8 The Distance from a Point to a Set

8.1 Recall that the Supremum Axiom, also known as the Axiom of Completeness, states that any non-empty set of real numbers that is bounded above has a least upper bound or supremum. A corollary of this axiom states that any non-empty set of real numbers that is bounded below has a greatest lower bound or infimum.

8.2 Definition
Suppose that \( \emptyset \neq A \subset M \) and that \( x \in M \). Then the set of all distances from \( x \) to a point in \( A \) is bounded below by \( 0 \). Therefore we can define the distance of \( x \) from \( A \), \( d(x;A) \), to be the greatest lower bound of the set of distances from \( x \) to a point in \( A \). That is,

\[
d(x;A) = \inf \{ d(x,a) \mid a \in A \}.
\]

8.3 \( d(x;A) \), as defined above, is sometimes called the least distance of \( x \) from \( A \).

8.4 Lemma
(i) For all \( x,y \in M \), \( d(x; \{ y \}) = d(x,y) \).

(ii) If \( \emptyset \neq A \subset B \subset M \) then, for all \( x \in M \), \( d(x;A) \geq d(x;B) \).

(iii) If \( a \in A \subset M \) then \( d(a;A) = 0 \).

Proof Exercise. \( \square \)

8.5 Problem
Evaluate each of the following distances in \( \mathbb{E}_1 \):

(i) \( d(0; [1,-\to)) \) where \( [1,-\to) = \{ x \in \mathbb{R} \mid x \geq 1 \} \),

(ii) \( d(0; (1,-\to)) \) where \( (1,-\to) = \{ x \in \mathbb{R} \mid x > 1 \} \),

(iii) \( d(\sqrt{2}; \mathbb{Q}) \).

\( \square \)

(i) \( d(0; [1,-\to)) = 1 \),

For all \( x \in [1,-\to) \), \( d(0,x) = |x-0| = |x| = x \). Therefore \( \{ d(0,x) \mid x \in [1,-\to) \} = [1,-\to) \). We must show that \( \inf[1,-\to) = 1 \).

But 1 is the minimum of \( [1,-\to) \).

Therefore \( \inf[1,-\to) = 1 \), that is, \( d(0,[1,-\to)) = 1 \).

(ii) \( d(0; (1,-\to)) = 1 \),

Here we must show that \( \inf(1,-\to) = 1 \). Clearly, since 1 is a lower bound of \( (1,-\to) \),

\[
\inf(1,-\to) \geq 1.
\]

Suppose that \( \epsilon > 0 \). Then \( 1+\epsilon \in (1,-\to) \) and thus every lower bound of \( (1,-\to) \) must be less than or equal to \( 1+\epsilon \). Therefore

\[
\inf(1,-\to) \leq 1+\epsilon.
\]

Since inequality (2) is true for all \( \epsilon > 0 \),

\[
\inf(1,-\to) \leq 1.
\]

Inequalities (1) and (3) imply that

\[
d(0;(1,-\to)) = \inf(1,-\to) = 1.
\]
8.72

(iii) \( d(\sqrt{2}; \mathbb{Q}) = 0. \)

Although \( \sqrt{2} \) is not a rational number we can always find a rational number that is as close to \( \sqrt{2} \) as we wish.

8.6 Notice that \( d(0; [1, \to)) = d(0, 1) = 1 \), that is, the distance from 0 to \([1, \to)\) is the smallest value of \( d(0, x) \) where \( x \in [1, \to) \).

It is also true that \( d(0; (1, \to)) = 1 \) but there is no \( x \in (0, \to) \) such that \( d(0, x) = 1 \).

8.7 Theorem
Suppose that \( \emptyset \neq A \subset M \). Then, for all \( x \in M \), \( d(x; A) = 0 \) if and only if \( x \in \text{Cl}(A) \).

Proof

(i) Suppose that \( x \in \text{Cl}(A) \) that is, \( x \in A \) or \( x \in \text{Bd}(A) \).

Lemma 8.4 (iii) implies that if \( x \in A \) then \( d(x; A) = 0 \). So we can take \( x \) to be a boundary point of \( A \).

Let \( \epsilon > 0 \). Since \( x \) is a boundary point of \( A \), there exists \( y(\epsilon) \in A \) such that \( y(\epsilon) \in B(x; \epsilon) \), that is, there exists \( y(\epsilon) \in A \) such that \( d(x, y(\epsilon)) < \epsilon \). Therefore

\[
d(x; A) = \inf \{ d(x, z) \mid z \in A \} < \epsilon \tag{1}
\]

Since (1) is true for all \( \epsilon > 0 \), \( d(x; A) = 0 \).

(ii) Suppose that \( x \notin \text{Cl}(A) \).

Theorem 5.39 implies that \( x \) is an exterior point of \( A \).

Therefore there exists \( \rho > 0 \) such that \( B(x; \rho) \cap A = \emptyset \).

Therefore, for all \( a \in A \), \( d(x, a) \geq \rho \).

Therefore

\[
d(x; A) = \inf \{ d(x, a) \mid a \in A \} \geq \rho > 0. \tag{\text{□}}
\]

8.8 Corollary
If \( A \) is a non-empty closed subset of \( M \) then, for all \( x \in M \), \( d(x; A) = 0 \) if and only if \( x \in A \). \tag{\text{□}}

8.9 Recall if \( A \) is a subset of \( M \) then

\[
\text{Bd}(A) = \text{Bd}(M \setminus A). \tag{5.37}
\]

Therefore

\[
\text{Cl}(A) \cap \text{Cl}(M \setminus A) = [\text{Int}(A) \cup \text{Bd}(A)] \cap [\text{Int}(M \setminus A) \cup \text{Bd}(M \setminus A)]
\]

\[
= [\text{Int}(A) \cup \text{Bd}(A)] \cap [\text{Ext}(A) \cup \text{Bd}(A)]
\]

\[
= \text{Bd}(A).
\]

In view of Theorem 8.7, this yields the following theorem.

8.10 Theorem
Suppose that \((M, d)\) is a metric space and that \( A \) is a subset of \( M \). Then \( x \in M \) is a boundary point of \( A \) if and only if

\[
d(x; A) = d(x; M \setminus A) = 0. \tag{\text{□}}
\]

8.11 This theorem gives us another definition of a boundary point.
Suppose that $(M,d)$ is a metric space and that $A \subset M$. Then $x \in M$ is a boundary point of $A$ if and only if the distance from $x$ to $A$ and the distance from $x$ to the complement of $A$ are both equal to zero.