21 The Fundamental Theorem of Algebra

21.1 Theorem (Cauchy’s Inequality)
Suppose that $a \in \mathbb{C}$, that $\rho > 0$, and that $\gamma = \kappa(a; \rho)$. Suppose that $f$ is a function that is holomorphic on $\mathbb{D}(\gamma)$, that is, on the closed disk of radius $\rho$, centred at $a$. Suppose also that $M \in \mathbb{R}$ is such that, for all $z \in \gamma^*$, $|f(z)| \leq M$. Then,

$$|f^{(n)}(a)| \leq \frac{n!M}{\rho^n}.$$ 

Proof

∗∗∗

21.2 Definition
A complex function $f : \mathbb{C} \to \mathbb{C}$ is said to be an entire function if and only if it is holomorphic (everywhere on $\mathbb{C}$).

21.3 Any polynomial function, the functions sin and cos, and the exponential function are all examples of entire functions.

21.4 Theorem (Liouville’s Theorem)
An entire function that is bounded on $\mathbb{C}$ is constant on $\mathbb{C}$.

Proof
Suppose that $f : \mathbb{C} \to \mathbb{C}$ is an entire function and that $M \in \mathbb{R}$ is such that

$$|f(z)| \leq M.$$ 

If $M = 0$ then $f$ is the zero constant function. Therefore we can assume that $M > 0$.

Suppose that $a \in \mathbb{C}$ and that $\varepsilon > 0$. Let $\rho = M/\varepsilon$. Then $\rho > 0$ and Cauchy’s Inequality, for $n = 1$, implies that

$$|f'(a)| \leq \frac{M}{\rho} = \varepsilon.$$ 

(1)

Since inequality (1) is true for all $\varepsilon > 0$, $|f''(a)| = 0$ and thus $f'(a) = 0$. Since $f'(a) = 0$ for all $a \in \mathbb{C}$, Theorem 13.27 implies that $f$ is constant on $\mathbb{C}$. □

21.5 Notice that Liouville’s Theorem implies that the complex trigonometric functions sin and cos are not bounded on $\mathbb{C}$.

21.6 Recall that if

$$p : \mathbb{C} \to \mathbb{C} : z \mapsto \sum_{k=0}^{n} a_n z^n$$

is a polynomial function where $a_n \neq 0$ then we say that $n$ is the degree of $p$. We denote the degree of $p$ by $\deg(p)$.

A polynomial function of degree 1 is called a linear function.

21.7 Definition
Suppose that $f : \mathbb{C} \to \mathbb{C}$. Then $a \in \text{dom}(f)$ is a zero of $f$ if and only if $f(a) = 0$.

21.8 Theorem (The Division Algorithm)
Suppose that $p$ and $q$ are polynomial functions and that the $\deg(q) \leq \deg(p)$. Then there exist polynomial functions $s$ and $r$ such that,
(i) for all \( z \in \mathbb{C} \), \( p(z) = q(z)s(z) + r(z) \)

(ii) \( \deg(r) < \deg(q) \).

**Proof** (in outline) We prove this theorem by treating \( p(z) \) and \( q(z) \) as polynomials in \( z \) and using algebraic long division to divide \( p(z) \) by \( q(z) \).

21.9 Example
Suppose that \( p(z) = z^3 + z^2 - z + 1 \) and \( q(z) = z + 2 \). We use algebraic long division to divide \( p(z) + q(z) \).

\[
\begin{array}{r|rrrr}
 & z^2 & + 2z & - 2z & - z + 1 \\
\hline
z + 2 | z^3 & + z^2 & - z & + 1 \\
& z^3 & + 2z^2 & & \\
\hline
& & -z^2 & - z & + 1 \\
& & -z^2 & - 2z & \\
\hline
& & & z & + 2 \\
\end{array}
\]

Therefore \( p(z) = q(z)s(z) + r(z) \) where \( s(z) = z^2 - z + 1 \) and \( r(z) = -1 \). Notice that the \( \deg(r) = 0 < 1 = \deg(s) \).

21.10 Corollary
Suppose that \( p \) is a polynomial function of degree \( n \) where \( n \geq 2 \) and that \( a \) is a zero of \( p \). Then there is a polynomial function \( s \) of degree \( n - 1 \) such that,

for all \( z \in \mathbb{C} \), \( p(z) = (z - a)s(z) \).

21.11 Lemma
If \( p : \mathbb{C} \to \mathbb{C} \) is a polynomial function of (strictly) positive degree then as \( |z| \to \infty \), \( |p(z)| \to \infty \).

**Proof** (in outline) Suppose that \( p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \). Then

\[
|p(z)| = |a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0| \\
= |a_n| |z|^n \left| 1 + \frac{a_{n-1}}{a_n z} + \frac{a_{n-2}}{a_n z^2} + \cdots + \frac{a_0}{a_n z^n} \right|.
\]

Therefore, for large values of \( z \), \( |p(z)| \approx |a_n| |z|^n \). Therefore as \( |z| \to \infty \), \( |p(z)| \to \infty \).

21.12 Suppose that \( p : \mathbb{C} \to \mathbb{C} \) is a polynomial function of strictly positive degree. Suppose also that \( p \) does not have a zero, that is, that

for all \( z \in \mathbb{C} \), \( p(z) \neq 0 \).
This allows us to define

\[ f : \mathbb{C} \to \mathbb{C} : z \mapsto \frac{1}{p(z)}. \]

Since \( f \) is a rational function it is a differentiable function, that is, \( f \) is holomorphic on \( \mathbb{C} \). In other words,

\[ f \text{ is an entire function.} \quad (2) \]

By Lemma 21.11, as \( |z| \to \infty \), \( |p(z)| \to \infty \) and thus \( |f(z)| \to 0 \). Therefore there exists \( M > 0 \) such that

\[ \text{for all } z \in \mathbb{C}, \text{ if } |z| > M \text{ then } |f(z)| < 1. \quad (3) \]

Let \( D \subset \mathbb{C} \) be the closed disk of radius \( M \) centred at \( 0 \). \( D \) is closed and bounded and \( f \), being holomorphic on \( \mathbb{C} \), is continuous on \( D \). Therefore we can use Theorem 9.20 to show that \( f \) is bounded on \( D \), that is, that there exists \( B > 0 \) such that,

\[ \text{for all } z \in \mathbb{C}, \text{ if } |z| \leq M \text{ then } |f(z)| \leq B. \quad (4) \]

Statements (3) and (4) imply that,

\[ \text{for all } z \in \mathbb{C}, \ |f(z)| \leq \max\{ 1, B \}. \]

In other words,

\[ f \text{ is bounded on } \mathbb{C}. \quad (5) \]

By Liouville’s Theorem, statements (2) and (5) imply that \( f \) is a constant function. But the degree of \( p \) is strictly positive and therefore \( f \) is not a constant function. Therefore statement (1) is false and we have proved the following theorem.

### 21.13 Theorem

Suppose that \( p : \mathbb{C} \to \mathbb{C} \) is a polynomial function of strictly positive degree. Then \( p \) has a zero, that is, there exists \( a \in \mathbb{C} \) such that \( p(a) = 0 \). \( \square \)

### 21.14 Theorem (The Fundamental Theorem of Algebra)

Suppose that \( p : \mathbb{C} \to \mathbb{C} \) is a polynomial function of degree \( n \) where \( n \geq 1 \). Then there exist \( c, a_1, a_2, \ldots, a_n \in \mathbb{C} \) such that

\[ \text{for all } z \in \mathbb{C}, \ p(z) = c \prod_{k=1}^{n} (z - a_k) \]

\[ = c(z - a_1)(z - a_2) \cdots (z - a_n). \]

**Proof** We prove this theorem by induction on \( n \). It is obvious that the theorem is true when \( n = 1 \).

Our inductive hypothesis is that if \( p_n \) is a polynomial function of degree \( n \) then there exist \( c, a_1, a_2, \ldots, a_n \in \mathbb{C} \) such that

\[ \text{for all } z \in \mathbb{C}, \ p_n(z) = c(z - a_1)(z - a_2) \cdots (z - a_n). \quad (1) \]

We must deduce that if \( p_{n+1} \) is a polynomial function of degree \( n + 1 \) then there exist \( c', a'_1, a'_2, \ldots, a'_{n+1} \in \mathbb{C} \) such that

\[ \text{for all } z \in \mathbb{C}, \]

\[ p_{n+1}(z) = c'(z - a'_1)(z - a'_2) \cdots (z - a'_{n+1}). \quad (2) \]

Suppose that \( p_{n+1} \) is a polynomial function of degree \( n + 1 \). Then, by Theorem 21.13, \( p_{n+1} \) has a zero \( a \in \mathbb{C} \). Therefore Corollary 21.10 implies that there is a polynomial function of degree \( n \), let us call it \( p_n \), such that

\[ \text{for all } z \in \mathbb{C}, \ p_{n+1}(z) = (z - a)p_n(z). \]
Therefore (1) implies that, for all $z \in \mathbb{C}$,
\[
p_{n+1}(z) = (z - a)c(z - a_1)(z - a_2) \cdots (z - a_n) \\
= c(z - a)(z - a_1)(z - a_2) \cdots (z - a_n).
\]

21.15 The Fundamental Theorem of Algebra can be restated as follows:

Every complex polynomial of degree $n$, where $n \geq 1$, has exactly $n$ linear factors.

21.16 Notice that Theorem 21.14 is not true if $\mathbb{C}$ is replaced by $\mathbb{R}$, that is, it is not true for real polynomials. For example, the real polynomial in $x$, $x^2 + 1$, does not have real linear factors.

21.17 Theorem 21.14 is an existence theorem: it does not contain an algorithm for constructing the linear factors of a given polynomial.