18 Cauchy’s Integral Formula

18.1 Lemma
Suppose that $A$ is an open subset of $\mathbb{C}$, that $f : A \to \mathbb{C}$, and that $a \in A$. Then $f$ is differentiable at $a$ if and only if there exist $g : A \to \mathbb{C}$ and $d \in \mathbb{C}$ that satisfy both of the following statements:

$\alpha$: for all $z \in A$, $f(z) - f(a) = (z - a)d + (z - a)g(z)$;

$\beta$: \( \lim_{z \to a} g(z) = 0. \)

If $f$ is differentiable at $a$ then $d = f'(a)$.

Proof

Suppose that $f$ is differentiable at $a$. Let

$$ g(z) = \begin{cases} \frac{f(z) - f(a)}{z - a} - f'(a) & \text{if } z \neq a, \\ 0 & \text{if } z = a. \end{cases} $$

Then $g$ satisfies $\alpha$ where $d = f'(a)$ and

$$ \lim_{z \to a} g(z) = \lim_{z \to a} \frac{f(z) - f(a)}{z - a} - \lim_{z \to a} f'(a) = f'(a) - f'(a) = 0. $$

Therefore $g$ satisfies $\beta$.

Now suppose that there exists $g : A \to \mathbb{C}$ and $d \in \mathbb{C}$ that satisfy $\alpha$ and $\beta$. Then

$$ \lim_{z \to a} \frac{f(z) - f(a)}{z - a} = \lim_{z \to a} [d + g(z)] = d. $$

Therefore $f$ is differentiable at $a$ and $f'(a) = d$. \qed

18.2 Lemma
Suppose that $A$ is a simply-connected region of $\mathbb{C}$ and that $f$ is holomorphic on $A$. Suppose also that $\gamma$ is a contour in $A$. Then, for all $a \in \mathbb{I}(\gamma)$,

$$ \int_{\gamma} \frac{f(z) - f(a)}{z - a} \, dz = 0. $$

Proof

Let $\epsilon > 0$

Since $f$ is differentiable at $a$, Lemma 18.1 implies that there exists $g : A \to \mathbb{R}$ such that

$$ f(z) - f(a) = (z - a)f'(a) + (z - a)g(z) \quad (1) $$

and

$$ \lim_{z \to a} g(z) = 0. $$

Therefore there exists $\delta > 0$ such that, for all $x \in A$.

$$ \text{if } 0 < |z - a| < \delta \text{ then } |g(z)| < \epsilon. \quad (2) $$

Since $a \in \mathbb{I}(\gamma)$ and $\mathbb{I}(\gamma)$ is an open set, there exists $\sigma > 0$ such that $B(a; \sigma) \subset \mathbb{I}(\gamma)$. Let

$$ \rho = \min \{ \delta/2, \sigma/2, 1 \} $$

and let

$$ \gamma^\rho : [0, 2\pi] \to \mathbb{C} : t \mapsto a + \rho e^{it}. $$

Then $\gamma^\rho$ is a contour and $\gamma^\rho$ is the circle of radius $\rho$ centred at $a$.

Since $\rho \leq \sigma/2 < \sigma$, $\gamma^\rho \subset \mathbb{I}(\gamma)$. 

\[ \square \]
Furthermore, for all \( z \in \gamma^\ast \rho \), \( 0 < |z - a| = \rho < \delta \) and thus for all \( z \in \gamma^\ast \rho \), \( |g(z)| < \varepsilon \).

By the Deformation Theorem (16.22),
\[
\int_{\gamma} \frac{f(z) - f(a)}{z - a} \, dz = \int_{\gamma^\rho} \frac{f(z) - f(a)}{z - a} \, dz
= \int_{\gamma^\rho} [f'(a) + g(z)] \, dz
= f'(a) \int_{\gamma^\rho} \, dz + \int_{\gamma^\rho} g(z) \, dz
= \int_{\gamma^\rho} g(z) \, dz
\]
Therefore
\[
\left| \int_{\gamma} \frac{f(z) - f(a)}{z - a} \, dz \right| = \int_{\gamma^\rho} |g(z)| \, dz
\]

Therefore, since \( \rho \leq 1 \), for all \( \varepsilon > 0 \),
\[
\left| \int_{\gamma} \frac{f(z) - f(a)}{z - a} \, dz \right| \leq 2\pi \varepsilon.
\]
Therefore
\[
\int_{\gamma} \frac{f(z) - f(a)}{z - a} \, dz = 0. \quad \Box
\]

**18.3 Theorem (Cauchy’s Integral Formula, CIF)**
Suppose that \( A \) is a simply-connected region of \( \mathbb{C} \) and that \( f \) is holomorphic on \( A \). Suppose that \( \gamma \) is a positively oriented contour in \( A \). Then, for all \( a \in \Pi(\gamma) \),
\[
f(a) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(z) \, dz}{z - a}.
\]

**Proof** For all \( z \in \Pi(\gamma) \),
\[
\int_{\gamma} \frac{f(z) \, dz}{z - a} = \int_{\gamma} \frac{f(a) + f(z) - f(a)}{z - a} \, dz
= f(a) \int_{\gamma} \frac{dz}{z - a} + \int_{\gamma} \frac{f(z) - f(a)}{z - a} \, dz
\]
Therefore Corollary 16.23 and Lemma 18.2 imply that
\[
\int_{\gamma} \frac{f(z) \, dz}{z - a} = f(a) \cdot 2\pi i. \quad \Box
\]

**18.4** For all \( a \in \mathbb{C} \) and \( \rho > 0 \), we denote the circular contour \( [0, 2\pi] \to \mathbb{C} : t \mapsto a + \rho e^{it} \) by \( \kappa(a; \rho) \).
18.5 Example
Evaluate each of following integrals:

(i) \( \int_{\gamma} \frac{\cos(z)}{z} \, dz \) where \( \gamma = \kappa(i; 2) \);

(ii) \( \int_{\gamma} \frac{z}{z^2 + 25} \, dz \) where \( \gamma = \kappa(3; 5) \);

(iii) \( \int_{\gamma} \frac{z}{z^2 + 25} \, dz \) where \( \gamma = \kappa(3i; 5) \).

(i)

Since \( 0 \in \mathbb{I}(\gamma) \), and cos is holomorphic on \( \mathbb{C} \), by CIF,
\[
\int_{\gamma} \frac{\cos(z)}{z} \, dz = 2\pi i \cos(0) = 2\pi i.
\]

(ii)

Let \( f(z) = \frac{z}{z^2 + 25} = \frac{z}{(z + 5i)(z - 5i)} \).
Since \( f \) is a rational function it is differentiable. Therefore, since \( 5i \) and \(-5i \notin \mathbb{I}^+(\gamma) \), \( f \) is holomorphic on \( \mathbb{I}^+(\gamma) \).
Therefore, by CIT,
\[
\int_{\gamma} \frac{z}{z^2 + 25} \, dz = 0.
\]

(iii)
Let $g(z) = z/(z + 5i)$. Then $f(z) = g(z)/(z - 5i)$.

Since $5i \in \Pi(\gamma)$ and $-5i \notin \Pi^*(\gamma)$, by CIF,

$$
\int_{\gamma} \frac{z}{z^2 + 25} \, dz = \int_{\gamma} \frac{g(z)}{z - 5i} \, dz = 2\pi i g(5i) = \pi i.
$$