ (this ensures that for $a \neq 0$ a limit, $g_{\frac{a}{\|a\|}}$, exists).
2. Every point on ∂B is extreme, namely, B is strictly convex (this ensures that the limit, $g_{\frac{a}{\|a\|}}$, is constant).

While property 2 does not hold for triples of rank > 1, we can ask if property 1 generalises to some such triples. The answer is negative, as the following shows that properties 1 and 2 are equivalent.

Proposition 2.2. *Let Z be a JB^*-triple with open unit ball B. The following are equivalent.*
(i) Z is (isometrically J^*-isomorphic to) a complex Hilbert space.
(ii) B is strictly convex.
(iii) Every $e \in \partial B$ is a tripotent (and is then automatically minimal and maximal at the same time).
(iv) For all $a \in B$, $Z_a \cong \mathsf{C}a$.
(v) For all $a \in B$, $g_a^n(0) \in \mathsf{C}a$. [Note that for $a = 0$, $g_a = I$.]

Proof. Implications (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv)$\Rightarrow$ (v) are straightforward.
(v) $\Rightarrow$ (iii): Let $e \in \partial B$. It suffices to prove that e is a tripotent. Let $a = \frac{e}{2} \in B$. By assumption $g_a^2(0) = g_a(a) \in \mathsf{C}a$. We may assume that $a \in C_0(S^+)$ is the map $a(s) \equiv s$ and $g_a(z) = \frac{z+a}{1+az}$, $z \in C_0(S^+)$. Then $g_a^2(0) = \frac{2a}{1+a^2} = \lambda a$, for some real constant $\lambda > 0$. This implies S^+ must be a singleton and therefore $e = \frac{a}{\|a\|}$ is a tripotent and we are done.

We return to arbitrary JB^*-triples. For the space of holomorphic functions on B, we note that nets, rather than sequences, are required to determine the topology. In particular, the set of all accumulation points of (g_a^n) with repect to the topology of local uniform convergence on B is precisely the set of limit points of all its locally uniformly convergent subnets. The following result shows that for (g_a^n) this is conveniently the same as the set of all its (locally uniform) subsequential limits.

THEOREM 2.3. *Let Z be a JB^*-triple with open unit ball B and $a \in B$. The set of accumulation points of (g_a^n) with respect to the toplogy of local uniform convergence on B is*

$$\{g_c : c \in \Gamma_a\},$$

where Γ_a is the set of all subsequential limits of $(g_a^n(0))$ in Z with respect to the norm topology. In particular, for (g_a^n) the set of accumulation points is exactly the set of its subsequential limits.

PROOF. Let h be an accumulation point of (g_a^n), namely, there is a subnet $(n_\alpha)_\alpha$ of N such that $g_a^{n_\alpha} \underset{\alpha}{\to} h$ locally uniformly on B. In particular then, $g_a^{n_\alpha}(0) \underset{\alpha}{\to} h(0)$ in Z. From (2.1) $g_a^{n_\alpha} = g_{g_a^{n_\alpha}(0)}$ and therefore (section 1.5) $g_a^{n_\alpha} = g_{g_a^{n_\alpha}(0)} \underset{\alpha}{\to} g_{h(0)}$ locally uniformly on B. Uniqueness of limits gives $h = g_{h(0)}$. Since the topology on Z is determined by sequences, the set of limits points of all (convergent) subnets of $(g_a^n(0))$ is the same as the set of all its subsequential limits. In other words, $h(0) \in \Gamma_a$ and $h = g_{h(0)} \in \{g_c : c \in \Gamma_a\}$. On the other hand, let $c \in \Gamma_a$, that is, $c = \lim_k g_a^{n_k}(0)$. As above $g_a^{n_k} = g_{g_a^{n_k}(0)} \underset{k}{\to} g_c$ locally uniformly on B, completing the proof.

Since for a tripotent e, direct calculation gives $g_{\lambda e}(\mu e) = t_\lambda(\mu)e$, $\lambda, \mu \in \Delta$, the next result follows exactly as in Theorem 2.1.

LEMMA 2.4. *Let e be a tripotent and $\lambda \in \Delta$. Then $g_{\lambda e}^n = g_{t_\lambda^n(0)e}$ for $n \in \mathsf{N}$ and $(g_{\lambda e}^n)$ converges locally uniformly on B to the boundary transvection $g_{\frac{\lambda}{|\lambda|}e}$, if $\lambda \neq 0$.*

Our next motivation comes from the fact that if $a \perp b$ then g_a is orthogonal to g_b in the sense that

$$g_a \circ g_b = g_b \circ g_a \text{ so that } g_{a+b} = g_a \circ g_b,$$

which will allow us to extend Lemma 2.4 above to finite linear combinations of tripotents, namely, to algebraic elements of Z. While this orthogonality result may be part of the folklore we supply a proof for completeness. (The following result for $n = 1$, a tripotent e and $v \perp e$ is used in the proof of Proposition 4.3 of [17], though a proof is not given there.)

LEMMA 2.5. *Let $a, b \in B$ be orthogonal. Then $g_{a+b} = g_a \circ g_b$. In particular, g_a and g_b commute.*

Proof. Let $a, b \in B$ be orthogonal. As in section 1, $g_a = \exp(X_\alpha)$, $g_b = \exp(X_\beta)$, for $\alpha = \tanh^{-1}(a)$, $\beta = \tanh^{-1}(b)$ and X_α is the vector field, $X_\alpha(z) = (\alpha - \{z, \alpha, z\}) \frac{\partial}{\partial z}$. Since $\tanh^{-1}(t) = \sum_{n=0}^{\infty} \frac{t^{2n+1}}{2n+1}$ is odd, $a \perp b$ implies $\tanh^{-1}(a + b) = \tanh^{-1}(a) + \tanh^{-1}(b) = \alpha + \beta$ and $\alpha \perp \beta$. As $[X_\alpha, X_\beta] = X_{\alpha \Box \beta - \beta \Box \alpha}$, $\alpha \perp \beta$ implies $[X_\alpha, X_\beta] = 0$, which gives $\exp(X_\alpha + X_\beta) = (\exp X_\alpha) \circ (\exp X_\beta)$. Therefore $g_{a+b} = \exp(X_{\tanh^{-1}(a+b)}) = \exp(X_{\alpha+\beta}) = \exp(X_\alpha + X_\beta) = (\exp X_\alpha) \circ (\exp X_\beta) = g_a \circ g_b$. In particular $g_a \circ g_b = g_b \circ g_a$.

The above lemmata combine to show that for algebraic elements a the dynamics of g_a on B are determined entirely by its support tripotent e.

Theorem 2.6. *Let Z be a JB^*-triple, $a \in B$ be algebraic and $e = supp(a)$. Then (g_a^n) converges locally uniformly on B to the holomorphic map g_e, where $g_e(B)$ is the boundary component, K_e, of e.*

Proof. The case $a = 0$ is trivial, as $g_a = g_e = I$. Let $a \in B \backslash \{0\}$ be algebraic, with $a = \lambda_1 e_1 + ... + \lambda_r e_r$, where $\|a\| = \lambda_1 > ... > \lambda_r > 0$, and $e_1, ..., e_r$ are mutually orthogonal tripotents. Fix $n \in \mathsf{N}$. Then

$$\begin{aligned} g_a^n &= g_{\lambda_1 e_1}^n \circ \cdots \circ g_{\lambda_r e_r}^n && \text{by Lemma 2.5} \\ &= g_{t_{\lambda_1}^n(0) e_1} \circ \cdots \circ g_{t_{\lambda_r}^n(0) e_r} && \text{by Lemma 2.4} \\ &= g_{t_{\lambda_1}^n(0) e_1 + \cdots + t_{\lambda_r}^n(0) e_r} && \text{by Lemma 2.5 again.} \end{aligned}$$

Since $t_{\lambda_i}^n(0) \underset{n}{\to} 1$, $1 \le i \le r$, $\lim_n t_{\lambda_1}^n(0) e_1 + \cdots + t_{\lambda_r}^n(0) e_r = e$ so that, as before, g_a^n converges locally uniformly on B to g_e and we are done.

Comments.
1. If Z is finite rank then all elements in Z are algebraic and, of course, every finite dimensional JB^*-triple is finite rank.
2. If Z is a JBW^*-triple then the algebraic elements are dense, cf. [15] section 2.
3. Every element $a \in Z$ is algebraic if, and only if, Z has the Radon-Nikodym property, [2].

Theorem 2.6 however merits a much closer look, even in finite dimensions. The transvection g_e maps B onto the holomorphic boundary component, K_e, of e. This yields our first major surprise for, as the following examples show, the boundary components of e and $\frac{a}{\|a\|}$ are generally different, so that Theorem 2.6 (and a later Theorem 3.1) diverges from the Hilbert space result in both expected (see (i) below) and unexpected (see (ii) and (iii) below) ways.

(i) (g_a^n) does not necessarily converge to a constant map. See Example 2.7 below.

(ii) Even if (g_a^n) does converge to a constant, that constant is not generally $\frac{a}{\|a\|}$. In fact, that constant is not necessarily in $K_{\frac{a}{\|a\|}}$. See Example 2.8 below.

(iii) Where it exists, the limit of (g_a^n) does not generally map into the boundary component $K_{\frac{a}{\|a\|}}$. See Example 2.7 below.

Example 2.7. Let Z be C^3 with ℓ_∞ norm, $\|(z_1, z_2, z_3)\| = \max_{1\le i\le 3} |z_i|$, so $B = \Delta^3$. Consider the rank 2 element $a = (\frac{1}{2}, \frac{1}{4}, 0) \in B$. Then

$$a = \frac{1}{2}e_1 + \frac{1}{4}e_2,\ e_1 = (1,0,0), e_2 = (0,1,0) \text{ and } e = \operatorname{supp}(a) = e_1 + e_2 = (1,1,0)$$

so that $g_e(B) = K_e = 1 \times 1 \times \Delta$. On the other hand,

$$\frac{a}{\|a\|} = e_1 + \frac{1}{2}e_2 \text{ and } K_{\frac{a}{\|a\|}} = K_{e_1} = 1 \times \Delta \times \Delta.$$

Clearly $K_e \cap K_{\frac{a}{\|a\|}} = \emptyset$. Note that $g_a^n(z) = (t_{\frac{1}{2}}^n(z_1), t_{\frac{1}{4}}^n(z_2), z_3)$ with $z = (z_1, z_2, z_3)$, where $t_{\frac{1}{2}}^n, t_{\frac{1}{4}}^n$ are Möbius maps on Δ that converge locally uniformly on Δ to 1. So (g_a^n) converges locally uniformly on B to g_e where $g_e(z) = (1, 1, z_3)$.

Example 2.8. Let Z be C^2 with ℓ_∞ norm, so $B = \Delta^2$. Take

$$a = \left(\frac{1}{2}, \frac{1}{4}\right) = \frac{1}{2}e_1 + \frac{1}{4}e_2, \text{where } e_1 = (1,0), e_2 = (0,1) \text{ and } e = \operatorname{supp}(a) = e_1 + e_2 = (1,1).$$

Here e is a complex extreme point, so $K_e = \{e\}$ and g_e is the constant map e. On the other hand

$$\frac{a}{\|a\|} = \left(1, \frac{1}{2}\right) = e_1 + \frac{1}{2}e_2 \text{ and } K_{\frac{a}{\|a\|}} = K_{(1,0)} = 1 \times \Delta.$$

Of course, (g_a^n) converges locally uniformly on Δ^2 to the constant map e.

The following proposition clarifies the situation.

Proposition 2.9. *Let $a \in B$ be algebraic and $e =$ supp(a). If $a \neq 0$ then*

$$K_{\frac{a}{\|a\|}} = K_e \text{ if, and only if, } \frac{a}{\|a\|} \text{ is a tripotent.}$$

Proof. Let $a \in B\backslash\{0\}$ be algebraic and $e = \text{supp}(a)$. From section 1.5, $K_{\frac{a}{\|a\|}} = K_e$ if, and only if, $\frac{a}{\|a\|} \in K_e = g_e(B) = e + B_0(e),\ B_0(e) = Z_0(e) \cap B$. In other words, $K_{\frac{a}{\|a\|}} = K_e$ if, and only if, $v := e - \frac{a}{\|a\|} \in B_0(e)$. Write $a = \lambda_1 e_1 + \cdots + \lambda_r e_r,\ \|a\| = \lambda_1 > \lambda_2 > ... > \lambda_r > 0$ and $e = e_1 + \cdots + e_r =$ supp(a). Then $v = \left(1 - \frac{\lambda_2}{\lambda_1}\right) e_2 + \cdots + \left(1 - \frac{\lambda_r}{\lambda_1}\right) e_r$. Clearly $v \in B$, so $v \in B_0(e)$ if, and only if, $v \square e = 0$. Since $v \square e = \left(1 - \frac{\lambda_2}{\lambda_1}\right) e_2 \square e_2 + \cdots + \left(1 - \frac{\lambda_r}{\lambda_1}\right) e_r \square e_r,$ we have $v \in B_0(e) \Leftrightarrow \frac{\lambda_i}{\lambda_1} = 1,\quad i = 2, ..., r \Leftrightarrow \|a\| = \lambda_1 = \lambda_2 = ... = \lambda_r \Leftrightarrow a = \|a\| e \quad \Leftrightarrow \frac{a}{\|a\|} = e \Leftrightarrow \frac{a}{\|a\|}$ is a tripotent and we are done.

From Proposition 2.2 every element of Z is a scalar multiple of a tripotent if, and only if, Z is a complex Hilbert space. Theorem 2.6 and Proposition 2.9 therefore yield the following (a precursor to a later result (Theorem 3.4) for arbitrary triples).

Corollary 2.10. *Let Z be a JB*-triple such that every element is algebraic. Then (g_a^n) converges locally uniformly on B to a constant, for all non-zero a in B, if, and only if, Z is a complex Hilbert space.*

3. Results: Regular Elements

We now use spectral theory to extend Theorem 2.6 above to those elements a of Z, for which the spectrum, Sp(a), does not contain 0. In [15, Lemma 4.1] $0 \notin \text{Sp}(a)$ is shown to be equivalent to several previously studied concepts of regularity, namely, $0 \notin \text{Sp}(a) \Leftrightarrow a$ is regular $\Leftrightarrow a$ is strongly regular $\Leftrightarrow a$ has a generalized inverse. For this reason, if $0 \notin \text{Sp}(a)$ we simply refer to a as being regular. Since 0 is never an isolated point of the spectrum, it follows that every algebraic element is regular. In particular, 0 is regular as $\text{Sp}(0) = \emptyset$. As noted in section 1, we have, for $S = \text{Sp}(a)$ and $S^+ = \{s \in S : s > 0\}$, triple isomorphisms $Z_a \cong C^-(S) \cong C_0(S^+)$, where a is identified with the the map $a(s) = s, \forall s \in S^+$.

The concept of support tripotent also exists for regular elements of Z. Let $a \in Z$ be regular and $S = \text{Sp}(a)$. Since $0 \notin S$ the map $e(s) = 1, \forall s \in S^+$ defines a tripotent e in $Z_a \cong C_0(S^+)$, which we refer to as the support tripotent of a, written $e =$

supp(a). Note that since $0 \notin S$, S^+ is, in this case, compact and $C_0(S^+) \cong C(S^+)$. We refer to Lemma 3.2 of [12] for properties of regular elements and to [1], [9] and [23] for complementary details. We also note that every element of Z is regular if, and only if, Z is finite rank [12]. Theorem 2.6 can now be generalised.

THEOREM 3.1. *Let Z be a JB^*-triple and $a \in B$ be regular. Then (g_a^n) converges locally uniformly on B to the holomorphic map g_e, where $e =$ supp(a) and $g_e(B) = K_e$.*

PROOF. The case $a = 0$ is trivial. Let $a \in B \setminus \{0\}$ be regular and $e = \text{supp}(a)$. Let $S = \text{Sp}(a)$, so that $S^+ \subset (0, \|a\|) \subset (0, 1)$. For $z \in Z_a \cong C(S^+)$, $g_a(z) = \frac{z+a}{1+\bar{a}z}$ hence

$$g_a(z)(s) = \frac{z(s) + a(s)}{1 + \overline{a(s)}z(s)} = t_{a(s)}(z(s)) = t_s(z(s)) \text{ for } s \in S^+.$$

By induction therefore

$$g_a^n(0)(s) = t_s^n(0) \quad \text{for } s \in S^+, n \in \mathsf{N}. \tag{3.1}$$

Now $(t_s^n(0))_n$ converges to 1 for all $s \in S^+$ and this is equivalent to saying that $(g_a^n(0))_n$ converges pointwise to e on S^+. On the other hand, it is easy to see by induction and (3.1) that $(g_a^n(0))_n$ is monotone increasing on S^+ and therefore, by Dini's theorem, $(g_a^n(0))_n$ converges uniformly on S^+ to e. In other words, $e = \lim_n g_a^n(0)$ in $C(S^+)$ and hence in Z. As before $g_a^n = g_{g_a^n(0)}$ then converges locally uniformly on B to the holomorphic map g_e and we are done.

A further look at the above proof reveals g_e as the only possible accumulation point of (g_a^n).

THEOREM 3.2. *Let Z be a JB^*-triple and $a \in B$. The set of (local uniform) accumulation points of the sequence of iterates (g_a^n) is non-empty if, and only if, a is regular. In particular, (g_a^n) converges locally uniformly on B if, and only if, a is regular.*

PROOF. Theorem 3.1 gives one direction. In the opposite direction, suppose that an accumulation point, $h = \lim_\alpha g_a^{n_\alpha}$, exists for the topology of local uniform convergence on B. In particular then $h(0) = \lim_\alpha g_a^{n_\alpha}(0) \in Z_a \cong C_0(S^+)$, where

$S = \mathrm{Sp}(a)$ and $S^+ = \{s \in S : s > 0\}$. From (3.1) $h(0)(s) = \lim_\alpha g_a^{n_\alpha}(0)(s) = \lim_\alpha t_s^{n_\alpha}(0) = 1$ for all $s \in S^+$. Since $h(0) \in C_0(S^+)$ this means 0 is not an accumulation point of S and, as 0 is never isolated in S, cf. section 1.3, it follows that $0 \notin S$. Hence a is regular and, by Theorem 3.1, $h = g_e$ for $e = \mathrm{supp}(a)$.

Theorems 3.1 and 3.2 therefore tell us that the only possible (local uniform) accumulation point of (g_a^n) is the holomorphic map g_e, where $e = \mathrm{supp}(a)$, and if a is not regular then such a support tripotent does not exist in Z. So our results above are somehow best possible.

The proofs of Theorems 3.1 and 3.2 also contain the proof of the following alternative characterisation of regularity.

Corollary 3.3. *Let Z be a JB^*-triple with open unit ball B and $a \in B$. Then a is regular if, and only if,* $\lim_n g_a^n(0)$ *exists in Z. In particular, a has a support tripotent in Z if, and only if,* $\lim_n g_a^n(0)$ *exists in Z.*

Since Z^{**} is a Banach dual space (a JBW*-triple) its closed unit ball is weak*-compact. This suggests a way to generalise the concept of support tripotent to arbitrary (non-regular) elements of Z by considering $\lim_n g_a^n(0)$ in Z^{**}, rather than in Z, and looking for weak*-accumulation points there. This has already been done in [1]. (We note that the maps g_a are not, in general, weak*-weak* continuous [8].)

Our final result extends Corollary 2.10 and is further evidence, if any is still required, that the study of the dynamics of a holomorphic map on the Hilbert ball does not generalise in an insightful way to the other bounded symmetric domains, as the strict convexity of the Hilbert ball makes it a natural outlier in this class.

Theorem 3.4. *Let Z be a JB^*-triple. The following are equivalent.*

(i) (g_a^n) *converges locally uniformly on B to a constant map, for all non-zero $a \in B$.*

(ii) *Z is (triple isomorphic to) a complex Hilbert space.*

Proof. (ii)$\Rightarrow$(i) is Theorem 2.1.

(i)$\Rightarrow$ (ii): Assume (i) and let $a \in B\backslash\{0\}$. By Theorem 3.2, a is regular and by

Theorem 3.1, (g_a^n) then converges to g_e, $e = \text{supp}(a)$. Since (i) holds, g_e must be a constant map, c say. Then $K_e = g_e(B) = \{c\}$ which happens if, and only if, $e = c$ is complex extreme, that is, e is a maximal tripotent. Therefore every non-zero tripotent is maximal. On the other hand, it is easy to see that every non-zero tripotent in Z is maximal only if every non-zero tripotent is also minimal and hence all non-zero tripotents are rank 1. In particular $e = \text{supp}(a)$ is rank 1, so $a = \|a\|e$. Proposition 2.2 now gives the result.

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