Solutions 2

2.1 We must prove that each of the following statements is true for all \( f, g \in C[a, b] \) and \( \lambda \in \mathbb{R} \):

- **N1** \( \| \lambda f \|_{\infty} = |\lambda| \| f \|_{\infty} \);
- **N2** \( \| f \|_{\infty} \geq 0 \);
- **N3** \( \| f \|_{\infty} = 0 \iff f = 0 \);
- **N4** \( \| f + g \|_{\infty} \leq \| f \|_{\infty} + \| g \|_{\infty} \).

The equation \( f = 0 \) means that \( f : [a, b] \to \mathbb{R} : x \mapsto 0 \), that is, that \( f \) is the zero function.

It is easy to show that statements **N1–N3** are true. Let us consider **N4**.

There exist \( u, v, w \in [a, b] \) such that

\[
\begin{align*}
\| f \|_{\infty} &= \max \{|f(x)| : x \in [a, b]\}, \\
\| g \|_{\infty} &= \max \{|g(x)| : x \in [a, b]\}, \\
\| f + g \|_{\infty} &= \max \{|(f + g)(x)| : x \in [a, b]\}.
\end{align*}
\]

Then

\[
\| f \|_{\infty} + \| g \|_{\infty} = |f(u)| + |g(v)| \geq |f(w)| + |g(w)| \geq |f(w) + g(w)| = \| f + g \|_{\infty}
\]

Therefore **N4** is true.

2.2 Suppose that \( \epsilon > 0 \). Then there exist \( J(\epsilon), K(\epsilon), \) and \( L(\epsilon) \in \mathbb{N} \) such that, for all \( n \in \mathbb{N} \),

\[
\begin{align*}
n > J(\epsilon) &\implies |b_n - b| < \frac{\epsilon}{\sqrt{3}} \quad (1) \\
n > K(\epsilon) &\implies |c_n - c| < \frac{\epsilon}{\sqrt{3}} \quad (2) \\
n > L(\epsilon) &\implies |d_n - d| < \frac{\epsilon}{\sqrt{3}} \quad (3)
\end{align*}
\]

Let \( N(\epsilon) = \max\{J(\epsilon), K(\epsilon), L(\epsilon)\} \). Then, for all \( n \in \mathbb{N} \), if \( n \geq N(\epsilon) \) then \( (1), (2), \) and \( (3) \) imply that

\[
\begin{align*}
\| a_n - a \|^2 &= \|(b_n, c_n, d_n) - (b, c, d)\|^2 \\
&= (b_n - b)^2 + (c_n - c)^2 + (d_n - d)^2 \\
&< \frac{\epsilon^2}{3} + \frac{\epsilon^2}{3} + \frac{\epsilon^2}{3} = \epsilon^2
\end{align*}
\]

Therefore \( \forall n \in \mathbb{N}, n \geq N(\epsilon) \implies \| a_n - a \| < \epsilon \).

Therefore \( \lim_{n} (a_n) = a \). \( \Box \)

2.3 Suppose that \( x, y, z \in M \). It is easy to see that, since \( d \) is a metric for \( M, e(x, y) \geq 0 \), that \( e(x, y) = 0 \iff x = y \), and that \( e(x, y) = e(y, x) \). To show that \( (M, e) \) is a metric space we must show that \( e(x, y) \leq e(x, z) + e(z, y) \), that is, that \( e(x, z) + e(z, y) - e(x, y) \geq 0 \).

Let \( a = d(x, z), b = d(z, y), \) and \( c = d(x, y) \). Then, since \( d \) is a metric for \( M, \)

\[
a, b, c \geq 0 \text{ and } a + b - c = d(x, z) + d(z, y) - d(x, y) \geq 0
\]

Please turn over.
Recall that

\[ e(x, z) + e(z, y) - e(x, y) = \frac{a}{1 + a} + \frac{b}{1 + b} - \frac{c}{1 + c} \]

\[ = \frac{a(1 + b)(1 + c) + b(1 + a)(1 + c) - c(1 + a)(1 + b)}{(1 + a)(1 + b)(1 + c)} \]

\[ = \frac{a + b - c + 2ab + abc}{(1 + a)(1 + b)(1 + c)} \] \hspace{1cm} (2)

Statements (1) and (2) imply that \( e(x, z) + e(z, y) - e(x, y) \geq 0 \).

2.4 \( B(0;1/4) \) appears in \( E_2 \) as an open disc of radius 1/3 centred at 0.
\( B(0;1/2) \) appears in \( E_2 \) as an open disc of radius 1 centred at 0.
\( B(0;3/4) \) appears in \( E_2 \) as an open disc of radius 3 centred at 0.
\( B(0;1) \) appears in \( E_2 \) as the entire real plane.

2.5 Recall that \((M,d)\) is a metric space iff, for all \( x, y, z \in M \),

\[ d(x, y) = d(y, x) \] \hspace{1cm} (M1)
\[ d(x, y) \geq 0 \] \hspace{1cm} (M2)
\[ d(x, y) = 0 \text{ iff } x = y \] \hspace{1cm} (M3)
\[ d(x, y) \leq d(x, z) + d(z, y) \] \hspace{1cm} (M4)

Let \((x_1, y_1), (x_2, y_2), (x_3, y_3) \in X \times Y \).

(i) It is easy to show that \( d_1 \) satisfies M1 and M2.

Since \( d(x_1, x_2) \) and \( d'(y_1, y_2) \geq 0 \)

\[ d_1((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d'(y_1, y_2) = 0 \]
\[ \Downarrow \]
\[ d(x_1, x_2) = 0 \text{ and } d'(y_1, y_2) = 0 \]
\[ \Downarrow \]
\[ x_1 = x_2 \text{ and } y_1 = y_2 \]
\[ \Downarrow \]
\[ (x_1, y_1) = (x_2, y_2) \]

Therefore \( d_1 \) satisfies M3.

\[ d_1((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) + d'(y_1, y_2) \]
\[ \leq d(x_1, x_3) + d(x_3, x_2) + d'(y_1, y_3) + d'(y_3, y_2) \]
\[ = d(x_1, x_3) + d'(y_1, y_3) + d(x_3, x_2) + d'(y_3, y_2) \]
\[ = d_1((x_1, y_1), (x_3, y_3)) + d_1((x_3, y_3), (x_2, y_2)) \]

Therefore \( d_1 \) satisfies M4. Therefore \( d_1 \) is a metric for \( X \times Y \).

(ii) It is easy to show that \( d_2 \) satisfies M1, M2, and M3. To show that \( d_2 \) satisfies M4 we observe that \( d_2((x_1, y_1), (x_2, y_2)) = \|x\| \) where \( x = (d(x_1, x_2), d'(y_1, y_2)) \in E_2 \).

Let \( a = (d(x_1, x_3), d'(y_1, y_3)) \) and \( b = (d(x_3, x_2), d'(y_3, y_2)) \in E_2 \). Then

\[ d_2((x_1, y_1), (x_3, y_3)) + d_2((x_3, y_3), (x_2, y_2)) = \|a\| + \|b\| \]
\[ \geq \|a + b\| \] \hspace{1cm} [Triangle law]
\[ = \sqrt{(d(x_1, x_3) + d(x_3, x_2))^2 + (d'(y_1, y_3) + d'(y_3, y_2))^2} \]
\[ \geq \sqrt{d(x_1, x_2)^2 + d'(y_1, y_2)^2} \] \hspace{1cm} [M4]
\[ = d_2((x_1, y_1), (x_2, y_2)) \]

Please turn over.
Therefore $d_2$ satisfies M4. Therefore $d_2$ is a metric for $X \times Y$.

(iii) It is easy to show that $d_\infty$ satisfies M1, M2, and M3.

Let $D = d_\infty((x_1, y_1), (x_3, y_3)) + d_\infty((x_3, y_3), (x_2, y_2))$. Then

$$D = \max\{d(x_1, x_3), d'(y_1, y_3)\} + \max\{d(x_3, x_2), d'(y_3, y_2)\}$$

Therefore $D \geq d(x_1, x_3) + d(x_3, x_2) \geq d(x_1, x_2)$ and $D \geq d'(y_1, y_3) + d'(y_3, y_2) \geq d'(y_1, y_2)$.

Therefore $D \geq \max\{d(x_1, x_2), d'(y_1, y_2)\} = d_\infty((x_1, y_1), (x_2, y_2))$. Therefore $d_\infty$ satisfies M4.

Therefore $d_\infty$ is a metric for $X \times Y$.

Please turn over.