**C. Complex Numbers**

**C.1 Definition**
Suppose that $F$ is a non-empty set with two binary operations

- addition: $F \times F \to F : (x, y) \mapsto x + y$
- and multiplication: $F \times F \to F : (x, y) \mapsto x \cdot y$.

Then we say that $(F, +, \cdot)$ is a field, or that $F$ is a field under the operations of addition and multiplication, iff each of the following axioms is satisfied.

**F1** For all $x,y,z \in F$, $x + (y + z) = (x + y) + z$

**F2** For all $x,y \in F$, $x + y = y + x$

**F3** There exists $0 \in F$ such that, for all $x \in F$, $x + 0 = 0 + x = x$.

**F4** For all $x \in F$, there exists $-x \in F$ such that $x + (-x) = (-x) + x = 0$

**F5** For all $x,y,z \in F$, $x \cdot (y \cdot z) = (x \cdot y) \cdot z$

**F6** For all $x,y \in F$, $x \cdot y = y \cdot x$

**F7** There exists $1 \in F$ such that, for all $x \in F$, $1 \cdot x = x \cdot 1 = x$

**F8** For all $x \in F$, if $x \neq 0$ then there exists $x^{-1} \in F$ such that $x \cdot x^{-1} = x^{-1} \cdot x = 1$

**F9** For all $x,y,z \in F$, $x \cdot (y + z) = x \cdot y + x \cdot z$ and $(x+y) \cdot z = x \cdot z + y \cdot z$

Therefore, for all $z = (x,y) \in \mathbb{C}$, since we identify $(x,0) \in \mathbb{C}$ with $x \in \mathbb{R}$ we can write

$$z = (x,y) = (x,0) + (0,y) = (x,0) + y(0,1) = x + y(0,1)$$

Now let us denote $(0,1) \in \mathbb{C}$ by $i$. Then

For all $z = (x,y) \in \mathbb{C}$, $z = x + iy$.

Now

$$i^2 = (0,1)^2 = (0,1) \cdot (0,1)$$

$$= (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0)$$

$$= (-1,0) = -1$$

and

$$(-i)^2 = (0,-1)^2 = (0,-1) \cdot (0,-1)$$

$$= (0 \cdot 0 - (-1) \cdot (-1), 0 \cdot (-1) + (-1) \cdot 0)$$

$$= (-1,0) = -1$$

Therefore

$$(-i)^2 = -1$$

Does this pair of equations define $i$?

**C.7 Lemma**
For all $z \in \mathbb{C}$, $z^2 = -1$ $\iff$ $z = \pm i$. □

**C.8** We have thus shown that any complex number $(x,y)$ can be written uniquely as $x + iy$ or $x + iy$ where $i^2 = -1$. If we write complex numbers in this way then addition and multiplication in $\mathbb{C}$ are merely standard arithmetic—as in any field—with the additional formula, $i^2 = -1$.

For all $(u,v), (x,y) \in \mathbb{C}$,

$$(u,v) + (x,y) = (u + vi) + (x + iy)$$

$$= u + v + (v + y)i$$

$$= (u + x, v + y)$$

$$(u,v) \cdot (x,y) = (u + vi)(x + iy)$$

$$= ux + uyi + vix + vyi$$

$$= (ux + vy + i(vy + ux))$$

$$= (ux - vy, uy + vx)$$

From now on we shall usually write complex numbers in this form. Remember that for all $u,v,x,y \in \mathbb{R}$,

$$u + vi = x + iy \iff [u = x \text{ and } v = y]$$.

**C.9** We shall take a statement such as

Let $z = x + iy \in \mathbb{C}$.

to imply that both $x$ and $y \in \mathbb{R}$.

There are many ways of writing $z = x + iy$ where $x,y,z \in \mathbb{C}$: for example,

$$3 + 5i = 4i + (1 - i)i = 7 + (5 + 4i)i \ldots$$

**C.10 Definition**
For all $z = x + iy \in \mathbb{C}$, we call $x$ the real part of $z$ and $y$ the imaginary part of $z$. These names are not well-chosen but they...
are always used. We write
\[ x = \Re(z) \quad \text{and} \quad y = \Im(z). \]

[Some authors use Gothic letters: \( x = \Re z \) and \( y = \Im z \).]

C.11 Suppose that \( z \in \mathbb{C} \). If \( \Im(z) = 0 \) then \( z \in \mathbb{R} \); if \( \Re(z) = 0 \) then we say that \( z \) is a pure imaginary number.

C.12 Definition
For all \( z = x + iy \in \mathbb{C} \), we define the \textit{(complex) conjugate of} \( z \),
\( \bar{z} \in \mathbb{C} \), by
\[ \bar{z} = x - iy. \]

C.13 Examples
\[
\begin{array}{ccc}
  z & 5 + 3i & 4 - 7i \\
  \bar{z} & 5 - 3i & 4 + 7i \\
\end{array}
\]

C.14 Theorem
For all \( w, z \in \mathbb{C} \),
(i) \( \bar{\bar{z}} = z \).
(ii) \( w + \bar{z} = w + \bar{z} \).
(iii) \( \bar{w} \cdot \bar{z} = \bar{wz} \).
(iv) if \( z \neq 0 \) then \( \bar{z}^{-1} = (z)^{-1} \).
(v) for all \( m \in \mathbb{Z} \), \( (\bar{z})^m = \bar{z}^m \).
(vi) \( z = \bar{z} \) iff \( z \in \mathbb{R} \).
(vii) \( z = -\bar{z} \) iff \( z \) is pure imaginary.
(viii) \( \Re(z) = (z + \bar{z})/2. \)

C.19 For all \( z \in \mathbb{C} \), where \( z \neq 0 \), the multiplicative inverse of \( z \),
\( z^{-1} \), is easily formulated in terms of the complex conjugate and
the modulus:
\[ z^{-1} = \frac{\bar{z}}{|z|^2}. \]

C.20 Theorem (The Triangle Law)
For all \( w, z \in \mathbb{C} \), \( |w + z| \leq |w| + |z| \). \( \square \)

C.21 Since \( \mathbb{C} \), considered simply as a set, is identical to \( \mathbb{R}^2 \) we
can represent \( \mathbb{C} \) by a plane using a Cartesian coordinate system.
The complex number \( x + iy \) is then represented by the point in
the plane with coordinates \( (x, y) \).

When we use the plane to represent complex numbers in this
way we call it the \textit{complex plane} or the \textit{Argand diagram}. We
usually talk simply about

“the point \( z \)” or “the point \( x + iy \)”

instead of

“the point that represents \( z \)” or “the point whose
coordinates are \( (x, y) \)”.

C.22 In the complex plane the first axis represents the set of
real numbers and is usually called the \textit{real axis}; the second axis is
called the \textit{imaginary axis}.

C.23 For all \( z \in \mathbb{C} \), the point \( -z \) is the reflection of \( z \) in
the origin and the point \( \bar{z} \) is the reflection of \( z \) in the real axis.
The modulus of \( z \), \( |z| \), is the distance of the point \( z \) from the
origin.

\[
\begin{array}{c}
-\bar{z} = -x + iy \\
\bullet \\
|z| = \sqrt{x^2 + y^2} \\
\end{array}
\]

C.24 Suppose that \( w = u + vi \), \( z = x + iy \in \mathbb{C} \); then
\( w + z = (u + x) + (v + y) \). It is easy to show that the four points
\( 0, w, z, \) and \( w + z \) are the corners of a parallelogram.
If we write $w + z$,

- The length of the side $|0, w| = |w|$;
- the length of $|0, w + z| = |w + z|$;
- the length of $|w, w + z|$ is the length of $|0, z| = |z|$.

Since the length of any side of a triangle cannot be greater than the sum of the lengths of the other two sides we see that

$$|w + z| \leq |w| + |z|.$$  

It can now be seen why this inequality is called the triangle law.

**C.25** We recall some theorems about circular functions. For all $\theta, \phi \in \mathbb{R}$,

(i) $\cos^2 \theta + \sin^2 \theta = 1$.

(ii) $\cos(\theta + \phi) = \cos \theta \cos \phi - \sin \theta \sin \phi$.

(iii) $\sin(\theta + \phi) = \sin \theta \cos \phi + \cos \theta \sin \phi$.

If $\theta \in [0, 2\pi)$ then $\theta = 5\pi/4$ but if $\theta \in (-\pi, \pi]$ then $\theta = -3\pi/4$.

Notice that the difference between these two values of $\theta$ is equal to $2\pi$.

**C.28 Theorem**

For all $z \in \mathbb{C}$, there exists $\theta \in \mathbb{R}$ such that

$$z = |z|(\cos \theta + i\sin \theta).$$

**C.29** Suppose that $z \in \mathbb{C}$.

(i) If $z = |z|(\cos \theta + i\sin \theta)$ where $\theta \in \mathbb{R}$ then we write $\theta = \arg(z)$; arg is an abbreviation for argument.

(ii) Clearly, for a given $z \in \mathbb{C}$, where $z \neq 0$, arg($z$) can take more than one value. But C.25 (iv) implies that, for all $\theta, \phi \in \mathbb{R}$,

$$z = |z|(\cos \theta + i\sin \theta) = |z|(\cos \phi + i\sin \phi)$$

if and only if there exists $n \in \mathbb{Z}$ such that $\theta - \phi = 2n\pi$.

(iv) $|\cos \theta = \cos \phi$ and $\sin \theta = \sin \phi|$ if and only if there exists $n \in \mathbb{Z}$ such that $\theta - \phi = 2n\pi$.

**C.26** Let $S = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$; $S$ is the unit circle centred the origin. For any $(x, y) \in S$, there exists $\theta \in \mathbb{R}$ such that

$$(x, y) = (\cos \theta, \sin \theta). \quad (*)$$

Clearly, there is more than one value of $\theta$ that satisfies $(*)$. But if we choose a half-open interval $I \subset \mathbb{R}$ of length $2\pi$ then, for any $(x, y) \in S$, there is a unique $\theta \in I$ such that $(x, y) = (\cos \theta, \sin \theta)$.

Two common choices for $I$ are the intervals $[0, 2\pi)$ and $(-\pi, \pi]$.

**C.27 Example**

The point $(-1/\sqrt{2}, 1/\sqrt{2}) \in S$. Suppose that $\theta \in \mathbb{R}$ satisfies

$$\left( \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) = (\cos \theta, \sin \theta).$$

(iii) If two real numbers $\theta$ and $\phi$ differ only by the addition of an integer multiple of $2\pi$ then we say that they are equal to each other modulo $2\pi$ and we write

$$\theta = \phi \pmod{2\pi}.$$  

(iv) Although arg($z$) is not unique for a given value $z$ it is said to be unique modulo $2\pi$.

(v) Suppose that $I$ is a half-open interval $I$ of length $2\pi$.

Then, for $z \in \mathbb{C}$, where $z \neq 0$, there is a unique $\theta \in I$ such that $\theta = \arg(z)$. We can choose $I$ as our principal range for arg and we then call the (unique) value of arg($z$) that belongs to $I$ the principal value of arg($z$).

(vi) In practice, finding arg($z$) for a given value of $z \in \mathbb{C}$ is best done by first making a rough sketch of the point $z$ in the complex plane.

**C.30** (i) For all $x \in \mathbb{R}$, $\tan^{-1}(x)$ is defined by the following two statements

(a) $\tan(\tan^{-1}(x)) = x$;  
(b) $-\frac{\pi}{2} < \tan^{-1}(x) < \frac{\pi}{2}$.

Thus if $x + iy \in \mathbb{C}$ and $x \neq 0$ then $\tan(\arg(z)) = y/x$ but it is not always true that $\arg(z) = \tan^{-1}(y/x)$.

(ii) If we write $z = r(\cos \theta + i\sin \theta) \in \mathbb{C}$, without comment, then we shall assume that $\theta, r \in \mathbb{R}$ where $r \geq 0$.

(iii) We shall use $z = \text{cis}(\theta)$ as an abbreviation for

$$z = r(\cos \theta + i\sin \theta).$$
In the notation of Theorem C.31, \( w = a \text{cis}(\alpha) \) and \( z = b \text{cis}(\beta) \in \mathbb{C} \),
\[
wz = ab \text{cis}(\alpha + \beta). \quad \square
\]

C.32 Example

Let \( w = -1 + iv\sqrt{3} \) and \( z = i \).

\[
\begin{align*}
-1 + iv\sqrt{3} & \quad \text{\( \sqrt{3} \)} \\
& \quad \text{\( i \)} \\
-1 & \quad \text{\( 2\pi/3 \)}
\end{align*}
\]

Since \( \tan(\pi/3) = \sqrt{3} \), it is easy to see that the principal value of \( \arg(w) = \pi - \pi/3 = 2\pi/3; |w| = \sqrt{1 + 3} = 2 \). Therefore
\[
w = -1 + iv\sqrt{3} = 2\text{cis}(2\pi/3).
\]

Clearly
\[
z = i = \text{cis}(\pi/2).
\]

Therefore
\[
wz = (-1 + iv\sqrt{3})i = -v\sqrt{3} - i = 2\text{cis}(2\pi/3 + \pi/2) = 2\text{cis}(7\pi/6).
\]

C.33 In the notation of Theorem C.31, \( a = |w|, b = |z| \), and \( \alpha \) and \( \beta \) are values of \( \arg(w) \) and \( \arg(z) \), respectively. Since we already know that \( |wz| = |w||z| \), it is tempting to present the conclusion of C.31 as
\[
\arg(wz) = \arg(w) + \arg(z). \quad (\dagger)
\]

But (\dagger) is not always true even when \( \arg(w) \) and \( \arg(z) \) are principal values. For example, let \( w = -i \) and \( z = -1 \). Then, if we use principal values,
\[
\begin{align*}
\arg(w) & = \arg(-i) = 3\pi/2. \\
\arg(z) & = \arg(-1) = \pi. \\
\arg(wz) & = \arg(-i) = \pi/2.
\end{align*}
\]

Therefore \( \arg(wz) \neq \arg(w) + \arg(z) \).

It is, however, easy to prove the following corollary of C.31.

C.34 Corollary

For all \( w, z \in \mathbb{C} \), where \( wz \neq 0 \),
\[
\arg(wz) = \arg(w) + \arg(z) \quad (\mod 2\pi). \quad \square
\]

C.35 If possible, we can avoid using \( \arg() \) by using phrases such as “if \( z = r \text{cis}(\theta) \in \mathbb{C} \) then \ldots”.

C.36 Lemma

For all \( z = r \text{cis}(\theta) \) where \( z \neq 0 \),
\[
\frac{1}{z} = \frac{1}{r} \text{cis}(-\theta). \quad \square
\]

C.37 Corollary

For all \( w = a \text{cis}(\alpha) \) and \( z = b \text{cis}(\beta) \in \mathbb{C} \) where \( z \neq 0 \),
\[
\frac{w}{z} = \frac{a}{b} \text{cis}(\alpha - \beta). \quad \square
\]

C.38 Theorem (de Moivre’s Formula)

For all \( \theta \in \mathbb{R}, n \in \mathbb{Z} \),
\[
(\cos \theta + i\sin \theta)^n = \cos(n\theta) + i\sin(n\theta). \quad \square
\]

C.39 Example

Express \( \cos(3\theta) \) in terms of \( \cos(\theta) \) and \( \sin(3\theta) \) in terms of \( \sin(\theta) \).

We use the binomial theorem and De Moivre’s Theorem.
\[
\cos(3\theta) + i\sin(3\theta) = (\cos \theta + i\sin \theta)^3
\]
\[
= \cos^3 \theta + 3\cos^2 \theta(i\sin \theta)
\]
\[
+ 3\cos \theta(i\sin \theta)^2 + (i\sin \theta)^3
\]
\[
= \cos^3 \theta - 3\cos \theta \sin^2 \theta
\]
\[
+ i[3\cos^2 \theta \sin \theta - \sin^3 \theta] \quad (1)
\]

Equating the real parts of (1) yields
\[
\cos(3\theta) = \cos^3 \theta - 3\cos \theta \sin^2 \theta
\]
\[
= \cos^3 \theta - 3\cos \theta(1 - \cos^2 \theta)
\]
\[
= 4\cos^3 \theta - 3\cos \theta
\]

Equating the imaginary parts of (1) yields
\[
\sin(3\theta) = 3\cos^2 \sin \theta - \sin^3 \theta
\]
\[
= 3(1 - \sin^2 \theta) \sin \theta - \sin^3 \theta
\]
\[
= 3\sin \theta - 4\sin^3 \theta
\]