**B  Equivalence Relations**

**B.1** The terms set and member of a set are accepted as undefined.

**B.2** We take as known the notation and standard theorems of elementary set theory. For example, \( x \in A, A \subseteq B, \) \( A = \{ x : x \text{ has property } P \} \), and so on.

**B.3** We normally use upper-case Roman letters to denote sets and lower-case Roman letters to denote elements of sets. We use upper-case script letters to denote sets of sets. For example, we could write \( a \in B \in \mathcal{C} \cdot \{ a \} \subseteq B, \) and \( \{ B \} \subseteq \mathcal{C} \).

**B.4** We also take as known the usual sets of numbers: 
\[ \mathbb{N} \subseteq \mathbb{Z}^+ \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \]
where 
\( \mathbb{N} = \{ 1, 2, 3, 4, \ldots \}, \mathbb{Z}^+ = \{ 0, 1, 2, 3, \ldots \}, \) and \( \mathbb{Z}, \mathbb{Q}, \text{ and } \mathbb{R} \) denote the integers, the rational numbers, and the real numbers, respectively. We also take as known the arithmetical operations and the order properties of the real numbers.

**B.5** A set with precisely two distinct members \( \{ x, y \} \), say, is called a doubleton. An ordered pair \( (x, y) \) is a doubleton where the order in which its members are presented is significant. Thus \( \{ x, y \} = \{ y, x \} \) but \( (x, y) \neq (y, x) \) unless \( x = y \). We can define \( (x, y) \) in purely set-theoretic terms by 
\[ (x, y) = \{ x, y, \{ x \} \}. \]

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**B.12** As usual, if \( x \sim y \) means “\( x \) is related to \( y \)” then \( x \not\sim y \) means “\( x \) is not related to \( y \)”.

**B.13 Definition**

Suppose that \( \sim \) is a binary relation on \( A \).

(i) We say that \( \sim \) is reflexive iff for all \( x \in A \), \( x \sim x \).

(ii) We say that \( \sim \) is symmetric iff for all \( x, y \in A \), \( x \sim y \) implies that \( y \sim x \).

(iii) We say that \( \sim \) is transitive iff for all \( x, y, z \in A \), \( x \sim y \) and \( y \sim z \) implies that \( x \sim z \).

**B.14 Example**

In each of the following examples \( \sim \) denotes a binary relation on \( \mathbb{N} \).

(i) \( x \sim y \) iff \( x \leq y \).

Clearly, \( \sim \) is reflexive and transitive but not symmetric.

(ii) \( x \sim y \) iff \( |x - y| \leq 3 \).

Clearly, \( \sim \) is reflexive and symmetric. But \( \sim \) is not transitive: for example, \( 1 \sim 4 \) and \( 4 \sim 6 \) but \( 1 \not\sim 6 \).

(iii) \( x \sim y \) iff \( x \) and \( y \) are both even numbers.

It is easy to see that \( \sim \) is symmetric and transitive. But \( \sim \) is not reflexive: for example, \( 1 \not\sim 1 \).

**B.15** The examples above show that none of the three properties, reflexivity, symmetry, or transitivity, can be inferred from the other two.

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**B.6 Definition**

The Cartesian product \( A \times B \) of the sets \( A \) and \( B \) is defined by
\[ A \times B = \{ (x, y) : x \in A \text{ and } y \in B \} \].

**B.7 Definition**

Suppose that \( A \) is a non-empty set. A binary relation or simply a relation on \( A \) is a subset \( R \subseteq A \times A \).

**B.8** Suppose that \( R \) is a binary relation on \( A \). Suppose that \( x, y \in A \). Then we say that \( x \) is related to \( y \) (with respect to \( R \)) iff \( (x, y) \in R \).

["iff" is an abbreviated form of “if and only if”.

**B.9 Example**

Let \( A \) be a non-empty set. Then two examples of a binary relation on \( A \) are \( \emptyset \)—no two elements of \( A \) are related—and \( A \times A \)—any two elements of \( A \) are related.

**B.10** Suppose that \( A = \{ 2, 3, \ldots, 8 \} \) and that the binary relation \( R \subseteq A \times A \) is defined by
\[ R = \left\{ (2, 2), (2, 4), (2, 6), (2, 8), (3, 3), (3, 6), (4, 4), (4, 8), (5, 5), (6, 6), (7, 7), (8, 8) \right\} \cdot \]

Then \((x, y) \in R \) iff \( x \mid y \).

**B.11** Suppose that \( R \subseteq A \times A \) is a binary relation on \( A \). We can then define an operator \( \sim \) (known as tild or twiddles) by the formula
\[ \forall x, y \in A, x \sim y \iff (x, y) \in R \]
and then use \( \sim \) instead of \( R \) to denote the binary relation on \( A \). This is how mathematicians usually describe binary relations in practice. For example, the binary relation \( R \) of the previous example could also be described as

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**B.16 Definition**

A binary relation \( \sim \) on a non-empty set \( A \) is said to be an equivalence relation on \( A \) iff it is reflexive, symmetric, and transitive.

**B.17** Clearly, equality is an equivalence relation on any non-empty set.

**B.18 Example**

Let \( T \) be the set of all triangles in the real plane. Then similarity and congruence, as defined in Euclidean geometry, are both equivalence relations on \( T \).

**B.19 Lemma**

Suppose that

- \( (M, d) \) is a metric space.
- \( y \in M \).
- \( f, g : [0, 1] \rightarrow M \) are both continuous on \( [0, 1] \subset \mathbb{E}_1 \).
- \( f(1) = g(0) = y \).
- \( h : [0, 1] \rightarrow M \) is defined by
  \[ h : [0, 1] \rightarrow M : t \rightarrow \begin{cases} f(2t) & \text{if } t \leq 1/2 \\ g(2t - 1) & \text{if } t \geq 1/2. \end{cases} \]

Then \( h \) is continuous at \( 1/2 \).

**Proof**

Let \( \varepsilon > 0 \).

Since \( f \) is continuous at \( 1 \), \( \exists \delta_1 > 0 \) such that
\[ \forall u \in [0, 1], |u - 1| < \delta_1 \implies d(f(u), y) < \varepsilon. \]
Since $g$ is continuous at 0, there exists $\delta_2 > 0$ such that
\[ \forall u \in [0, 1], \ |u| < \delta_2 \implies d(g(u), y) < \varepsilon. \] (2)

Let
\[ \delta = \frac{1}{2} \min\{\delta_1, \delta_2\}. \]

Suppose that $t \in [0, 1]$ satisfies $|t - 1/2| < \delta$.

If $t \leq 1/2$ then
\[ |t - 1/2| < \delta \implies |2t - 1| < 2\delta \]
\[ \implies |2t - 1| < \delta_1 \]
\[ \implies d(f(2t), y) < \varepsilon \]
\[ \iff d(h(t), y) < \varepsilon. \]

If $t \geq 1/2$ then
\[ |t - 1/2| < \delta \implies |2t - 1| < 2\delta \]
\[ \implies |2t - 1| < \delta_2 \]
\[ \implies d(g(2t - 1), y) < \varepsilon \]
\[ \iff d(h(t), y) < \varepsilon. \]

Therefore $h$ is continuous at 1/2. □