4.3 The Black-Scholes Partial Differential Equation

Let $S$ be the price at time $t$ of a particular asset. After a (short) time interval of length $dt$, the asset price changes by $dS$, to $S + dS$. Rather than measuring the absolute change $dS$, we measure the return on the asset which is defined to be

$$\frac{dS}{S}.$$ 

Note: This return expresses the change in the asset price as a proportion of the original asset price.

One common mathematical model of the return has two components. The first is a predictable, deterministic component (similar to the return on a risk-free investment in a bank). This is

$$\mu dt.$$ 

The parameter $\mu$ is called the drift. It is a measure of the average rate of growth of the asset price.

The second contribution to the return $\frac{dS}{S}$ is

$$\sigma dX.$$ 

Here $\sigma$ is called the volatility and is a measure of the standard deviation of the returns. The quantity $dX$ is a random variable having a normal distribution with mean 0 and variance $dt$:

$$dX \sim N(0,(\sqrt{dt})^2).$$

This component is a random contribution to the return. For each interval $dt$, $dX$ is a sample drawn from the distribution $N(0,(\sqrt{dt})^2)$ - this is multiplied by $\sigma$ to produce the term $\sigma dX$. The value of the parameters $\sigma$ and $\mu$ may be estimated from historical data.

We obtain the following stochastic differential equation (stochastic analysis is the study of functions of random variables).

$$\frac{dS}{S} = \mu dt + \sigma dX. \quad (4.1)$$

Notes

1. If $\sigma = 0$ then the behaviour of the asset price is totally deterministic and we have the ordinary differential equation

$$\frac{dS}{S} = \mu dt.$$ 

This can be solved to give

$$S = S_0 e^{\mu t}$$

where $S_0$ is the asset price at time $t = 0$.

2. The equation 4.1 is an example of a random walk. It cannot be solved to give a deterministic path for the share price but it gives probabilistic information about the behaviour of $S$. 

39
3. The equation 4.1 can be considered to be a scheme for constructing time series that may be realised by share prices.

**Example 4.3.1** The price $S$ of a particular share today is €10. Construct a time series for the share price over three intervals if

$$
\mu = 0.4, \quad \sigma = 0.2, \quad dt = \frac{1}{250}.
$$

(This value of $dt$ is basically one day, assuming 250 business days in a year.)

**Solution:** We have

$$
\frac{dS}{S} - \mu dt + \sigma dX = 0.4 \left( \frac{1}{250} \right) + 0.2 dX.
$$

Here $dX$ is drawn (at each step) from a normal distribution with mean 0 and standard deviation $1/\sqrt{250} \approx 0.063$.

**Step 1** $S_0 = 10$. A value for $dX$ is chosen from $N(0,1/250)$ - choose $dX = -0.05$. Then

$$
\begin{align*}
\frac{dS}{10} & = 0.4/250 + 0.2(-0.05) \\
\frac{dS}{9.16} & = 10(0.4/250 + 0.2(-0.05)) \\
S_1 & = 10 - 0.084 \\
S_1 & = 9.916.
\end{align*}
$$

**Step 2** $S_1 = 9.916$, take $dX = 0.12$.

$$
\begin{align*}
\frac{dS}{9.916} & = 0.4/250 + 0.2(0.12) \\
\frac{dS}{10} & = 9.916(0.4/250 + 0.2(0.12)) \\
S_2 & = 9.916 + 0.254 \\
S_2 & = 10.17.
\end{align*}
$$

**Step 3** $S_2 = 10.17$, take $dX = 0.08$.

$$
\begin{align*}
\frac{dS}{10.17} & = 0.4/250 + 0.2(0.08) \\
\frac{dS}{10} & = 10.17(0.4/250 + 0.2(0.08)) \\
S_3 & = 10.17 + 0.179 \\
S_3 & = 10.35.
\end{align*}
$$

The following is a graphical representation of this time series.
In real life asset prices are quoted at discrete intervals of time, and so there is a practical lower bound for the basic time step $dt$ of our random walk. If this time step were used in practice however, the sheer quantity of data involved would be unmanageable. One approach is to develop a continuous model by taking a limit as $dt \to 0$. We finish these lecture notes now with a brief outline of such a model. We need Itô’s Lemma, which is a version of Taylor’s Theorem for functions of random variables.

**Recall:** Taylor Series

Let $f$ be a function with derivatives of all orders on an interval $I$ containing a point $a$. The Taylor Series of $f$ at $x = a$ is

$$
\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.
$$

If this series converges to $f$ on $I$ then

$$
f(x) = f(a) + \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n
\implies f(x) - f(a) = \sum_{n=1}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n.
$$

Now replace $x$ with $x + \Delta x$ and $a$ with $x$ to obtain

$$
\Delta f = f(x + \Delta x) - f(x) = f'(x) + \frac{f''(x)}{2!} (\Delta x)^2 + \ldots
$$

This relates the small change $\Delta f$ in the function $f$ to the small change $\Delta x$ in $x$.

We now return to our consideration of what happens to

$$
\frac{dS}{S} = \mu dt + \sigma dX
$$

41
as \( t \to 0 \). We need the following fact which we state without proof:

With probability 1 \( (dX)^2 \to dt \) as \( dt \to 0 \).

Suppose that \( f(S) \) is a function of the asset price \( S \). If we change \( S \) by a small amount \( dS \) then by Taylor’s Theorem we have

\[
 df = \frac{df}{dS} dS + \frac{1}{2} \frac{d^2 f}{dS^2} (dS)^2 + \ldots
\]

Now \( dS = S(\mu dt + \sigma dS) \) and

\[
 (dS)^2 = S^2 (\mu^2 (dt)^2 + 2 \mu \sigma dtdX + \sigma^2 (dX)^2)
\]

Now since \( (dX)^2 \to dt \) as \( dt \to 0 \), the term \( S^2 \sigma^2 (dX)^2 \) dominates the above expression for \( (dS)^2 \) as \( dt \) becomes small. Retaining only this term we use

\[
 S^2 \sigma^2 dS
\]

as an approximation for \( (dS)^2 \) as \( dt \to 0 \). We then have

\[
 df = \frac{df}{dS} dS + \frac{1}{2} \frac{d^2 f}{dS^2} (S^2 \sigma^2 dt)
\]

\[
 = \frac{df}{dS} (S \mu dt + \sigma dX) + \frac{d^2 f}{dS^2} (S^2 \sigma^2 dt)
\]

\[
 df = \sigma S \frac{df}{dS} dX + \left( \mu S \frac{df}{dS} + \frac{1}{2} \sigma^2 S^2 \frac{d^2 f}{dS^2} + \frac{\partial f}{\partial t} \right) dt.
\]

This is Itô’s Lemma relating a small change in a function of a random variable to a small change in the variable itself. There is a deterministic component \( dt \) and a random component \( dX \).

In fact we need a version of Itô’s Lemma for a function of more than one variable: if \( f \) is a function of two variables \( S, t \) we have

\[
 df = \sigma S \frac{\partial f}{\partial S} dX + \left( \mu S \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t} \right) dt.
\]

**The Black-Scholes PDE**

Let \( V(S, t) \) be the value of an option (this is usually called \( C(S, t) \) for a call and \( P(S, t) \) for a put). Let \( r \) be the interest rate and let \( \mu \) and \( \sigma \) be as above. Using Itô’s Lemma we have

\[
 dV = \sigma S \frac{\partial V}{\partial S} dX + \left( \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt.
\]

Consider a portfolio containing one option and \(-\Delta \) units of the underlying stock. The value of the portfolio is

\[
 \Pi = V - \Delta S.
\]
Thus $d\Pi = dV - \Delta dS$.

$$d\Pi = \sigma S \frac{\partial V}{\partial S} dX + \left( \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt - \Delta S \mu dt - \Delta S \sigma dX$$

$$= \sigma S \left( \frac{\partial V}{\partial S} - \Delta \right) dX + \left( \mu S \frac{\partial V}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \Delta S \right) dt$$

Choose $\Delta = \frac{\partial V}{\partial S}$ to get

$$d\Pi = \left( \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt.$$

Now if $\Pi$ was invested in riskless assets it would see a growth of $r\Pi dt$ in the interval of length $dt$. Then for a fair price we should have

$$r \Pi dt = \left( \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt$$

$$\Rightarrow r \left( V - \frac{\partial V}{\partial S} S \right) = \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}$$

Thus we obtain the Black-Scholes PDE.

$$\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + r S \frac{\partial V}{\partial S} - rV = 0.$$