15 Complex Integrals

15.1 Definition
Suppose that $a$ and $b \in \mathbb{R}$ where $a \leq b$. Then we call a continuous function $\gamma : [a, b] \to \mathbb{C}$ a path in $\mathbb{C}$. We call the set
\[ \{ \gamma(t) \mid t \in [a, b] \} \]
the track of $\gamma$ and we denote it by $\gamma^*$.

We say that the function $\gamma$ parametrises $\gamma^*$ and we call the interval $[a, b]$ the parameter interval of $\gamma$.

We call $\gamma(a)$ the initial point and $\gamma(b)$ the final point of $\gamma$.

15.2 In practice, we often speak of “the path $\gamma$” when we actually mean “the track of the path $\gamma$”.

15.3 Definition
A path $\gamma$ in $\mathbb{C}$ is said to be oriented positively in the direction that $\gamma(t)$ moves along $\gamma^*$ at $t$ increases.

\[ \gamma(a) \quad \gamma^* \quad \gamma(b) \]
\[ \{ \gamma(t) \mid t \in [a, b] \} \]

15.4 Definition
If the initial and final points of a path $\gamma$ are equal then we say that $\gamma$ is a closed path.

15.5 Definition
We say that a path $\gamma : [a, b] \to \mathbb{C}$ is simple if and only if $a \leq s < t \leq b$ implies that $\gamma(s) \neq \gamma(t)$ unless $\gamma$ is a closed curve, $s = a$, and $t = b$.

15.6 In other words, if a curve is not closed then it is simple if and only if its track does not intersect itself. If the curve is closed then it is simple if and only if its track intersects itself only where the initial and final points of the curve meet and at no other point.

\[ \gamma \]
\[ \gamma \text{ is not simple.} \]

15.7 Definition
Suppose that $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [c, d] \to \mathbb{C}$ are two paths in $\mathbb{C}$ and that
\[ \gamma_1(b) = \gamma_2(c). \]

Then we can define a path $\gamma = \gamma_1 \ast \gamma_2$, called the join of $\gamma_1$ and $\gamma_2$ by
\[ \gamma : [a, b + d - c] \to \mathbb{C} : t \mapsto \begin{cases} 
\gamma_1(t) & \text{if } t \in [a, b], \\
\gamma_2(t + c - b) & \text{if } t \in [b, b + d - c].
\end{cases} \]
15.8 The proof that $\gamma$ is a path, that is, that $\gamma$ is continuous, is similar to the proof of Lemma B.19 in Appendix B.

15.9 We can join any finite number of paths $\gamma_1, \gamma_2, \ldots, \gamma_n$ to form the path
$$\gamma_1 \ast \gamma_2 \ast \cdots \ast \gamma_n$$
provided that, for all $k = 2, 3, \ldots, n$, the final point of $\gamma_{k-1}$ equals the initial point of $\gamma_k$.

15.10 Definition
Suppose that $\gamma : [a, b] \to \mathbb{C}$ is a path. Then we define a path $\tilde{\gamma}$ by
$$\tilde{\gamma} : [a, b] \to \mathbb{C} : t \mapsto \gamma(a + b - t).$$

15.11 Clearly $\gamma$ and $\tilde{\gamma}$ have the same track but different orientations.

15.12 Many authors write $-\gamma$ instead of $\tilde{\gamma}$ and $\gamma_1 + \gamma_2$ instead of $\gamma_1 \ast \gamma_2$.

15.13 Definition
Let $a, b \in \mathbb{R}$ where $a < b$.

(i) If $f \in C^1[a, b]$, that is, if $f'$ is continuous on $[a, b]$ then we say that $f$ is smooth on $[a, b]$.

(ii) Suppose that $f$ is continuous on $[a, b]$ and that there exist $x_0, x_1, \ldots, x_n \in [a, b]$ where
$$a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$$
such that, for all $i = 1, 2, \ldots, n$, $f'$ is continuous on the open interval $(x_{i-1}, x_i)$.
Then we say that $f$ is piecewise-smooth on $[a, b]$.  
Clearly, if $f$ is smooth on $[a, b]$ then $f$ is piecewise-smooth on $[a, b]$.

**15.14 Definition**
Suppose that $\gamma : [a, b] \to \mathbb{C}$ is a path and that $\gamma = \xi + i\eta$, that is to say that for all $t \in [a, b]$, $\gamma(t) = \xi(t) + i\eta(t)$.

Then we say that $\gamma$ is a smooth path if and only if $\xi$ and $\eta$ are both smooth on $[a, b]$.

If $\xi$ and $\eta$ are both piecewise-smooth on $[a, b]$ then we say that $\gamma$ is a piecewise-smooth path in $\mathbb{C}$.

**15.15 Definition**
Suppose that $\gamma : [a, b] \to \mathbb{C}$ is a piecewise-smooth path and that $f : \mathbb{C} \to \mathbb{C}$ is defined and continuous on $\gamma^\ast$. Then we define the complex path-integral

$$\int_\gamma f = \int_\gamma f(z) \, dz$$

by

$$\int_\gamma f(z) \, dz = \int_a^b f(\gamma(t)) \gamma'(t) \, dt.$$ 

**15.16** Suppose that $\gamma$ and $f$ are as in 15.15, that $f = u + iv$, and that, for all $t \in [a, b]$, we set

$$\gamma(t) = z(t) = x(t) + iy(t).$$

Then

$$\int_\gamma f(z) \, dz = \int_a^b f(z(t)) \frac{dz}{dt} \, dt$$

$$= \int_a^b [u(x,y) + iv(x,y)] \frac{d(x+iy)}{dt} \, dt$$

$$= \int_a^b [u(x,y) + iv(x,y)] \left[ \frac{dx}{dt} + i \frac{dy}{dt} \right] \, dt$$

$$= \int_a^b \left[ u(x(t),y(t)) \frac{dx}{dt} - v(x(t),y(t)) \frac{dy}{dt} \right] \, dt$$

$$+ i \int_a^b \left[ v(x(t),y(t)) \frac{dx}{dt} + u(x(t),y(t)) \frac{dy}{dt} \right] \, dt.$$
15.17 Example
Suppose that $\rho > 0$ and that

$$
\gamma : [0, 2\pi] \rightarrow \mathbb{C} : t \mapsto \rho e^{it} = \rho (\cos(t) + i\sin(t)).
$$

Then $\gamma^*$ is the circle of radius $\rho$, centred at 0, oriented or described positively, that is, in the anti-clockwise sense.

Evaluate the path integral

$$
\int_{\gamma} \frac{dz}{z}.
$$

Evaluate

$$
\int_{\gamma} \frac{dz}{z} = \int_{\gamma} \frac{1}{z(t)} \frac{dz}{dt} dt
= \int_{0}^{2\pi} \left( \frac{1}{\rho e^{it}} \right) \frac{d(\rho e^{it})}{dt} dt
= \int_{0}^{2\pi} \left( \frac{1}{\rho e^{it}} \right) i\rho e^{it} dt
$$

Therefore

$$
\int_{\gamma} \frac{dz}{z} = 2\pi i.
$$

15.18 Theorem
Suppose that $n \in \mathbb{N}$, and that

$$
\gamma = \gamma_1 * \gamma_2 * ... \gamma_n
$$

where $\gamma_1, \gamma_2, \ldots, \gamma_n$ are piecewise-smooth paths in $\mathbb{C}$ such that $\gamma$ is well-defined. Suppose also that $f : \mathbb{C} \rightarrow \mathbb{C}$ is continuous on $\gamma^*$. Then

(i) $\gamma$ is a piecewise-smooth path.

(ii) $\int_{\gamma} f(z) dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z) dz$. □

15.19 Theorem
Suppose that $\gamma$ is a piecewise-smooth path in $\mathbb{C}$ and that $f : \mathbb{C} \rightarrow \mathbb{C}$ is continuous on $\gamma^*$. Then

$$
\int_{\gamma} f(z) dz = -\int_{\gamma} f(z) dz. \quad □
$$

15.20 Theorem
Suppose that $\gamma$ is a piecewise-smooth path in $\mathbb{C}$ and that $f : \mathbb{C} \rightarrow \mathbb{C}$ is continuous on $\gamma^*$. Then

$$
\left| \int_{\gamma} f(z) dz \right| \leq \int_{\gamma} |f(z)| dz. \quad □
$$
15.21 Definition
Suppose that $\gamma : [a, b] \to \mathbb{C}$ is a piecewise-smooth path in $\mathbb{C}$. Then we define the length of $\gamma$, $|\gamma|$ by

$$|\gamma| = \int_{\gamma} |dz| = \int_{a}^{b} |\gamma'(t)| \, dt.$$ 

15.22 Example
The circle of radius $\rho > 0$, centred at $0$, and described positively is parametrised by the path $\gamma : [0, 2\pi] \to \mathbb{C} : t \mapsto \rho e^{it}$.
Therefore the length of the circle is $|\gamma|$ where

$$|\gamma| = \int_{\gamma} |dz| = \int_{0}^{2\pi} |\rho e^{it}| \, dt = \int_{0}^{2\pi} \rho \, dt = 2\pi \rho.$$ 

15.23 Theorem
Suppose that $\gamma$ is a piecewise-smooth path in $\mathbb{C}$ and that $f : \mathbb{C} \mapsto \mathbb{C}$ is continuous on $\gamma^*$. Suppose also that there exists $M \in \mathbb{R}$ such that, for all $z \in \gamma^*$, $|f(z)| < M$. Then

$$\left| \int_{\gamma} f(z) \, dz \right| \leq M |\gamma|. \quad \square$$

15.24 Example
Suppose that $\alpha$ is the line segment $[0, 1+i]$, that is, that $\alpha$ parametrises the line segment $[0, 1+i]$ positively, that is, from $0$ to $1+i$. Suppose also that $\beta$ is the path that consists of the line segment $[0, 1]$ followed by the line segment $[1, 1+i]$.
Evaluate each of the following integrals:

(i) $\int_{\alpha} z^2 \, dz,$  (ii) $\int_{\beta} z^2 \, dz,$
(iii) $\int_{\alpha} z \, dz,$  (iv) $\int_{\beta} z \, dz.$

\[ \begin{array}{c}
\text{We can parametrise } \alpha^* \text{ and } \beta^* \text{ as follows:} \\
\alpha : [0, 1] \to \mathbb{C} : t \mapsto (1+i)t; \\
\beta : [0, 2] \to \mathbb{C} : t \mapsto \begin{cases} 
 t & \text{if } 0 \leq t \leq 1 \\
 1 + i(t-1) & \text{if } 1 \leq t \leq 2.
\end{cases}
\end{array} \]
\[ (1+i)^3 \int_0^1 t^2 \, dt = \frac{(1+i)^3}{3}. \]

(ii) \[ \int_{\beta} z^2 \, dz = \int_0^1 t^2 \, dt + \int_1^2 [1+i(t-1)]^2 \, dt \]
\[ = \int_0^1 t^2 \, dt + \int_1^2 [1+i(t-1)]^2 i \, dt \]
\[ = \frac{1}{3} + \int_0^1 (1+iu)^2 i \, du \quad \text{where } u = t-1 \]
\[ = \frac{1}{3} + \left[ \frac{(1+iu)^3}{3} \right]_0 = \frac{(1+i)^3}{3}. \]

(iii) \[ \int_{\alpha} \overline{z} \, dz = \int_0^1 (1-i)t(1+i) \, dt \]
\[ = 2 \int_0^1 t \, dt = 1. \]

(iv) \[ \int_{\beta} \overline{z} \, dz = \int_0^1 t \, dt + \int_1^2 (1-i(t-1))i \, dt \]
\[ = \frac{1}{2} + i \int_0^1 (1-iu) \, du \quad \text{where } u = t-1 \]
\[ = \frac{1}{2} + i \left[ u - \frac{iu^2}{2} \right]_0 \]
\[ = \frac{1}{2} + i \left( 1 - \frac{i}{2} \right) = 1+i. \]

15.25 The function \( z \mapsto z^2 \) is holomorphic on \( \mathbb{C} \). The function \( z \mapsto \overline{z} \) is not holomorphic at any point of \( \mathbb{C} \).

15.26 If \( \alpha \) and \( \beta \) are as in Example 15.24 then
\[ \int_{\alpha} z^2 \, dz = \int_{\beta} z^2 \, dz = \left. \frac{(1+i)^3}{3} \right|^{1+i}_0 = \frac{z^3}{3}. \]

15.27 Suppose that \( \gamma = \beta \circ \bar{\alpha} \).

Then
\[ \int_{\gamma} z^2 \, dz = \int_{\beta} z^2 \, dz - \int_{\alpha} z^2 \, dz = 0. \]

and
\[ \int_{\gamma} \bar{z} \, dz = \int_{\beta} \bar{z} \, dz - \int_{\alpha} \bar{z} \, dz = 1+i - 1 = i. \]

15.28 Lemma
Suppose that \( \rho > 0 \), that \( z_0 \in \mathbb{C} \), and that
\[ \gamma: [0,2\pi] \to \mathbb{C} : t \mapsto z_0 + \rho e^{it}. \]
γ parametrises the circle of radius ρ, centred at z₀, and oriented positively. Then, for all \( m \in \mathbb{Z} \),

\[
\int_{\gamma} (z-z_0)^m \, dz = \begin{cases} 
2\pi i & \text{if } m = -1, \\
0 & \text{if } m \neq -1.
\end{cases}
\]

**Proof**

For all \( m \in \mathbb{Z} \),

\[
\int_{\gamma} (z-z_0)^m \, dz = \int_0^{2\pi} (\rho e^{it})^m \frac{d(z_0 + \rho e^{it})}{dt} \, dt = i\rho^{m+1} \int_0^{2\pi} \exp(i(m+1)t) \, dt.
\]

If \( m = -1 \) then

\[
\int_{\gamma} (z-z_0)^m \, dz = i \int_0^{2\pi} \, dt = 2\pi i.
\]

If \( m \neq -1 \) then

\[
\int_{\gamma} (z-z_0)^m \, dz = i\rho^{m+1} \left[ \exp((m+1)2\pi i) - \exp(0) \right] = 0. \quad \square
\]

[12.40]