13 Holomorphic Functions

13.1 Since \( \mathbb{C} \) is a metric space and, as such, is identical to \( \mathbb{E}_2 \), each of the following terms can be applied to \( \mathbb{C} \): open ball, open set, closed set, boundary, interior, exterior, closure, connected, path-connected, polygonally-connected.

13.2 Definition
Suppose that \( A \) is a non-empty subset of \( \mathbb{C} \), that \( f : A \rightarrow \mathbb{C} \), and that \( a \in A \). Then we say that \( f \) is continuous at \( a \) if and only if, for all \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that, for all \( z \in A \),

if \( |z - a| < \delta \) then \( |f(z) - f(a)| < \varepsilon \).

13.3 Since \( \mathbb{C} \) is a field, many of the theorems about real continuous functions make sense and are true for complex continuous functions. For example, if \( f \) and \( g \) are complex functions that are both continuous at \( a \in \mathbb{C} \) then \( f + g \), \( fg \), and, if \( g(a) \neq 0 \), \( f/g \) are continuous at \( a \).

Complex polynomial functions and rational functions are continuous as are the complex functions \( \exp, \cos, \) and \( \sin \).

13.4 The limit of a complex function is also defined in much the same way that the limit of a real function is defined. The limit of a function \( f \) at \( a \) may exist when \( a \notin \text{dom}(f) \) provided that \( f(z) \) is defined for values of \( z \) arbitrarily close to \( a \).

13.5 Definition
Suppose that \( f : A \rightarrow \mathbb{C} \) where \( \emptyset \neq A \subset \mathbb{C} \), that \( a \in \text{Cl}(A) \), and that \( l \in \mathbb{C} \). Then we say that the limit of \( f \) at \( a \) equals \( l \) and we write

\[
\lim_{z \to a} f(z) = l
\]

if and only if, for all \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that, for all \( z \in A \),

if \( 0 < |z - a| < \delta \) then \( |f(z) - l| < \varepsilon \).

13.6 Sometimes we write \( f(z) \to l \) as \( z \to a \) instead of \( \lim_{z \to a} f(z) = l \).

13.7 Since \( \mathbb{C} \) is a field, many of the theorems about real limits make sense and are true for complex limits. For example, a complex function \( f \) has at most one limit as \( z \to a \). Furthermore, if

\[
\lim_{z \to a} f(z) = l \quad \text{and} \quad \lim_{z \to a} g(z) = m
\]

then

(i) \( \lim_{z \to a} f(z) + g(z) = l + m \).

(ii) \( \lim_{z \to a} f(z)g(z) = lm \).

(iii) If \( m \neq 0 \) then \( \lim_{z \to a} \left( \frac{f(z)}{g(z)} \right) = \frac{l}{m} \).

13.8 Theorem
Suppose that \( f : \mathbb{C} \rightarrow \mathbb{C} \) and that \( a \in \text{dom}(f) \). Then \( f \) is continuous at \( a \) if and only if

\[
\lim_{z \to a} f(z) = f(a). \quad \square
\]
13.9 Definition
Suppose that \( A \) is an open subset of \( \mathbb{C} \), that \( f : A \to \mathbb{C} \), and that \( z \in A \). Then we say that \( f \) is differentiable at \( z \) if and only if the limit
\[
\lim_{h \to 0} \frac{f(z + h) - f(z)}{h}
\]
exists. If \( f \) is differentiable at \( z \) then we call this limit the derivative of \( f \) at \( z \) and we denote it by \( f'(z) \). Thus, if \( f \) is differentiable at \( a \) then
\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}.
\]

13.10 Since \( A \) is open, for all \( z \in A \) there is an open ball \( B(z; \rho) \subset A \). This implies that in 13.9 the number \( z + h \) can approach \( z \) in any direction and along any path through \( z \). Therefore if \( f \) is differentiable at \( z \) then the value of the limit
\[
\lim_{h \to 0} \frac{f(z + h) - f(z)}{h}
\]
always takes the same value, \( f'(z) \), if we calculate it as \( h \) approaches 0 along different paths to 0.

13.11 Theorem
If a complex function \( f \) is differentiable at \( a \) then it is continuous at \( a \). □

13.12 The well-known formulas for the derivative of a sum, a product, and a quotient are valid for complex functions. The derivative of a composite complex function is found by using the chain rule.

13.13 Lemma
Suppose that \( f : \mathbb{C} \to \mathbb{C} : z \mapsto \text{Re}(z) \). Then, for all \( z \in \mathbb{C} \), \( f \) is not differentiable at \( z \).

Proof
\[
\text{For all } z \text{ and } h \in \mathbb{C}, \text{ where } h \neq 0, \quad \frac{f(z + h) - f(z)}{h} = \frac{\text{Re}(z + h) - \text{Re}(z)}{h} = \frac{\text{Re}(h)}{h}.
\]
Suppose that \( h \) is a real number. Then \( \text{Re}(h) = h \) and, as \( h \to 0 \),
\[
\frac{f(z + h) - f(z)}{h} = h \to 1.
\]
Now suppose that \( h \) is a pure imaginary number, that is, that \( \text{Re}(h) = 0 \). Then, as \( h \to 0 \),
\[
\frac{f(z + h) - f(z)}{h} = 0 \to 0.
\]
Therefore
\[
\lim_{h \to 0} \frac{f(z + h) - f(z)}{h} = 0.
\]
Suppose that $z$ all derivatives differentiable on $A$. It can be shown that if $f$ such that, for all $z$

Furthermore, since $i k \to 0$ if and only if $k \to 0$, $f'(z) = \lim_{k \to 0} \frac{f(z+k) - f(z)}{k}$

and

Furthermore, for all $x + iy \in A$,

$$f'(x + iv) = \frac{\partial u}{\partial x} + iv \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}.$$  

**Proof** (in outline)  
**Suppose that** $z = x + iy \in A$. Then

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}.$$  

**Suppose that** $k \in \mathbb{R}$ **is such that** $k \neq 0$, $z + k \in A$, and $z + ik \in A$. **Then**, in view of 13.10,

$$f'(z) = \lim_{k \to 0} \frac{f(z+k) - f(z)}{k} = \lim_{k \to 0} \frac{f(x+k+iy) - f(x+iy)}{k} = \lim_{k \to 0} \frac{u(x+k,y) + iv(x+k,y) - [u(x,y) + iv(x,y)]}{k} = \lim_{k \to 0} \frac{u(x+k,y) - u(x,y)}{k} + i \lim_{k \to 0} \frac{v(x+k,y) - v(x,y)}{k} = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}.$$  

Furthermore, since $ik \to 0$ if and only if $k \to 0$,

$$f'(z) = \lim_{ik \to 0} \frac{f(z+ik) - f(z)}{ik}$$  

and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$  

This implies that the four partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

exist on $A \subset \mathbb{R}^2$.

**13.15 Theorem**

Suppose that $A$ is a non-empty open subset of $\mathbb{C}$, that $f : \mathbb{C} \to \mathbb{C}$ is differentiable on $A$, and that $f = u + iv$. Then, at every point of $A \subset \mathbb{R}^2$,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$  

(1)
\[
\lim_{k \to 0} f(x + i(y + k)) - f(x + iy) = f(x + iy)
\]

\[
= \lim_{k \to 0} u(x, y + k) + iv(x, y + k) - [u(x, y) + iv(x, y)]
\]

\[
= \frac{1}{i} \lim_{k \to 0} \frac{u(x, y + k) - u(x, y)}{k} + \lim_{k \to 0} \frac{v(x, y + k) - v(x, y)}{k}
\]

\[
= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}
\]

\[
= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}
\]

Equations (3) and (4) imply that

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.
\]

13.17 Equations (1) and (2) of Theorem 13.16 are known as the Cauchy-Riemann Equations.

13.18 Example

We have seen (12.23) that if \( f : z \to z^3 \) then

\[
f(x + iy) = u(x, y) + iv(x, y)
\]

where \( u(x, y) = x^3 - 3xy^2 \) and \( v(x, y) = 3x^2y - y^3 \).

\[
\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = 3x^2 - 3y^2 + i(6xy)
\]

(1)

13.19 Example

Suppose that \( f(z) = z \) and that \( f = u + iv \). Then

\[
f(x + iy) = x - iy
\]

and thus

\[
u(x, y) = x \quad \text{and} \quad v(x, y) = -y.
\]

Therefore

\[
\frac{\partial u}{\partial x} = 1 \quad \text{and} \quad \frac{\partial v}{\partial y} = -1.
\]

Therefore, for all \((x, y) \in \mathbb{R}^2\), \( u \) and \( v \) do not satisfy the Cauchy-Riemann equations at \((x, y)\). Therefore \( f \) is not differentiable at any point in \( \mathbb{C} \).

13.20 Example

Suppose that

\[
f : \mathbb{C} \to \mathbb{C} : z = x + iy \mapsto \begin{cases} 1 & \text{if } xy = 0, \\ 0 & \text{if } xy \neq 0. \end{cases}
\]
13.126

Suppose that \( f = u + iv \). Then \( v(x, y) = 0 \) and
\[ u(x, y) = f(x + iy). \]

\[ \left. \frac{\partial u}{\partial x} \right|_{(0,0)} = \lim_{h \to 0} \frac{u(0+h,0) - u(0,0)}{h} = \lim_{h \to 0} \frac{1 - 1}{h} = \lim_{h \to 0} 0 = 0. \]

Similarly,
\[ \left. \frac{\partial u}{\partial y} \right|_{(0,0)} = 0. \]

Therefore, since \( v(x, y) = 0 \), \( u \) and \( v \) satisfy the Cauchy-Riemann equations at \((0,0)\).

Let \( h = (1+i)k \) where \( k \in \mathbb{R} \). Then, for \( k \neq 0 \),
\[ \frac{f(0+h) - f(0)}{h} = \frac{f((1+i)k) - f(0)}{(1+i)k} = -\frac{1}{(1+i)k}. \]

Therefore
\[ \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} \]
does not exist. Therefore \( f \) is not differentiable at \( 0 \).

13.21 This example shows that the converse of Theorem 13.16 is not true. The following theorem, which we state without proof, is true.

13.22 Theorem
Suppose that \( A \) is a non-empty open subset of \( \mathbb{C} \), that \( f : A \to \mathbb{C} \), and that, for all \( x + iy \in A \),

(i) \( f(x + iy) = u(x, y) + iv(x, y) \) where \( u \) and \( v \) are real-valued functions;

(ii) \( \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \) and \( \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \);

(iii) All the partial derivatives of \( u \) and \( v \) are continuous on \( A \subset \mathbb{R}^2 \).

Then \( f \) is differentiable on \( A \). \( \square \)

13.23 Sometimes we denote the partial derivatives \( \frac{\partial u}{\partial x} \) and \( \frac{\partial u}{\partial y} \) by \( u_x \) and \( u_y \), respectively.

So far, we have used Leibniz notation for partial derivatives. But Leibniz notation is not suitable for denoting the value of a partial derivative at a specific point. It is simpler to write
\[ D_1 f(a, b) \]
rather than
\[ \left. \frac{\partial f}{\partial x} \right|_{(a, b)}. \]

13.24 Recall the Mean Value Theorem.
Suppose that \( a \) and \( b \in \mathbb{R} \) where \( a < b \) and that \( f \) is a function that has the following properties:

- \( f \) is continuous on the closed interval \([a, b]\);
• $f$ is differentiable on the open interval $(a, b)$; Then there exists $c \in \mathbb{R}$ such that
• $c \in (a, b)$, that is, $a < c < b$;
• $f(b) - f(a) = (b - a)f'(c)$.

13.25 Suppose that $A$ is a non-empty open subset of $\mathbb{R}^2$, that $a = (b, c) \in A$, and that $h = (k, l) \in \mathbb{R}^2$ is such that the line segment $[a, a + h] \subset A$.

Suppose that $u : A \to \mathbb{R}$ is differentiable on $A$ and suppose that
$$
\phi : [0, 1] \to \mathbb{R}^2 : t \mapsto a + th.
$$
Suppose also that
$$
g = u \circ \phi : [0, 1] \to \mathbb{R} : t \mapsto u(a + th) = u(b + tk, c + tl).
$$
Then $\phi$ is continuous on $[0, 1]$ and differentiable on $(0, 1)$. Therefore the Mean Value Theorem implies that there exists $0 < \lambda < 1$ such that
$$
g(1) - g(0) = g'(\lambda)
$$
(1)

By the Chain Rule,
$$
g'(t) = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}
= k \cdot D_1 u(a + th) + l \cdot D_2 u(a + th).
$$
(2)

Statements (1) and (2) imply that there exists $0 < \lambda < 1$ such that
$$
 u(a + h) - u(a) = k \cdot D_1 u(a + \lambda h) + l \cdot D_2 u(a + \lambda h).
$$
(3)
Now suppose that, for all $(x, y) \in A$, $D_1 u(x, y) = D_2 u(x, y) = 0$.
In other words, suppose that both (first-order) partial derivatives of $u$ vanish on $A$.
Then statement (3) implies that $u(a + h) = u(a)$.

We have proved that if both (first-order) partial derivatives of $u$ vanish on $A$ then $u$ has the same value on any two points of $A$ that are connected by a line-segment in $A$. 
Therefore if both partial derivatives of $u$ vanish on $A$ then if $a$ and $b \in A$ are connected by a polygonal path in $A$ then $u(a) = u(b)$.

We can now invoke 11.21 to prove the following lemma.

**13.26 Lemma**

Suppose that $A$ is a non-empty open connected subset of $\mathbb{R}^2$, that is, of $\mathbb{E}_2$. Suppose also that $u : A \to \mathbb{R}$ is such that both of its partial derivatives are defined on $A$. Then if both partial derivatives of $u$ vanish on $A$ then $u$ is constant on $A$. $\square$

**13.27 Theorem**

Suppose that $A$ is a non-empty open connected subset of $\mathbb{C}$ and that $f : A \to \mathbb{C}$ is differentiable on $A$. Then if, for all $z \in A$, $f'(z) = 0$ then $f$ is constant on $A$.

**Proof**

Suppose that $f = u + iv$. By Theorem 13.16, for all $x + iy \in A$,

$$f'(x + iy) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}.$$  

Therefore, on $A \subset \mathbb{R}^2$,

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = 0 \quad \text{and} \quad \frac{\partial v}{\partial y} + i \frac{\partial u}{\partial y} = 0.$$  

Therefore

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0 \quad \text{and} \quad \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0.$$  

Lemma 13.26 implies that $u$ and $v$ are constant on $A$. Therefore $f$ is constant on $A$. $\square$

13.28 From now on we shall call a non-empty open connected subset of $\mathbb{C}$ a region of $\mathbb{C}$ and we shall use the word “region” only in this sense.

**13.29 Definition**

A function $f : \mathbb{C} \to \mathbb{C}$ is said to be holomorphic at $a \in \text{dom}(f)$ if and only if there exists $\rho > 0$ such that

(i) $B(a; \rho) \subset \text{dom}(f)$;

(ii) $f$ is differentiable on $B(a; \rho)$.

13.30 A function $f$ is holomorphic on a set $A$ if and only if, for all $z \in A$, $f$ is holomorphic at $z$. If $A$ is open then $f$ is holomorphic on $A$ if and only if $f$ is differentiable on $A$.

13.31 Some authors use regular or analytic instead of holomorphic.

13.32 From now on when we say that a function $f$ is holomorphic on a set $A$ we shall take it to imply that $A$ is a region. Notice that if $A$ is a region then $f$ is differentiable on $A$ if and only if it is holomorphic on $A$.

13.33 If $A$ is an open disconnected subset of $\mathbb{C}$ then $A$ has an open partition. We can study a function $f$ that is differentiable on $A$ by studying the restriction of $f$ to each set in this partition.
13.34 Because we usually demand that the domain of a holomorphic function is a region some authors call a region a domain.

13.35 Corollary
Suppose that $f : A \to \mathbb{C}$ is a holomorphic function that takes only real values. Then $f$ is a constant function.

Proof
Suppose that $f = u + iv$. Since $f$ takes only real values, for all $z = x + iy \in A$, $v(x, y) = 0$. Since $f$ is differentiable on $A$, $u$ and $v$ satisfy the Cauchy-Riemann equations on $A$. Therefore, for all $(x, y) \in A$,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 0.$$

Therefore Theorem 13.27 implies that $u$ is constant on $A$. Therefore $f$ is constant on $A$. 