8 The Distance from a Point to a Set

8.1 Recall that the Supremum Axiom, also known as the Axiom of Completeness, states that any non-empty set of real numbers that is bounded above has a least upper bound or supremum. A corollary of this axiom states that any non-empty set of real numbers that is bounded below has a greatest lower bound or infimum.

8.2 Definition
Suppose that \( \emptyset \neq A \subseteq M \) and that \( x \in M \). Then the set of all distances from \( x \) to a point in \( A \) is bounded below by 0.
Therefore we can define the distance of \( x \) from \( A \), \( d(x;A) \), to be the greatest lower bound of the set of distances from \( x \) to a point in \( A \). That is,
\[
d(x;A) = \inf \{ d(x,a) \mid a \in A \}.
\]

8.3 \( d(x;A) \), as defined above, is sometimes called the least distance of \( x \) from \( A \).

8.4 Lemma
(i) For all \( x, y \in M \), \( d(x, \{ y \}) = d(x,y) \).
(ii) If \( \emptyset \neq A \subseteq B \subseteq M \) then, for all \( x \in M \), \( d(x;A) \geq d(x;B) \).
(iii) If \( a \in A \subseteq M \) then \( d(a;A) = 0 \).
Proof Exercise.

8.5 Problem
Evaluate each of the following distances in \( E_1 \):
(i) \( d(0; [1, \rightarrow)) \) where \( [1, \rightarrow) = \{ x \in \mathbb{R} \mid x \geq 1 \} \).
(ii) \( d(0; (1, \rightarrow)) \) where \( (1, \rightarrow) = \{ x \in \mathbb{R} \mid x > 1 \} \).
(iii) \( d(\sqrt{2}; \mathbb{Q}) \).

\[
(i) d(0; [1, \rightarrow)) = 1.
\]

\[
(ii) d(0; (1, \rightarrow)) = 1. \quad \text{Clearly since 1 is a lower bound of (1, \rightarrow)}
\]
\[
\inf(1, \rightarrow) = 1.
\]

\[
(iii) d(\sqrt{2}; \mathbb{Q}) = 0.
\]

Although \( \sqrt{2} \) is not a rational number we can always find a rational number that is as close to \( \sqrt{2} \) as we wish.

8.6 Notice that \( d(0; [1, \rightarrow)) = d(0,1) = 1 \), that is, the distance from 0 to \( [1, \rightarrow) \) is the smallest value of \( d(0,x) \) where \( x \in [1, \rightarrow) \).

It is also true that \( d(0; (1, \rightarrow)) = 1 \) but there is no \( x \in (0, \rightarrow) \) such that \( d(0,x) = 1 \).

8.7 Theorem
Suppose that \( \emptyset \neq A \subseteq M \). Then, for all \( x \in M \), \( d(x;A) = 0 \) if and only if \( x \in \text{Cl}(A) \).
Proof

\[
(i) \text{ Suppose that } x \in \text{Cl}(A) \text{ that is, that } x \in A \text{ or } x \in \text{Bd}(A).
\]

Lemma 8.4(iii) implies that if \( x \in A \) then \( d(x;A) = 0 \). So we can take \( x \) to be a boundary point of \( A \).

Let \( \varepsilon > 0 \). Since \( x \) is a boundary point of \( A \), there exists \( y(\varepsilon) \in A \) such that \( y(\varepsilon) \in B(x;\varepsilon) \), that is, there exists \( y(\varepsilon) \in A \) such that \( d(x,y(\varepsilon)) < \varepsilon \). Therefore
\[
d(x;A) = \inf \{ d(x,z) \mid z \in A \} < \varepsilon \quad (1)
\]

Since (1) is true for all \( \varepsilon > 0 \), \( d(x;A) = 0 \).

(ii) Suppose that \( x \notin \text{Cl}(A) \).

8.8 Corollary
If \( A \) is a non-empty closed subset of \( M \) then, for all \( x \in M \), \( d(x;A) = 0 \) if and only if \( x \in A \).

8.9 Recall if \( A \) is a subset of \( M \) then
\[
\text{Bd}(A) = \text{Bd}(M \setminus A). \quad [5.37]
\]

Therefore
\[
\text{Cl}(A) \cap \text{Cl}(M \setminus A) = \text{Int}(A) \cup \text{Bd}(A) \cap \text{Int}(M \setminus A) \cup \text{Bd}(M \setminus A) = \text{Bd}(A).
\]

In view of Theorem 8.7, this yields the following theorem.

8.10 Theorem
Suppose that \( (M,d) \) is a metric space and that \( A \) is a subset of \( M \). Then \( x \in M \) is a boundary point of \( A \) if and only if
\[
d(x;A) = d(x;M \setminus A) = 0.
\]

8.11 This theorem gives us another definition of a boundary point.
Suppose that \((M, d)\) is a metric space and that \(A \subset M\). Then \(x \in M\) is a boundary point of \(A\) if and only if the distance from \(x\) to \(A\) and the distance from \(x\) to the complement of \(A\) are both equal to zero.