19 Higher-Order Derivatives

19.1 Lemma
Suppose that \( \gamma \) is a path in \( \mathbb{C} \) and that \( f \) is holomorphic on \( \gamma' \), that is, that \( f \) is holomorphic on an open superset of \( \gamma' \). Then there exists \( M \in \mathbb{R} \) such that,

\[
|f(z)| < M.
\]
for all \( z \in \gamma' \). \( f \) is holomorphic on \( \Gamma' \). Therefore, by CIF,

\[
\lim_{h \to 0} \int_{\gamma} \frac{hf(z)\,dz}{(z-a)^2} = 0.
\]

Proof (in outline)
Let \( f = u + iv \). Since \( f \) is differentiable on \( \gamma' \), it is continuous on \( \gamma' \) and therefore \( u \) and \( v \) are continuous real-valued functions on the compact set \( \gamma' \subset \mathbb{E} \). Therefore Corollary 9.23 implies that \( u \) and \( v \) attain maximum and minimum values on \( \gamma' \) and thus both of them are bounded on \( \gamma' \). Therefore \( |f| = |u + iv| \) is bounded on \( \gamma' \), that is, there exists \( M \in \mathbb{R} \) such that,

\[
|f(z)| < M.
\]
for all \( z \in \gamma' \). \( f \) is holomorphic on \( \Gamma' \). Therefore, by CIF,

\[
\lim_{h \to 0} \int_{\gamma} \frac{hf(z)\,dz}{(z-a)^2} = 0.
\]

19.2 Lemma
Suppose that \( a \in \mathbb{C} \), that \( \rho > 0 \), and that \( \gamma = \kappa(a; \rho) \). Suppose also that \( f \) is holomorphic on \( \Gamma' (\gamma) \). Then

\[
\lim_{h \to 0} \int_{\gamma} \frac{hf(z)\,dz}{(z-a)^2(z-a-h)} = 0.
\]

Proof (in outline)
Lemma 19.1 implies that there exists \( M \in \mathbb{R} \) such that,

\[
|f(z)| < M.
\]
for all \( z \in \gamma' \). \( f \) is holomorphic on \( \Gamma' \). Therefore, by CIF,

\[
\lim_{h \to 0} \int_{\gamma} \frac{hf(z)\,dz}{(z-a)^2(z-a-h)} = 0.
\]

Since we are trying to evaluate a limit as \( h \to 0 \), we can assume that \( |h| < \rho/2 \).

19.3 Theorem (CFI-1)
Suppose that \( \gamma \) is a contour in \( \mathbb{C} \) and that \( f \) is holomorphic on \( \Gamma' (\gamma) \). Then, for all \( a \in \Gamma (\gamma) \),

\[
f'(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)\,dz}{z-a}.
\]

Proof (in outline)
Let \( a \in \Gamma (\gamma) \). Since \( \Gamma (\gamma) \) is open there exists \( h \in \mathbb{C} \) such that \( h \neq 0 \) and \( a + h \in \Gamma (\gamma) \). Therefore, by CIF,

\[
f(a + h) - f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)\,dz}{z-a} - \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)\,dz}{z-a-h} = \frac{h}{2\pi i} \int_{\gamma} \frac{f(z)\,dz}{(z-a)(z-a-h)}.
\]

Therefore

\[
f(a + h) - f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)\,dz}{(z-a)^2} + \frac{g(h)}{2\pi i}.
\]

19.4 Problem
Evaluate the following integral where \( \gamma = \kappa(0; 2) \):

\[
\int_{\gamma} \frac{z\exp(\pi z)}{(z-i)^2} \,dz
\]

where

\[
g(h) = \int_{\gamma} \left[ \frac{f(z)}{(z-a)(z-a-h)} - \frac{f(z)}{(z-a)^2} \right] \,dz = \int_{\gamma} \frac{hf(z)\,dz}{(z-a)^2(z-a-h)}.
\]

Since \( \Gamma (\gamma) \) is open, there exists \( \rho > 0 \) such that

\[
\gamma' = \kappa(a; \rho) \subset \Gamma (\gamma).
\]

By Theorem 16.22, if \( |h| < \rho \) then

\[
\int_{\gamma} \frac{hf(z)\,dz}{(z-a)^2(z-a-h)} = \int_{\gamma} \frac{hf(z)\,dz}{(z-a)^2}.
\]

Lemma 19.2 and equations (1), (2), and (3) imply that

\[
\lim_{h \to 0} \frac{f(a + h) - f(a)}{h} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)\,dz}{(z-a)^2},
\]

that is,

\[
f'(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)\,dz}{(z-a)^2}.
\]
Suppose that CIF-2 is a remarkable theorem. CIF-1 gives us a formula

\[ f'(z) = e^{z^2} + z \pi e^{\pi z} = e^{z^2}(1 + \pi z). \]

Therefore, by CIF,

\[ \oint_{\gamma} z e^{z^2} \, dz = 2\pi i f'(i) = 2\pi i e^{\pi}(1 + \pi i). \]

19.5 Suppose that \( \gamma \) is a contour in \( \mathbb{C} \) and that \( f \) is
holomorphic on \( \Gamma(\gamma) \). Suppose also that \( a \in \Gamma(\gamma) \) and that \( h \in \mathbb{C}^* \) is such that \( a + h \in \Gamma(\gamma) \). Then

\[
\frac{f(a + h) - f(a)}{h} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z) - f(a)}{(z - a)^2} \, dz
\]

where

\[
g(h) = \int_{\gamma} \frac{2h}{(z - a)^3} \, dz.
\]

By using methods similar to, but more complicated than, those
used in the proof of Lemma 19.2, we can show that

\[ \lim_{h \to 0} g(h) = 0 \]

and thus prove the following theorem.

19.6 Theorem (CIF-2)

Suppose that \( \gamma \) is a contour in \( \mathbb{C} \) and that \( f \) is holomorphic on \( \Gamma(\gamma) \). Then \( f' \) is differentiable on \( \Gamma(\gamma) \) and, for all \( a \in \Gamma(\gamma) \),

\[ f''(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{2f(z) \, dz}{(z - a)^3}. \]

19.7 CIF-2 is a remarkable theorem. CIF-1 gives us a formula
for the derivative of a holomorphic function, that is, a function
already known to be differentiable. CIF-2, however, states that
if \( f \) is holomorphic at \( a \) then its derivative \( f' \) is differentiable at \( a \). In other words, since any point in a region \( U \) can be
strictly encircled by a contour in \( U \), CIF-2 implies that if a
complex function is differentiable on a region \( U \) then it is
twice differentiable on \( U \).

19.8 Suppose that \( f \) is holomorphic on a region \( U \). Then, by
CIF-2, \( f' \) is holomorphic on \( U \). Therefore, by CIF-2 again,
\( f'' = (f')' \) is holomorphic on \( U \). Therefore, by CIF-2 yet
again, \( f^{(3)} = f''' = (f'')' \) is holomorphic on \( U \).

19.9 Lemma

Suppose that \( U \) is a region of \( \mathbb{C} \) and that \( f \) is holomorphic on
\( U \). Then, for all \( n \in \mathbb{N} \), \( f \) is \( n \)-times differentiable on \( U \).

Proof Since, for all \( n \in \mathbb{N} \) and all \( z \in U \),
\( f^{(n+1)}(z) = (f^{(n)})'(z) \), the lemma is easily proved by induction
on \( n \).

19.10 Let us list the equations of CIF-1, and CIF-2 in order.

\[ f(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{f(z) \, dz}{z - a}. \]  

19.11 Theorem CIF-2 not only tells us that \( f' \) is differentiable
but also gives us a formula for \( f''(a) \). We have seen that if \( f \) is
holomorphic on a region \( A \) then \( f \) is infinitely differentiable on
\( A \), that is, \( f \) has derivatives of every order on \( A \). Is there
a formula for \( f^{(n)}(a) \), the \( n \)-th order derivative of \( f \) at \( a \), for
every \( n \in \mathbb{N} \)?

19.12 (CIF-n)

Suppose that \( \gamma \) is a contour in \( \mathbb{C} \) and that \( f \) is holomorphic on
\( \Gamma(\gamma) \). Then \( f \) is infinitely differentiable on \( \Gamma(\gamma) \) and, for all
\( a \in \Gamma(\gamma) \) and \( n \in \mathbb{N} \),

\[ f^{(n)}(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{n! f(z) \, dz}{(z - a)^{n+1}}. \]

19.13 There is a proof of CIF-n that is similar to the proofs
of CIF-1 and CIF-2. But this proof is long and tedious. There is
a more elegant proof but it belongs to a more advanced course
than this one. We shall not prove this theorem but we shall
show how its formula can be proved by induction on \( n \) if we
accept differentiation under the integral sign as a valid
operation.

19.14 CIF-1 tells that, under the conditions of CIF-n, the formula,

\[ f^{(n)}(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{n! f(z) \, dz}{(z - a)^{n+1}}, \]

is true when \( n = 1 \). We shall take (\ast) to be our inductive
hypothesis. We must deduce that

\[ f^{(n+1)}(a) = \frac{1}{2\pi i} \oint_{\gamma} \frac{(n + 1)! f(z) \, dz}{(z - a)^{n+2}}. \]

Using differentiation under the integral sign, we see that

\[ f^{(n+1)}(a) = \frac{df^{(n)}(a)}{da}. \]
Theorem (17.5),

\[
\int \frac{n! f(z)}{(z-a)^{n+1}} \, dz
\]

Therefore

\[
\int \frac{n! f(z) \partial}{\partial a} (z-a)^{-n-1} \, dz
\]

Therefore

\[
\int \frac{n! f(z) (-n+1)(z-a)^{-n-2} (-1)}{\partial a} \, dz
\]

Therefore

\[
\frac{1}{2\pi i} \int \left( \frac{n+1}{\partial a} f(z) \right) \, dz
\]

This completes the "proof".

19.15 Example
Evaluate the integral

\[
\int_{\gamma} \frac{z^5}{(z-2)^4} \, dz
\]

where \( \gamma = \kappa(1;2) \).

Let \( f(z) = z^5 \). Since \( 2 \in \mathbb{H}(\gamma) \) and \( f \) is holomorphic on \( \mathbb{C} \), we can use CIF-3 to evaluate this integral.

\[
f'(z) = 5z^4; \quad f''(z) = 20z^3; \quad f'''(z) = 60z^2.
\]

By CIF-3,

\[
f'''(2) = \frac{1}{2\pi i} \int_{\gamma} \frac{3! f(z) \, dz}{(z-2)^4}.
\]

Therefore

\[
\int_{\gamma} \frac{z^5}{(z-2)^4} \, dz = \frac{2\pi i}{3!} f'''(2) = \frac{2\pi i}{6} \cdot 60 \cdot 2^2 = 80\pi i.
\]

19.16 Theorem
Suppose that \( \alpha, \beta, \) and \( \gamma \) are three positively oriented contours in \( \mathbb{C} \) such that

\[
\mathbb{I}'(\alpha) \cap \mathbb{I}'(\beta) = \emptyset, \quad \mathbb{I}'(\alpha) \subset \mathbb{I}(\gamma), \quad \text{and} \quad \mathbb{I}'(\beta) \subset \mathbb{I}(\gamma).
\]

Suppose also that \( f : \mathbb{C} \to \mathbb{C} \) is holomorphic on that part of \( \mathbb{C} \) that lies inside \( \gamma \) and outside both \( \alpha \) and \( \beta \), where inside and outside are not interpreted strictly, that is, that \( f \) is holomorphic on

\[
\mathbb{I}''(\gamma) \setminus (\mathbb{I}(\alpha) \cup \mathbb{I}(\beta)).
\]

Then

\[
\int_{\gamma} f = \int_{\alpha} f + \int_{\beta} f.
\]

Proof (in outline) Construct two paths \( \sigma \) and \( \tau \) connection \( \alpha \) and \( \beta \), respectively, to \( \gamma \) as shown below.

Now let \( \gamma = \gamma_1 \ast \gamma_2 \) as shown below.

Let

\[
\Gamma = \gamma_1 \ast \sigma \ast \sigma^{-1} \ast \gamma_2 \ast \tau \ast \beta \ast \tau^{-1}.
\]

[ Notice that as we move along \( \Gamma \) the area of the figure on our left-hand side is shaded. In other words the area inside \( \Gamma \) is shaded. ]

Since the \( f \) is holomorphic on \( \mathbb{I}'(\Gamma) \), by Cauchy’s Integral Theorem (17.5),

\[
\int_{\Gamma} F = 0.
\]

Therefore

\[
(\int_{\gamma} + \int_{\sigma} - \int_{\gamma_1} - \int_{\sigma^{-1}} + \int_{\gamma_2} - \int_{\tau} - \int_{\beta} - \int_{\tau^{-1}}) f = 0.
\]

Therefore

\[
\int_{\gamma} f + \int_{\sigma} f - \int_{\gamma_1} f - \int_{\sigma^{-1}} f = 0.
\]

Therefore

\[
\int_{\gamma} f = \int_{\sigma} f + \int_{\beta} f. \quad \Box
\]

19.17 Problem
Evaluate the following integral

\[
\int_{\gamma} \frac{z^5}{(z-2)^4} \, dz
\]

where \( \gamma \) is a positively-oriented contour that strictly encloses both 2 and -2.
In Theorem 19.16 we put two separated contours inside a contour. In a stronger version of this theorem we can put any finite number of separated contours inside a contour and obtain a similar result.

**19.19 Theorem**

Suppose that \( \gamma \) is a positively oriented contour, that \( n \in \mathbb{N} \), and that, for all \( f = 1, 2, \ldots, n \), there is a positively oriented contour \( \gamma_f \) such that

(i) \( \gamma_f (\gamma) \subset \gamma \).

(ii) if \( i \neq j \) then \( \gamma_i (\gamma) \cap \gamma_j (\gamma) = \emptyset \).

Then \( \alpha, \sigma, \tau \) are separated positively-oriented contours that lie strictly inside \( \gamma \).

\[ \int_\gamma f(z) dz = \int_\alpha f(z) dz + \int_\sigma f(z) dz + \int_\tau f(z) dz = 2\pi i \sum_{i=1}^n g_i(0) = 2\pi i. \quad \square \]

**19.20 Problem**

Evaluate the following integral

\[ \int_\gamma \frac{(z - 1)dz}{z^2 + z^2} \]

where \( \gamma = \kappa(0;2) \), the circular contour of radius 2, described positively and centred at the origin.

Let \( f(z) = \frac{z - 1}{z^2 + z^2} \). Then

\[ f(z) = \frac{z - 1}{z^2 + z^2} = \frac{g_1(z)}{z^2} = \frac{g_2(z)}{z + 1} = \frac{g_3(z)}{z - 1} \]

where

\[ g_1(z) = \frac{z - 1}{z^2 + 1}, \quad g_2(z) = \frac{z - 1}{z(z - i)}, \quad \text{and} \quad g_3(z) = \frac{z - 1}{z(z + i)}. \]

Let \( \alpha = \kappa(0;0.4), \quad \sigma = \kappa(-i;0.4), \quad \text{and} \quad \tau = \kappa(i;0.4) \).

Then \( \alpha, \sigma, \tau \) are separated positively-oriented contours that lie strictly inside \( \gamma \).

Since we are going to use CIF-1, we must differentiate \( g_1 \).

\[ g_1'(z) = \frac{z^2 + 1 - (z - 1)2z}{(z^2 + 1)^2} = \frac{-z^2 - 2z + 1}{(z^2 + 1)^2} \]

Since \( g_1 \) is holomorphic on \( \Gamma^*(\alpha) \), by CIF-1,

\[ \int_\alpha f(z) dz = \int_\alpha \frac{g_1(z)}{z} dz = 2\pi i g_1(0) = 2\pi i. \quad (1) \]
By CIF,
\[ \int_\sigma f(z) \, dz = \int_\sigma \frac{g_2(z) \, dz}{z - (-i)} = 2\pi i g_2(-i) \]
\[ = 2\pi i \left( \frac{-i - 1}{-(-1 - i)} \right) = -\pi(1 + i) \] (2)

and
\[ \int_\tau f(z) \, dz = \int_\tau \frac{g_3(z) \, dz}{z - i} = 2\pi i g_3(i) \]
\[ = 2\pi i \left( \frac{1 - 1}{1 + i} \right) = \pi(1 - i). \] (3)

Therefore, by Theorem 19.19 and equations (1), (2), and (3),
\[ \int_\gamma \frac{(z - 1) \, dz}{z^4 + z^2} = \int_\gamma f(z) \, dz \]
\[ = \int_\alpha f(z) \, dz + \int_\sigma f(z) \, dz + \int_\tau f(z) \, dz \]
\[ = 2\pi i - \pi - \pi + \pi - \pi i \]
\[ = 0. \]