Problems 5

5.1 Suppose that 3 paths in \( \mathbb{C} \), \( \alpha \), \( \beta \), and \( \gamma \), are defined as follows:

\[
\begin{align*}
\alpha : [0, 2] & \to \mathbb{C} : t \mapsto t + it^3; \\
\beta : [0, 1] & \to \mathbb{C} : t \mapsto 2t; \\
\gamma : [0, 1] & \to \mathbb{C} : t \mapsto 2 + 8it.
\end{align*}
\]

Make a rough sketch of \( \alpha^* \), \( \beta^* \), and \( \gamma^* \); show how each path is oriented.

Evaluate each of the following integrals:

\[
\begin{align*}
(i) & \int_{\alpha} \overline{z} \, dz, \\
(ii) & \int_{\beta} \overline{z} \, dz, \\
(iii) & \int_{\gamma} \overline{z} \, dz, \\
(iv) & \int_{\beta \ast \gamma} \overline{z} \, dz
\end{align*}
\]

5.2 Working directly from the definition of a complex path integral prove that if \( \gamma \) is a contour then, for all \( n \in \mathbb{N} \),

\[
\int_{\gamma} z^n \, dz = 0.
\]

Hence show that, for any polynomial function, \( p \),

\[
\int_{\gamma} p(z) \, dz = 0.
\]

[ In this problem you are asked to verify Cauchy’s Integral Theorem for polynomial functions. ]

5.3 Suppose that \( \alpha \), \( \beta \), and \( \gamma \) are defined as in Problem 5.1. Evaluate each of the following integrals:

\[
\begin{align*}
(i) & \int_{\alpha} z^3 \, dz, \\
(ii) & \int_{\beta} z^3 \, dz, \\
(iii) & \int_{\gamma} z^3 \, dz, \\
(iv) & \int_{\beta \ast \gamma} z^3 \, dz
\end{align*}
\]

5.4 For each of the following real-valued functions \( u \),

- Show that \( u \) is a harmonic function.
- Construct a function \( v : \mathbb{R}^2 \to \mathbb{R} \) such that if \( f = u + iv \) then \( f \) is holomorphic on \( \mathbb{C} \).
- Describe \( f \) by writing an expression in \( z \) for \( f(z) \).

\[
\begin{align*}
(i) & \quad u : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto x^3 - 3xy^2. \\
(ii) & \quad u : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto e^x \cos(y) + e^{-y} \cos(x).
\end{align*}
\]

Please turn over.
5.5 Evaluate each of the following integrals.

(i) \( \int_{\gamma} \frac{z \, dz}{z^2 - 2iz + 3} \) \quad \text{where} \ \gamma = \kappa(i; 1).

(ii) \( \int_{\gamma} \frac{z \, dz}{z^2 - 2iz + 3} \) \quad \text{where} \ \gamma = \kappa(0; 2).

(iii) \( \int_{\gamma} \frac{z \, dz}{z^2 - 2iz + 3} \) \quad \text{where} \ \gamma = \kappa(2i; 2).

(iv) \( \int_{\gamma} \frac{z \, dz}{z^2 - 2iz + 3} \) \quad \text{where} \ \gamma = \kappa(0; 4).

(v) \( \int_{\gamma} \frac{ze^{\pi z} \, dz}{(z - i)^3} \) \quad \text{where} \ \gamma = \kappa(0; 3).

(vi) \( \int_{\gamma} \frac{z \, dz}{(z - 2iz + 8)^2} \) \quad \text{where} \ \gamma = \kappa(0; 3).

Recall that \( \kappa(a; p) \) is the positively-oriented circular contour of radius \( p \) centred at \( a \).