14 The Logarithm

14.1 In Chapter 12, we used a power series to define the (complex) exponential function, \( \exp : \mathbb{C} \to \mathbb{C} \):
\[
\exp(z) = e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots.
\]

The complex exponential function has most of the properties of the real exponential function.

14.2 Theorem
(i) For all \( z \in \mathbb{C} \), \( \exp'(z) = d(e^z)/dz = e^z \).
(ii) \( e^0 = 1 \).
(iii) For all \( w \) and \( z \in \mathbb{C} \), \( e^{w+z} = e^w e^z \).
(iv) For all \( z \in \mathbb{C} \), \( e^z \neq 0 \).

\[\text{Proof} \]
(i) We can use term-by-term differentiation to differentiate a function represented by a power series. Therefore, for all \( z \in \mathbb{C} \),
\[
\exp'(z) = \frac{d(e^z)}{dz} = \sum_{n=0}^{\infty} \frac{d}{dz} \left( \frac{z^n}{n!} \right)
= \sum_{n=1}^{\infty} \frac{n}{n!} z^{n-1} = \sum_{n=0}^{\infty} \frac{e^{n-1}}{(n-1)!}
= \sum_{k=0}^{\infty} \frac{z^k}{k!} \quad \text{(where } k = n - 1 \text{)}
\]

14.3 The (real) logarithm function \( \log : \mathbb{R} \to (0, -\infty) \). Can we define a complex logarithm function as the inverse of the complex exponential function? We shall use “\( \ln \)” to denote the real logarithm function and, if the answer to our question is ‘Yes’, we shall use “\( \log \)” for the complex logarithm function.

14.4 Suppose that \( z = x + iy \) and \( w = u + iv \in \mathbb{C} \) are such that
\[z = e^w. \quad (1)\]

We want to know whether or not equation (1) has a unique solution for \( w \) in terms of \( z \).
\[z = e^w \iff x + iy = e^{u+iv} \iff x + iy = e^u(\cos(v) + i\sin(v)). \quad (2)\]

For all \( u \) and \( v \in \mathbb{R} \),
\[|\cos(v) + i\sin(v)| = \sqrt{\cos^2(v) + \sin^2(v)} = 1 \]
and \( e^u > 0 \). Therefore equation (2) implies that
\[|z| = |x + iy| = |e^u||\cos(v) + i\sin(v)| = e^u. \]
Therefore, if \(|z| \neq 0\), that is, if \( z \neq 0 \), then
\[u = \ln(|z|). \quad (3)\]
and equation (2) can be rewritten as
\[z = |z|(\cos(v) + i\sin(v)). \quad (4)\]

14.4.3 Equation (4) implies that \( v = \text{Arg}(z) \). For any \( z \in \mathbb{C} \), any two values of \( \text{Arg}(z) \) differ only by an integer multiple of \( 2\pi \). Therefore we can define a multifunction \( \text{Log} \) on \( \mathbb{C}^* \) by
\[\text{Log}(z) = \ln|z| + i\text{Arg}(z) \pmod{2\pi i}, \quad (5)\]
where \( \mathbb{C}^* \) is the set of all non-zero complex numbers.

We have not found a function \( \log \) that is the inverse of \( \exp \) but we have found a multifunction \( \text{Log} \) such that,

\[\text{for all } z \in \mathbb{C}^*, \text{exp(Log}(z)) = z. \]

14.5 Suppose that \( z = x + iy \in \mathbb{C} \). Then
\[\exp(z) = \exp(x + iy) = e^x(\cos(y) + i\sin(y)). \]
Therefore \[|\exp(z)| = e^x \]
and \[\text{Arg}(\exp(z)) = y \pmod{2\pi}. \]
Therefore \[\text{Log}(\exp(z)) = \ln|\exp(z)| + i\text{Arg}(z) \pmod{2\pi} \]
\[= \ln(e^x) + iy \pmod{2\pi} \]
\[= x + iy \pmod{2\pi}. \]
Therefore

\[
\text{for all } z \in \mathbb{C}, \text{Log}(\exp(z)) = z \pmod{2\pi i}.
\]

**14.6** We can define a function \( \log : \mathbb{C}^* \to \mathbb{C} \) if we replace \( \text{Arg} \) in the definition of \( \text{Log} \) by its principal value \( \arg \). In other words

\[
\log : \mathbb{C}^* \to \mathbb{C} : z \mapsto \ln|\text{Re}(z)| + i\arg(z)
\]

where \(-\pi < \arg(z) \leq \pi\).

**14.7** Notice that \( \ln(-1) \) is not defined but

\[
\log(-1) = \ln|-1| + i\arg(-1) = \ln(1) + i\pi = i\pi.
\]