Third Arts [B.Sc. (Economics and Finance) III]  

Problems 1

If \( f : \mathbb{R} \to \mathbb{R} \) then the statement “\( f \) is continuous at \( a \)” can be defined in terms of convergent sequences. Use this form of the definition to solve Problems 1, 2, and 3.

1.1 Suppose that \( f \) and \( g : \mathbb{R} \to \mathbb{R} \). Then we define

\[
f g = f \cdot g : \text{dom}(f) \cap \text{dom}(g) \to \mathbb{R} : x \mapsto f(x)g(x).
\]

Prove that if \( f \) and \( g \) are both continuous at \( a \) then \( f g \) is continuous at \( a \).

1.2 Suppose that \( f \) and \( g : \mathbb{R} \to \mathbb{R} \) and that

\[
A = \{ x \in \text{dom}(f) \cap \text{dom}(g) \mid g(x) \neq 0 \}.
\]

Then we define \( f / g : A \to \mathbb{R} : x \mapsto f(x) / g(x) \).

Prove that if \( f \) and \( g \) are both continuous at \( a \) and \( g(a) \neq 0 \) then \( f / g \) is continuous at \( a \).

1.3 Suppose that \( f \) and \( g : \mathbb{R} \to \mathbb{R} \) are both continuous at \( a \). Working directly from Cauchy’s \((\epsilon, \delta)\)-definition of continuity, prove that \( f + g \) is continuous at \( a \).

1.4 For all \( x \in \mathbb{R} \), we define \( \lfloor x \rfloor \), the floor of \( x \) to be the largest integer that is less than or equal to \( x \).

For example, \( \lfloor 3 \rfloor = 3 \), \( \lfloor 3.7 \rfloor = 3 \), and \( \lfloor -2.5 \rfloor = -3 \).

Suppose that \( f : \mathbb{R} \to \mathbb{R} : x \mapsto \lfloor x \rfloor \).

(i) Draw the graph of \( f \) on the interval \([−3, 3]\).

(ii) Decide what are the points of discontinuity of \( f : \mathbb{R} \to \mathbb{R} \).

(iii) Justify your decision in (ii) for one of the points of discontinuity.

1.5 Construct a function \( f : [0, 2] \to \mathbb{R} \) that has all the following properties.

(i) \( \min \{ f(x) \mid x \in [0, 2] \} = 0 \);

(ii) \( \max \{ f(x) \mid x \in [0, 2] \} = 1 \);

(iii) for all \( 0 < \gamma < 1 \), there exists \( c \in [0, 2] \) such that \( f(c) = \gamma \);

(iv) \( f \) is not continuous at \( 1 \).

Hint: Begin by drawing the graph of \( f \).

\*Some authors call the floor of \( x \) the integer part of \( x \) and denote it by \( \lfloor x \rfloor \).

Please turn over.
1.6 Suppose that 

\[ f : [-1, 1] \to \mathbb{R} : x \mapsto \begin{cases} 
\frac{1}{x} & \text{if } x \neq 0 \\
0 & \text{if } x = 0.
\end{cases} \]

Prove that 

(i) \( f \) is neither bounded above nor bounded below on \([-1, 1]\).

(ii) \( f \) is not continuous at 0.

**Hint:** Begin by drawing the graph of \( f \).

1.7 Prove that the function \( \cos : \mathbb{R} \to \mathbb{R} \) has a unique fixed point.

1.8 Suppose that \( f(x) = \tan(\sin(x)) \). Use the iteration sequence for \( f \) to find two non-zero numbers that are fixed points of \( f \) correct to 6 decimal places.

[ You will need to use a calculator for this problem. ]