19 Higher-Order Derivatives

19.1 Lemma
Suppose that \( \gamma \) is a path in \( \mathbb{C} \) and that \( f \) is holomorphic on \( \gamma^* \), that is, that \( f \) is holomorphic on an open superset of \( \gamma^* \). Then there exists \( M \in \mathbb{R} \) such that,

\[
\text{for all } z \in \gamma^*, \ |f(z)| < M.
\]

**Proof (in outline)**
Let \( f = u + iv \). Since \( f \) is differentiable on \( \gamma^* \) it is continuous on \( \gamma^* \) and therefore \( u \) and \( v \) are continuous real-valued functions on the compact set \( \gamma^* \subset \mathbb{E}_2 \). Therefore Corollary 9.23 implies that \( u \) and \( v \) attain maximum and minimum values on \( \gamma^* \) and thus both of them are bounded on \( \gamma^* \). Therefore \( |f| = |u + iv| \) is bounded on \( \gamma^* \), that is, there exists \( M \in \mathbb{R} \) such that,

\[
\text{for all } z \in \gamma^*, \ |f(z)| < M. \quad \square
\]

19.2 Lemma
Suppose that \( a \in \mathbb{C} \), that \( \rho > 0 \), and that \( \gamma = \kappa(a; \rho) \). Suppose also that \( f \) is holomorphic on \( \mathbb{D}^*(\gamma) \). Then

\[
\lim_{h \to 0} \int_{\gamma} \frac{hf(z)}{(z-a)^2(z-a-h)} \, dz = 0.
\]

**Proof**

\( \text{Lemma 19.1 implies that there exists } M \in \mathbb{R} \text{ such that,} \)

\[
\text{for all } z \in \gamma^*, \ |f(z)| < M. \quad (1)
\]

Since we are trying to evaluate a limit as \( h \to 0 \), we can assume that \( |h| < \rho/2 \).

Then, for all \( z \in \gamma^* = \kappa(a; \rho)^* \),

\[
|z - a - h| \geq |z - a| - |h| > \rho - \frac{\rho}{2} = \frac{\rho}{2}.
\]

Therefore, for all \( z \in \gamma^* \), if \( |h| < \rho/2 \) then

\[
\frac{1}{|z-a|^2|z-a-h|} < \frac{2}{\rho^3}
\]

and thus

\[
\left| \frac{hf(z)}{(z-a)^2(z-a-h)} \right| < \frac{2|h|M}{\rho^3}. \quad [1]
\]

Therefore, for all \( z \in \gamma^* \), if \( |h| < \rho/2 \) then

\[
\int_{\gamma} \frac{hf(z)}{(z-a)^2(z-a-h)} \, dz < \frac{2|h|M}{\rho^3} |\gamma| = \frac{2|h|M}{\rho^3} \cdot 2\pi \rho = \frac{4M\pi}{\rho^2} |h|.
\]

Therefore, the Squeezing Rule implies that

\[
\lim_{h \to 0} \int_{\gamma} \frac{hf(z)}{(z-a)^2(z-a-h)} \, dz = 0.
\]

Therefore

\[
\lim_{h \to 0} \int_{\gamma} \frac{hf(z)}{(z-a)^2(z-a-h)} \, dz = 0. \quad \square
\]
19.3 Theorem (CFI-1)
Suppose that $\gamma$ is a contour in $\mathbb{C}$ and that $f$ is holomorphic on $\mathbb{I}^*(\gamma)$. Then, for all $a \in \mathbb{I}(\gamma)$,

$$f'(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)\,dz}{z-a}.$$  

Proof (in outline)
Let $a \in \mathbb{I}(\gamma)$. Since $\mathbb{I}(\gamma)$ is open there exists $h \in \mathbb{C}$ such that $h \neq 0$ and $a + h \in \mathbb{I}(\gamma)$. Therefore, by CIF,

$$f(a+h) - f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)\,dz}{z-a} - \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)\,dz}{z-(a+h)} = \frac{h}{2\pi i} \int_{\gamma} \frac{f(z)\,dz}{(z-a)(z-a-h)}.$$  

Therefore

$$\frac{f(a+h) - f(a)}{h} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)\,dz}{(z-a)^2} + \frac{g(h)}{2\pi i}$$  

(1)

where

$$g(h) = \int_{\gamma} \left[ \frac{f(z)}{(z-a)(z-a-h)} - \frac{f(z)}{(z-a)^2} \right] dz$$

$$= \int_{\gamma} \frac{hf(z)\,dz}{(z-a)^2(z-a-h)}.$$  

(2)

Since $\mathbb{I}(\gamma)$ is open, there exists $\rho > 0$ such that $\gamma^\rho = \kappa(a;\rho)^* \subset \mathbb{I}(\gamma)$.

By Theorem 16.22, if $|h| < \rho$ then

$$\int_{\gamma} \frac{hf(z)\,dz}{(z-a)^2(z-a-h)} = \int_{\gamma^\rho} \frac{hf(z)\,dz}{(z-a)^2(z-a-h)}.$$  

(3)

Lemma 19.2 and equations (1), (2), and (3) imply that

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)\,dz}{(z-a)^2},$$

that is,

$$f'(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z)\,dz}{(z-a)^2}. \quad \square$$

19.4 Problem
Evaluate the following integral where $\gamma = \kappa(0;2)$:

$$\int_{\gamma} \frac{z\exp(\pi z)\,dz}{(z-1)^2}.$$
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Clearly, \(i \in \mathbb{I}(\gamma)\). Let \(f : \mathbb{C} \rightarrow \mathbb{C} : z \mapsto z e^{\pi z}\). Then \(f\) is holomorphic and, for all \(z \in \mathbb{C}\),
\[
f'(z) = e^{\pi z} + z \pi e^{\pi z} = e^{\pi z}(1 + \pi z).
\]
Therefore, by CIF,
\[
\int_{\gamma} \frac{ze^{\pi z} \, dz}{(z-i)^2} = 2\pi i f'(i) = 2\pi i e^{\pi i}(1 + \pi i) = 2\pi(\pi - i).
\]

19.5 Suppose that \(\gamma\) is a contour in \(\mathbb{C}\) and that \(f\) is holomorphic on \(I^*(\gamma)\). Suppose also that \(a \in \mathbb{I}(\gamma)\) and that \(h \in \mathbb{C}^*\) is such that \(a + h \in \mathbb{I}(\gamma)\). Then
\[
\frac{f'(a+h) - f'(a)}{h} = \frac{1}{2\pi i} \int_{\gamma} \left[ \frac{f(z) - f(z)}{(z-a-h)^2} \right] \, dz
\]
\[
= \frac{1}{2\pi i} \int_{\gamma} \frac{2f(z) \, dz}{(z-a)^3} + \frac{g(h)}{2\pi i}
\]
where
\[
g(h) = \int_{\gamma} \frac{(2h - 3(z-a)) \, dz}{(z-a)^3(z-a-h)^2}.
\]
By using methods similar to, but more complicated than, those used in the proof of Lemma 19.2, we can show that
\[
\lim_{h \to 0} g(h) = 0
\]
and thus prove the following theorem.

19.6 Theorem (CIF-2)
Suppose that \(\gamma\) is a contour in \(\mathbb{C}\) and that \(f\) is holomorphic on \(I^*(\gamma)\). Then \(f'\) is differentiable on \(I(\gamma)\) and, for all \(a \in I(\gamma)\),
\[
f''(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{2f(z) \, dz}{(z-a)^3}. \quad \Box
\]

19.7 CIF-2 is a remarkable theorem. CIF-1 gives us a formula for the derivative of a holomorphic function, that is, a function already known to be differentiable. CIF-2, however, states that if \(f\) is holomorphic at \(a\) then its derivative \(f'\) is differentiable at \(a\). In other words, since any point in a region \(U\) can be strictly encircled by a contour in \(U\), CIF-2 implies that if a complex function is differentiable on a region \(U\) then it is twice differentiable on \(U\).

19.8 Suppose that \(f\) is holomorphic on a region \(U\). Then, by CIF-2, \(f'\) is holomorphic on \(U\). Therefore, by CIF-2 again, \(f'' = (f')'\) is holomorphic on \(U\). Therefore, by CIF-2 yet again, \(f''' = f''' = (f'')'\) is holomorphic on \(U\).

19.9 Lemma
Suppose that \(U\) is a region of \(\mathbb{C}\) and that \(f\) is holomorphic on \(U\). Then, for all \(n \in \mathbb{N}\), \(f\) is \(n\)-times differentiable on \(U\).

**Proof** Since, for all \(n \in \mathbb{N}\) and all \(z \in U\), \(f^{(n+1)}(z) = (f^{(n)})'(z)\), the lemma is easily proved by induction on \(n\). \(\Box\)

19.10 Let us list the equations of CIF, CIF-1, and CIF-2 in order.
\[
f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z) \, dz}{z-a}. \quad (1)
\]
19.12 Theorem (CIF-n)
Suppose that \( \gamma \) is a contour in \( \mathbb{C} \) and that \( f \) is holomorphic on \( \mathbb{I}^*(\gamma) \). Then \( f \) is infinitely differentiable on \( \mathbb{I}(\gamma) \) and, for all \( a \in \mathbb{I}(\gamma) \) and \( n \in \mathbb{N} \),

\[
f^{(n)}(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{n! f(z) \ dz}{(z-a)^{n+1}}. \quad \square
\]

19.13 There is a proof of CIF-n that is similar to the proofs of CIF-1 and CIF-2. But this proof is long and tedious. There is a more elegant proof but it belongs to a more advanced course than this one. We shall not prove this theorem but we shall show how its formula can be proved by induction on \( n \) if we accept differentiation under the integral sign as a valid operation.

19.14 CIF-1 tells that, under the conditions of CIF-n, the formula,

\[
f^{(n)}(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{n! f(z) \ dz}{(z-a)^{n+1}}, \quad (*)
\]

is true when \( n = 1 \). We shall take (*) to be our inductive hypothesis. We must deduce that

\[
f^{(n+1)}(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{(n+1)! f(z) \ dz}{(z-a)^{n+2}}.
\]

Using differentiation under the integral sign, we see that

\[
f^{(n+1)}(a) = \frac{df^{(n)}(a)}{da}
\]
\[
\int_\gamma \frac{z^5 \, dz}{(z - 2)^4} = \frac{2\pi i}{3!} f^{(3)}(2) = \frac{2\pi i}{6} \cdot 60 \cdot 2^2 = 80\pi i.
\]

This completes the “proof”.

19.16 Theorem
Suppose that \(\alpha\), \(\beta\), and \(\gamma\) are three positively oriented contours in \(\mathbb{C}\) such that
\[
\mathbb{I}^*(\alpha) \cap \mathbb{I}^*(\beta) = \emptyset, \mathbb{I}^*(\alpha) \subset \mathbb{I}(\gamma), \text{ and } \mathbb{I}^*(\beta) \subset \mathbb{I}(\gamma).
\]
Suppose also that \(f : \mathbb{C} \to \mathbb{C}\) is holomorphic on that part of \(\mathbb{C}\) that lies inside \(\gamma\) and outside both \(\alpha\) and \(\beta\), where inside and outside are not interpreted strictly, that is, that \(f\) is holomorphic on
\[
\mathbb{I}^*(\gamma) \setminus (\mathbb{I}(\alpha) \cup \mathbb{I}(\beta)).
\]
Then
\[ \int_{\gamma} f = \int_{\alpha} f + \int_{\beta} f. \]

**Proof** (in outline) Construct two paths \( \sigma \) and \( \tau \) connection \( \alpha \) and \( \beta \), respectively, to \( \gamma \) as shown below.

Now let \( \gamma = \gamma_1 \ast \gamma_2 \) as shown below.

Let
\[ \Gamma = \gamma_2 \ast \sigma \ast \hat{\sigma} \ast \gamma_1 \ast \tau \ast \hat{\beta} \ast \hat{\tau}. \]

[Notice that as we move along \( \Gamma \) the area of the figure on our left-hand side is shaded. In other words the area inside \( \Gamma \) is shaded.]

Since the \( f \) is holomorphic on \( \mathbb{I}^*(\Gamma) \), by Cauchy’s Integral Theorem (17.5),
\[ \int_{\Gamma} F = 0. \]

Therefore
\[ \left( \int_{\gamma_2} + \int_{\sigma} - \int_{\alpha} - \int_{\hat{\sigma}} + \int_{\gamma_1} + \int_{\tau} - \int_{\beta} - \int_{\hat{\tau}} \right) f = 0. \]

Therefore
\[ \int_{\gamma_2} f + \int_{\gamma_1} f - \int_{\alpha} f - \int_{\beta} f = 0. \]

Therefore
\[ \int_{\gamma} f = \int_{\alpha} f + \int_{\beta} f. \quad \square \]

**19.17 Problem**
Evaluate the following integral
\[ \int_{\gamma} \frac{z^3}{z^2 - 4} dz. \]

where \( \gamma \) is a positively-oriented contour that strictly encloses both 2 and \(-2\).
Since $-2$ and $2$ both lie strictly inside $\gamma$ there are two positively oriented contours $\alpha$ and $\beta$ such that

(i) $-2$ lies inside $\alpha$ and $2$ lies inside $\beta$.

(ii) $\mathbb{I}^*(\alpha) \subset \mathbb{I}(\gamma)$ and $\mathbb{I}^*(\beta) \subset \mathbb{I}(\gamma)$.

(iii) $\mathbb{I}^*(\alpha) \cap \mathbb{I}^*(\beta) = \emptyset$.

Let $f(z) = \frac{z^3}{z^2 - 4}$.

Then, if $z \neq \pm 2$ then $f(z) = g(z)/(z+2) = h(z)/(z-2)$ where

$$g(z) = \frac{z^3}{z-2} \quad \text{and} \quad h(z) = \frac{z^3}{z+2}.$$ 

Since $g$ is holomorphic on $\mathbb{I}^*(\alpha)$ and $h$ is holomorphic on $\mathbb{I}^*(\beta)$, by CIF,

$$\int_{\alpha} f(z) dz = \int_{\alpha} \frac{g(z)}{z-2} = 2\pi i g(-2) = 4\pi i$$

and

$$\int_{\beta} f(z) dz = \int_{\beta} \frac{h(z)}{z-2} = 2\pi i h(2) = 4\pi i.$$ 

Therefore

$$\int_{\gamma} \frac{z}{z^2 - 4} dz = \int_{\gamma} f(x) dx = \int_{\alpha} f(x) dx + \int_{\beta} f(x) dx$$

$$= 4\pi i + 4\pi i = 8\pi i.$$ 

19.18 In Theorem 19.16 we put two separated contours inside a contour. In a stronger version of this theorem we can put any finite number of separated contours inside a contour and obtain a similar result.

19.19 Theorem

Suppose that $\gamma$ is a positively oriented contour, that $n \in \mathbb{N}$, and that, for all $j = 1, 2, \ldots, n$, there is a positively oriented contour $\gamma_j$ such that

(i) $\mathbb{I}^*(\gamma_j) \subset \mathbb{I}(\gamma)$.

(ii) if $i \neq j$ then $\mathbb{I}^*(\gamma_i) \cap \mathbb{I}^*(\gamma_j) = \emptyset$.

Suppose that

$$A = \mathbb{I}^*(\gamma) - \bigcup_{j=1}^{n} \mathbb{I}(\gamma_j).$$

and that $f : \mathbb{C} \to \mathbb{C}$ is holomorphic on $A$. Then

$$\int_{\gamma} f = \sum_{j=1}^{n} \int_{\gamma_j} f. \quad \square$$
19.20 Problem
Evaluate the following integral
\[
\int_{\gamma} \frac{(z-1)\,dz}{z^4 + z^2}
\]
where \(\gamma = \kappa(0;2)\), the circular contour of radius 2, described positively and centred at the origin.

Let \(f(z) = \frac{(z-1)}{(z^4 + z^2)}\). Then

\[
f(z) = \frac{z-1}{z^2(z+i)(z-i)} = \frac{g_1(z)}{z^2} = \frac{g_2(z)}{z+i} = \frac{g_3(z)}{z-i}
\]

where

\[
g_1(z) = \frac{z-1}{z^2+1}, \quad g_2(z) = \frac{z-1}{z^2(z-i)}, \quad \text{and} \quad g_3(z) = \frac{z-1}{z^2(z+i)}.
\]

Let \(\alpha = \kappa(0;0.4)\), \(\sigma = \kappa(-i;0.4)\), and \(\tau = \kappa(i;0.4)\).

Then \(\alpha\), \(\sigma\), and \(\tau\) are separated positively-oriented contours that lie strictly inside \(\gamma\).

Since we are going to use CIF-1, we must differentiate \(g_1\).

\[
g_1'(z) = \frac{z^2 + 1 - (z-1)2z}{(z^2+1)^2} = \frac{-z^2 - 2z + 1}{(z^2+1)^2}
\]

Since \(g_1\) is holomorphic on \(\Pi^*(\alpha)\), by CIF-1,

\[
\int_{\alpha} f(z)\,dz = \int_{\alpha} \frac{g_1(z)}{z^2}\,dz = 2\pi i g_1'(0) = 2\pi i. \quad (1)
\]
By CIF,

\[
\int_{\sigma} f(z) \, dz = \int_{\sigma} \frac{g_2(z) \, dz}{z - (-i)} = 2\pi i g_2(-i) \\
= 2\pi i \left( \frac{-i - 1}{-i - (-i)} \right) = -\pi(1 + i)
\]

(2)

and

\[
\int_{\tau} f(z) \, dz = \int_{\tau} \frac{g_3(z) \, dz}{z - i} = 2\pi i g_3(i) \\
= 2\pi i \left( \frac{i - 1}{i + i} \right) = \pi(1 - i).
\]

(3)

Therefore, by Theorem 19.19 and equations (1), (2), and (3),

\[
\int_{\gamma} \frac{z - 1}{z^4 + 2} \, dz = \int_{\gamma} f(z) \, dz \\
= \int_{\alpha} f(z) \, dz + \int_{\sigma} f(z) \, dz + \int_{\tau} f(z) \, dz \\
= 2\pi i - \pi - \pi i + \pi - \pi i \\
= 0.
\]