6 Continuous Functions

6.1 Suppose that \( f : M \rightarrow N \) where \( M \) and \( N \) are metric spaces. Suppose also that \( A \) is a non-empty subset of \( \text{dom}(f) \). Then we say that \( f \) is continuous on \( A \) if and only if, for all \( a \in A \), \( f \) is continuous at \( a \). If \( f \) is continuous on \( \text{dom}(f) \) then we say that \( f \) is continuous.

6.2 Definition
Suppose that \( f : A \rightarrow B \) and that \( X \) is a set. Then we define \( f(X) \), the image of \( X \) by \( f \) by
\[
 f(X) = \{ y \in B \mid \text{for some } x \in X \cap A, y = f(x) \}.
\]
In other words, \( f(X) \) is the set of all the values that \( f \) assigns to those elements of \( X \) on which \( f \) is defined.

6.3 If \( f : A \rightarrow B \) and \( X \cap A = \emptyset \) then \( f(X) = \emptyset \).

6.4 Example
Suppose that \( f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto x^2 \). Then
- \( f(\mathbb{R}) = [0, \rightarrow) \);
- \( f[1,2] = f([1,2]) = [1,4] \);
- \( f[-1,3] = [0,9] \).

6.5 Definition
Suppose that \( f : A \rightarrow B \) and that \( Y \) is a set. Then we define \( f^{-1}(Y) \), the inverse image of \( Y \) by \( f \) by
\[
 f^{-1}(Y) = \{ x \in A \mid f(x) \in Y \}.
\]
In other words, \( f^{-1}(Y) \) is the set of all elements of \( A \) to which \( f \) assigns values in \( Y \).

6.6 If \( f : A \rightarrow B \) and \( f(A) \cap Y = \emptyset \) then \( f^{-1}(Y) = \emptyset \).

6.7 Example
Suppose that \( f : \mathbb{R} \rightarrow \mathbb{R} : x \mapsto x^2 \). Then
- for all \( a \in (0, \rightarrow) \), \( f^{-1}([a]) = \{-\sqrt{a}, \sqrt{a}\} \);
- \( f^{-1}[0,9] = [-3,3] \);
Recall that in 5.25 we presented the following definition: Suppose that \( f : X \rightarrow Y \) where \((X, d)\) and \((Y, d')\) are metric spaces, and that \( a \in \text{dom}(f) \). Then \( f \) is continuous at \( a \) if and only if, for every open ball \( B' \subset Y \) centred at \( f(a) \) there exists an open ball \( B \subset X \) centred at \( a \) such that,

\[
\text{for all } x \in \text{dom}(f), \text{ if } x \in B \text{ then } f(x) \in B'.
\]

6.8 Theorem
Suppose that \((M, d_1)\) and \((N, d_2)\) are metric spaces and that \( f : M \rightarrow N \). Then \( f \) is continuous (on \( M \)) if and only if, for every open subset \( V \) of \( N \), the set \( f^{-1}(V) \) is an open subset of \( M \).

Proof

(i) Suppose that \( f \) is continuous and that \( V \) is an open subset of \( N \).

Let \( x \in f^{-1}(V) \), that is, suppose that \( x \in M \) is such that \( f(x) \in V \). We shall show that \( x \) is an interior point of \( f^{-1}(V) \) thus proving that \( f^{-1}(V) \) is open.

Let \( y = f(x) \). Since \( V \) is open and \( y \in V \), there exists \( \varepsilon > 0 \) such that

\[
B(y; \varepsilon) \subset V. \tag{1}
\]

Since \( f \) is continuous at \( x \), there exists \( \delta > 0 \) such that, for all \( u \in M \),

\[
u \in B(x; \delta) \implies f(u) \in B(y; \varepsilon)\]

Therefore \( B(x; \delta) \subset f^{-1}(V) \). Therefore \( x \) is an interior point of \( f^{-1}(V) \).

(ii) Suppose that the inverse image of every open subset of \( N \) by \( f \) is open in \( M \).

Let \( x \in M \). We shall show that \( f \) is continuous at \( x \) and thus continuous on \( M \).

Let \( B' \) be any open ball in \( N \) that is centered at \( f(x) \). Let

\[
U = f^{-1}(B').
\]

Since \( f(x) \in B' \), \( x \in U \).

Since \( B' \) is an open subset of \( N \), \( U \) is an open subset of \( M \).

Since \( x \in U \) and \( U \) is open, \( x \in \text{Int}(U) \). Therefore there is an open ball \( B \) centred at \( x \) that is a subset of \( U \). Furthermore, for all \( z \in M \),

\[
z \in B \implies z \in U \implies f(z) \in B'.
\]

Therefore \( f \) is continuous at \( x \). \qed

6.10 Corollary
Suppose that \((M, d)\) is a metric space and that \( f : M \rightarrow \mathbb{E}_1 \) is continuous. Then, for all \( a \in \mathbb{R} \), \( \{x \in M : f(x) < a\} \) is an open subset of \( M \).

Proof
For all $a \in \mathbb{R}$, $(\leftarrow, a)$ is an open subset of $\mathbb{E}_1$. Now

\[
\{ x \in M \mid f(x) < a \} = \{ x \in M \mid f(x) \in (\leftarrow, a) \}
= f^{-1}(\leftarrow, a)
\]

Therefore, since $f$ is continuous, Theorem 6.9 implies that

\[
\{ x \in M \mid f(x) < a \}
\]

is an open subset of $M$. \hfill \Box

6.11 A closed interval of finite length is called a compact interval. If $f : \mathbb{R} \to \mathbb{R}$ is continuous on the compact interval $[a, b]$ then $f$ has both a minimum point and a maximum point in $[a, b]$.

In the above diagram $a$ is a maximum point and $c$ is a minimum point of $f$ on $[a, b]$.

Can we extend the definition of compact to subsets of a metric space so that if $(M, d)$ is a metric space, $f : M \to \mathbb{R}$ is continuous, and $A$ is a compact subset of $M$ then $f$ has both a minimum point and a maximum point on $A$?