Solutions 3

3.1 Let \( x \in U \). Since \( U \) is open, \( x \) is an interior point of \( U \). Therefore there exists \( \rho > 0 \) such that \( B(x; \rho) \subset U \). Since \( U \subset A \), \( B(x; \rho) \subset A \). Therefore \( x \) is an interior point of \( A \). Therefore \( U \subset \text{Int}(A) \).

3.2 Suppose that \( a \in A \). Since \( A \) is an open set, there exists \( \rho_a > 0 \) such that \( B_a = B(a; \rho_a) \subset A \).

Since \( a \in B_a \),
\[
A \subset \bigcup \{B_a : a \in A\} \tag{1}
\]

Since \( B_a \subset A \),
\[
\bigcup \{B_a : a \in A\} \subset A \tag{2}
\]

Statements (1) and (2) imply that
\[
A = \bigcup \{B_a : a \in A\}.
\]

3.3 (i) Suppose that \( x \in \text{Int}(A \cap B) \). Then there exists \( \rho > 0 \) such that \( B(x; \rho) \subset A \cap B \). Therefore \( B(x; \rho) \subset A \) and \( B(x; \rho) \subset B \). Therefore \( x \in \text{Int}(A) \) and \( x \in \text{Int}(B) \), that is, \( x \in \text{Int}(A) \cap \text{Int}(B) \).

Therefore
\[
\text{Int}(A \cap B) \subset \text{Int}(A) \cap \text{Int}(B) \tag{1}
\]

Now suppose that \( x \in \text{Int}(A) \cap \text{Int}(B) \), that is, \( x \in \text{Int}(A) \) and \( x \in \text{Int}(B) \). Then there exists \( \alpha > 0 \) and \( \beta > 0 \) such that \( B(x; \alpha) \subset A \) and \( B(x; \beta) \subset B \). Let \( \rho = \min \{\alpha, \beta\} \). Then
\[
B(x; \rho) \subset B(x; \alpha) \subset A \quad \text{and} \quad B(x; \rho) \subset B(x; \beta) \subset B.
\]

Therefore \( B(x; \rho) \subset A \cap B \) and so \( x \in \text{Int}(A \cap B) \). Therefore
\[
\text{Int}(A) \cap \text{Int}(B) \subset \text{Int}(A \cap B) \tag{2}
\]

Statements (1) and (2) imply that \( \text{Int}(A) \cap \text{Int}(B) = \text{Int}(A \cap B) \).

(ii) Let \( A = [1, 2] \) and \( B = [2, 3] \subset \mathbb{E}_1 \). Then \( \text{Cl}(A) = [1, 2] \) and \( \text{Cl}(B) = [2, 3] \) and so \( \text{Cl}(A) \cap \text{Cl}(B) = \{2\} \). But \( A \cap B = \emptyset \) and thus \( \text{Cl}(A \cap B) = \emptyset \). Therefore it is not always true that \( \text{Cl}(A) \cap \text{Cl}(B) = \text{Cl}(A \cap B) \).

3.4 Suppose that \( (a_n) \) is a sequence in \( M \) and that \( a \in M \).

(i) Suppose that, for any open ball \( B \) centred at \( a \), there exists \( N \in \mathbb{N} \) such that
\[
\forall n \in \mathbb{N}, \text{ if } n \geq N \text{ then } a_n \in B.
\]

We must show that \( \lim_n(a_n) = a \).

Let \( \epsilon > 0 \). Then there exists \( N \in \mathbb{N} \) such that
\[
\forall n \in \mathbb{N}, \text{ if } n \geq N \text{ then } a_n \in B(a; \epsilon),
\]
that is,
\[
\forall n \in \mathbb{N}, \text{ if } n \geq N \text{ then } d(a_n, a) < \epsilon.
\]

Therefore \( \lim_n(a_n) = a \).

Please turn over.
(ii) Suppose that \( \lim_n (a_n) = a \) and that \( B \) is an open ball centred at \( a \). Let \( \rho \) be the radius of \( B \), that is, let \( B = B(a; \rho) \). Since \( \rho > 0 \), there exists \( N = N(\rho) \in \mathbb{N} \) such that

\[
\forall n \in \mathbb{N}, \text{ if } n \geq N \text{ then } d(a_n, a) < \rho,
\]

that is,

\[
\forall n \in \mathbb{N}, \text{ if } n \geq N \text{ then } a_n \in B(a; \rho) = B.
\]

3.5 Suppose that \( f : X \to Y \) and that \( a \in \text{dom}(f) \).

(i) Suppose that for any open ball \( B' \subset Y \) centred at \( f(a) \) there exists an open ball \( B \subset X \) centred at \( a \) such that, for all \( x \in \text{dom}(f) \), if \( x \in B \) then \( f(x) \in B'(f(a)) \). Let \( \epsilon > 0 \). Then there exists an open ball \( B \subset X \) centred at \( a \) such that, for all \( x \in \text{dom}(f) \),

\[
\text{if } x \in B \text{ then } f(x) \in B(f(a); \epsilon)
\]

Let \( \delta \) be the radius of \( B \). Then (1) can be written as

\[
\text{if } x \in B(a; \delta) \text{ then } f(x) \in B(f(a); \epsilon),
\]

that is,

\[\text{if } d(x, a) < \delta \text{ then } d'(f(x), f(a)) < \epsilon.\]

Therefore \( f \) is continuous at \( a \).

(ii) Suppose that \( f \) is continuous at \( a \) and that \( B' \subset Y \) is an open ball centred at \( f(a) \). Let \( \rho \) be the radius of \( B' \). Since \( \rho > 0 \) and \( f \) is continuous at \( a \), there exist \( \delta > 0 \) such that, for all \( x \in \text{dom}(f) \),

\[
\text{if } d(x, a) < \delta \text{ then } d'(f(x), f(a)) < \rho,
\]

that is,

\[
\text{if } x \in B(a; \delta) \text{ then } f(x) \in B'. \quad \Box
\]

3.6 Suppose that \( (a_n) \) is a sequence in \( M \) and that \( a \in M \). Then, for all \( n \in \mathbb{N} \),

\[
|d(a_n, a) - 0| = |d(a_n, a)| = d(a_n, a)
\]

since a metric does not take a negative value. Therefore

\[
\lim_n (a_n) = a
\]

\[
\exists \epsilon > 0, \exists N(\epsilon) \in \mathbb{N} \text{ such that } n \geq N(\epsilon) \implies d(a_n, a) < \epsilon
\]

\[
\exists \epsilon > 0, \exists N(\epsilon) \in \mathbb{N} \text{ such that } n \geq N(\epsilon) \implies |d(a_n, a) - 0| < \epsilon
\]

\[
\lim_n (d(a_n, a)) = 0. \quad \Box
\]

3.7 Suppose that \( A \) is a non-empty subset of \( M \). Let \( a \in A \). Then, since \( (M, d) \) is a discrete metric space, for all \( x \in M \),

\[
d(x, a) < 0.5 \iff d(x, a) = 0 \iff x = a.
\]

Therefore \( B(a; 0.5) = \{a\} \) and thus \( B(a; 0.5) \subset A \). Therefore \( A \) is open.

For any \( A \subset M \), \( M \setminus A \subset M \). Therefore \( M \setminus A \) is open. Therefore \( A = M \setminus (M \setminus A) \) is closed.

Please turn over.
3.8 The constant function \( f : E_1 \to E_1 : x \to 1 \) is continuous. If \( A \) is the open interval \((0, 1)\) then \( f(A) = \{1\} \) and \( \{1\} \), being a singleton, is closed. Therefore, the answer to the question in the problem is “No.”

3.9 For all \( x \in A \),
\[
x \in \phi^{-1}(\theta^{-1}(S)) \iff \phi(x) \in \theta^{-1}(S) \iff \theta(\phi(x)) \in S
\]
Therefore \( (\theta \circ \phi)^{-1}(S) = \phi^{-1}(\theta^{-1}(S)) \).

3.10 Suppose that \( U \) is an open subset of \( P \). Then, since \( g \) is continuous, \( g^{-1}(U) \) is an open subset of \( N \). Therefore, since \( f \) is continuous, \( f^{-1}(g^{-1}(U)) \) is an open subset of \( M \). Therefore, by the equation in 4.9, \( (g \circ f)^{-1}(U) \) is an open subset of \( M \). Therefore \( g \circ f \) is continuous.

3.11 Let \( B = B(0; \rho) \). For all \( x, y \in B \),
\[
d(x, y) \leq d(x, 0) + d(0, y) < 2\rho \tag{1}
\]
Therefore the set of distances in \( B \) is bounded above by \( 2\rho \). Therefore the diameter of \( B \) is defined. Let \( \lambda \) be the diameter of \( B \). Inequality (1) implies that
\[
\lambda \leq 2\rho \tag{2}
\]
Suppose that
\[
\lambda < 2\rho \tag{3}
\]
Let \( \alpha = (\lambda + 2\rho)/2 \). Then \( \lambda/2 < \alpha < \rho \). Since \( \alpha < \rho \),
\[
(-\alpha, 0, \ldots, 0) \text{ and } (\alpha, 0, \ldots, 0) \in B \tag{4}
\]
Since \( \alpha > \lambda/2 \),
\[
d((-\alpha, 0, \ldots, 0), (\alpha, 0, \ldots, 0)) = 2\alpha > \lambda \tag{5}
\]
Since \( \lambda \) is an upper bound of the set of distances in \( B \), in view of (4), (5) cannot be true. Therefore (3) must be false. Therefore \( \lambda = 2\rho \).

3.12 Suppose that \( (a_n) \) is an infinite sequence in \( K \cup L \). We consider two possibilities, one of which must hold.
(i) Only a finite number of the terms of \( (a_n) \) belong to \( K \).

Therefore an infinite number of the terms of \( (a_n) \) must belong to \( L \), in other words, the terms of \( (a_n) \) that belong to \( L \) form an infinite sequence in \( L \). Since \( L \) is compact, this sequence has a subsequence \( (a_{k_n}) \) that converges in \( L \). But \( (a_{k_n}) \) is a subsequence of \( (a_n) \) and, since it converges in \( L \), it converges in \( K \cup L \).

(ii) An infinite number of the terms of \( (a_n) \) belong to \( K \).

These terms form an infinite sequence in \( K \). Since \( K \) is compact, this sequence has a subsequence \( (a_{k_n}) \) that converges in \( K \). But \( (a_{k_n}) \) is a subsequence of \( (a_n) \) and, since it converges in \( K \), it converges in \( K \cup L \).

Therefore any infinite sequence in \( K \cup L \) has a subsequence that converges in \( K \cup L \). Therefore \( K \cup L \) is compact.

3.13 Use induction on \( n \) and the proposition in Problem 4.12

Please turn over.
3.14 Suppose that $K$ is a compact subset of a metric space $M$. Suppose that $(a_n)$ is an infinite sequence in $K$ that converges to $a \in M$. To show that $K$ is closed we must show that $a \in K$.
Since $K$ is compact, $(a_n)$ has a subsequence $(a_{k_n})$ that converges in $K$. But, since $(a_n)$ converges to $a$, $(a_{k_n})$ must converge to $a$. Therefore $a \in K$. Therefore $K$ is closed.

3.15 Suppose that $A \cup B$ is disconnected. Then there is a continuous two-valued function $f : A \cup B \to \mathbb{E}_1$. The function $f|A$ is continuous and $f(A) \subseteq \{0, 1\}$. Since $A$ is connected, $f|A$ cannot be two-valued and therefore it must be constant. Since $B$ is connected, $f|B$ must also be constant. Since $f$ is two-valued on $A \cup B$ its value on $A$ must be different from its value on $B$.
But this is impossible because $A \cap B \neq \emptyset$. Therefore $A \cup B$ is connected.

3.16 Clearly, $A$ is bounded. We must show that $A$ is closed.
Suppose that $(x_n)$ is an infinite sequence in $A$ that converges in $(\mathbb{Q}, d)$ to $x$. We must show that $x \in A$. If we treat $(x_n)$ as an infinite sequence in $\mathbb{E}_1$ then, since, for all $n \in \mathbb{N}$, $0 \leq x_n < e$ the squeezing (or sandwich) rule implies that $0 \leq x \leq e$. But $x \in \mathbb{Q}$ and $e \notin \mathbb{Q}$. Therefore $0 \leq x < e$, that is, $x \in A$. Therefore $A$ is a closed subset of $(\mathbb{Q}, d)$.
Notice that $A$ is not a closed subset of $\mathbb{E}_1$.
For all $n \in \mathbb{N}$, let $e_n = (1 + 1/n)^n$. Then $(e_n)$ is a sequence in $A$. If $(e_n)$ is treated as a sequence in $\mathbb{E}_1$ then $\lim_n (e_n) = e$. Therefore every subsequence of $(e_n)$ converges to $e$. Therefore, since $e \notin A$, no subsequence of $(e_n)$ converges in $A$. Therefore $A$ is not a compact subset of $(\mathbb{Q}, d)$.

3.17 (i) Suppose that $M$ is disconnected. Then $M$ has a proper non-empty subset that is both an open subset of $M$ and a closed subset of $M$. Suppose that
$$\text{Bd}(A) \neq \emptyset$$

Let $x \in \text{Bd}(A)$. If $x \in A$ the $A$ is not open: if $x \notin A$ then $A$ is not closed. Therefore (1) must be false, that is, $\text{Bd}(A) = \emptyset$.

(ii) Suppose that $A$ is a proper non-empty subset of $M$ and that $\text{Bd}(A) = \emptyset$. Then
$$\text{Cl}(A) = A \cup \text{Bd}(A) = A \cup \emptyset = A$$
and $\text{Int}(A) = A \setminus \text{Bd}(A) = A \setminus \emptyset = a$.

Therefore $A$ is both open and closed. Therefore $M$ is disconnected.

3.18 Suppose that $A \subset M$. Then $x \in \text{Bd}(A)$ iff every open ball centred at $x$ contains a point of $A$ and a point of $M \setminus A$. It is easy to see that this implies that $\text{Bd}(A) = \text{Bd}(M \setminus A)$. Therefore
$$\text{Cl}(A) \cap \text{Cl}(M \setminus A) = [A \cup \text{Bd}(A)] \cap [(M \setminus A) \cup \text{Bd}(M \setminus A)] = [A \cup \text{Bd}(A)] \cap [(M \setminus A) \cup \text{Bd}(A)] = \text{Bd}(A)$$
since $A \cap (M \setminus A) = \emptyset$. Therefore
$$\text{Cl}(A) \cap \text{Cl}(M \setminus A) = \emptyset \text{ iff } \text{Bd}(A) = \emptyset.$$ 

The proof is completed by using the proposition of Problem 3.17.