5 Closed Sets and Open Sets

5.1 Recall that
\[(0, 1] = \{ x \in \mathbb{R} \mid 0 < x \leq 1 \}.\]
Suppose that, for all \(n \in \mathbb{N}\), \(a_n = 1/n\). Then \((a_n)\) is an infinite sequence in \((0, 1]\) that converges in \(E^1\) but its limit 0 does not belong to \((0, 1]\). Thus \((0, 1]\) is not closed under taking the limit of a convergent sequence.

5.2 Definition
Suppose that \((M, d)\) is a metric space. Then a set \(F \subset M\) is said to be closed (in \((M, d)\)) if and only if the limit of every infinite sequence in \(F\) that converges in \((M, d)\) is an element of \(F\).

5.3 From now on, unless we say otherwise,
- we shall always take \((M, d)\) to be a metric space;
- we shall always take the terms sequence and subsequence to mean infinite sequence and infinite subsequence, respectively.

5.4 Theorem
If \((a_n)\) is a sequence in \(M\) that converges to \(a \in M\) then every subsequence of \((a_n)\) converges to \(a\).

Proof Exercise. □

5.5 Lemma
If \(F\) and \(G \subset M\) are closed then \(F \cup G\) is closed.

Proof
Suppose that \(F\) and \(G \subset M\) are closed, that \((a_n) \subset F \cup G\), that is, that \((a_n)\) is a sequence in \(F \cup G\), and that \(\lim_n(a_n) = a\). We must prove that \(a \in F \cup G\).

We consider two possibilities, one of which must be true.
(i) There are only a finite number of terms of \((a_n)\) in \(F\). This implies that there is a subsequence of \((a_n)\) in \(G\). By Theorem 5.4, this subsequence converges to \(a\). Therefore, since \(G\) is closed, \(a \in G\) and thus \(a \in F \cup G\).
(ii) There is a subsequence of \((a_n)\) in \(F\). Since \(F\) is closed, Theorem 5.4 implies that \(a \in F\) and thus that \(a \in F \cup G\). Therefore \(F \cup G\) is closed. □

5.6 The only way to show directly that a subset \(X\) of \(M\) is not closed is to show that there is a convergent sequence in \(X\) whose limit does not belong to \(X\).

5.7 Theorem
Suppose that \((M, d)\) is a metric space.

(i) \(M\) and \(\emptyset\) are closed subsets of \(M\).
(ii) The union of any finite collection of closed subsets of \(M\) is closed.
(iii) The intersection of any collection of closed sets is closed.
5.40

Proof

(i) $M$ is obviously closed. In view of 5.6 it is easy to see that $\emptyset$ is closed.

(ii) Exercise. Use Lemma 5.5 and induction on the number of subsets.

(iii) Suppose that $\Lambda$ is a non-empty set and that, for all $\lambda \in \Lambda$, $F_\lambda$ is a closed subset of $M$. Let

$$F = \bigcap \{ F_\lambda \mid \lambda \in \Lambda \}.$$ 

Suppose that $(a_n)$ is a sequence in $F$ and that

$$\lim_{n} (a_n) = a.$$ 

Now $(a_n) \subset F$ means that, for all $\lambda \in \Lambda$, $(a_n) \subset F_\lambda$.

Therefore, for all $\lambda \in \Lambda$, since $F_\lambda$ is closed, $a \in F_\lambda$.

Therefore $a \in F$. Therefore $F$ is closed. □

5.8 Lemma

Any singleton in $M$ is a closed set.

Proof The only sequence in a singleton is constant and thus converges to a limit in the singleton. □

5.9 Corollary

Any finite subset of $M$ is closed.

Proof A finite set is a finite union of singletons. □

5.10 Example

For all $n \in \mathbb{N}$, the singleton $\{1/n\}$ is a closed subset of $\mathbb{E}_1$. Let

$$X = \bigcup_{n \in \mathbb{N}} \left\{ \frac{1}{n} \right\}.$$ 

Then, since the sequence $(1/n) \subset X$ but $0 \notin X$, $X$ is not closed. This example shows that it is not always true that the union of an infinite collection of closed sets is closed.

5.11 Definition

Suppose that $a \in M$ and that $\rho > 0$. Then we define $B(a; \rho) \subset M$ by

$$B(a; \rho) = \{ x \in M \mid d(x,a) < \rho \}.$$ 

5.12 The elements of $\mathbb{E}_1$ are real numbers and, for all $x \in \mathbb{E}_1$, $\|x\| = |x|$. Therefore, for all $a, \rho \in \mathbb{R}$, where $\rho > 0$,

$$B(a; \rho) = \{ x \in \mathbb{R} \mid |x - a| < \rho \},$$ 

that is, $B(a; \rho)$ is the open interval of half-length $\rho$ centred at $a$.

5.13 The elements of $\mathbb{E}_2$ represent points in the plane. If $a$ and $x \in \mathbb{E}_2$ then $\|x - a\|$ is the distance from $a$ to $x$. Thus the $B(a; \rho) \subset \mathbb{E}_2$ is the set of all points whose distance from $a$ is strictly less than $\rho$, that is, $B(a; \rho)$ is the set of points that lie inside but not on the circle of radius $\rho$ centred at $a$.

5.14 In the following diagram a broken line is used to represent the circle $\|x - a\| = \rho$ to show that it is not part of $B(a, \rho)$. 
5.15 In any diagram that represents a set in $\mathbb{E}_2$—the only metric space in which diagrams can be easily drawn—we shall use a broken curve to show that the points on the circumference of the set do not belong to the set and an unbroken curve to show that they do belong to the set.

5.16 In $\mathbb{E}_3$ $B(a; \rho)$ is the set of points that lie inside but not on the hollow sphere of radius $\rho$ centred at $a$.

5.17 Definition
Suppose that $A \subset M$. Then $a \in M$ is said to be an interior point of $A$ if and only if there exists $\rho > 0$ such that $B(a; \rho) \subset A$.

5.18
(i) Clearly if $a$ is an interior point of $A$ then $a \in A$.
(ii) If $a$ is an interior point of $A$ then we say that $a$ is strictly inside $A$.

5.19 Definition
$U \subset M$ is said to be open if and only if every point of $U$ is an interior point of $U$.

5.20 It is easy to see that $U \subset M$ is open if and only if, for all $u \in U$, there exists $\rho > 0$ such that $B(u; \rho) \subset U$.

5.21 It is not true that if a subset of a metric space is not open then it must be closed. In general, most subsets of a metric space are neither open nor closed. For example, consider the interval
$$I = (0, 1] = \{ x \in \mathbb{R} \mid 0 < x \leq 1 \}.$$ as a subset of the metric space $\mathbb{E}_2$. We have seen already that $I$ is not closed. It is easy to see that no nbd of 1 is a subset of $I$ and therefore $I$ is not an interior point of $I$. Therefore $I$ is not open.

5.22 Theorem
For all $a \in M$ and $\rho > 0$, $B(a; \rho)$ is open.

Proof

Suppose that $b \in B(a; \rho)$. Let
$$\delta = \rho - d(b, a).$$ Since $b \in B(a; \rho)$, $\delta > 0$. 
Let $x \in B(b; \delta)$. Then, by M4,

\[ d(x, a) \leq d(x, b) + d(b, a) < \delta + d(b, a) = \rho \]

Therefore $x \in B(a; \rho)$. Therefore $B(b; \delta) \subset B(a; \rho)$.

Therefore $b$ is an interior point of $B(a; \rho)$. Therefore $B(a; \rho)$ is open. □

5.23 For all $a \in M$, $\rho > 0$, the set $B(a; \rho)$ is called the open ball of radius $\rho$ centred at $a$.

5.24 $U \subset M$ is open if and only if every point of $U$ is the centre of an open ball that is contained in $U$.

5.25 Lemma

(i) Suppose that $(a_n)$ is a sequence in $M$. Then $(a_n)$ converges to $a$ if and only if, for every open ball $B$ centred at $a$, there exists $N \in \mathbb{N}$ such that,

\[ \forall n \in \mathbb{N}, \text{ if } n \geq N \text{ then } a_n \in B. \]

(ii) Suppose that $f : X \to Y$ where $(X, d)$ and $(Y, d')$ are metric spaces, and that $a \in \text{dom}(f)$. Then $f$ is continuous at $a$ if and only if, for every open ball $B' \subset Y$ centred at $f(a)$ there exists an open ball $B \subset X$ centred at $a$ such that,

\[ \forall x \in \text{dom}(f), \text{ if } x \in B \text{ then } f(x) \in B'. \]

Proof Exercise. □

5.26 Lemma
If $U$ and $V \subset M$ are open then $U \cap V$ is open.

Proof

Let $x \in U \cap V$.

Since $U$ is open, there exists $\alpha > 0$ such that $B(x; \alpha) \subset U$.

Since $V$ is open, there exists $\beta > 0$ such that $B(x; \beta) \subset V$.

Let $\rho = \min \{ \alpha, \beta \}$. Then $\rho > 0$ and, for all $y \in M$,

\[ d(y, x) < \rho \text{ implies that } d(y, x) < \alpha \text{ and } d(y, x) < \beta, \]

that is

\[ y \in B(x; \rho) \text{ implies that } y \in B(x; \alpha) \text{ and } y \in B(x; \beta). \]

Therefore

\[ B(x; \rho) \subset B(x; \alpha) \cap B(x; \beta) \subset U \cap V. \]

Therefore $U \cap V$ is open. □

5.27 The only way to show directly that a subset $X$ of $M$ is not open is to show that there exists $x \in X$ such that no open ball centred at $x$ is a subset of $X$.

5.28 Theorem
Suppose that $(M, d)$ is a metric space.

(i) $M$ and $\emptyset$ are open subsets of $M$.

(ii) The union of any collection of open sets is open.
(iii) The intersection of any finite collection of open subsets of $M$ is open.

**Proof**

(i) $M$ is obviously open. In view of 5.27 it is easy to see that $\emptyset$ is open.

(ii) Suppose that $\Lambda$ is a non-empty set and that, for all $\lambda \in \Lambda$, $U_\lambda$ is an open subset of $M$. Let

$$U = \bigcup \{ U_\lambda \mid \lambda \in \Lambda \}.$$

Suppose that $u \in U$. Then there exists $\kappa \in \Lambda$ such that $u \in U_\kappa$. Since $U_\kappa$ is an open set, there exists $\rho > 0$ such that $B(u, \rho) \subset U_\kappa$. Therefore, since $U_\kappa \subset U$, $B(u; \rho) \subset U$. Therefore $U$ is open.

(iii) Exercise. Use Lemma 5.26 and induction on the number of subsets. □

**5.29 Lemma**

Suppose that $(a_n)$ is a sequence in $M$. Then $(a_n)$ converges to $a$ in $(M, d)$ if and only if $(d(a_n, a))$ converges to 0 in $\mathbb{E}_1$, that is,

$$\lim_{n} (a_n) = a \text{ if and only if } \lim_{n} (d(a_n, a)) = 0.$$

**Proof** Exercise. □

**5.30 Theorem**

$U \subset M$ is open if and only if $M \setminus U$ is closed.

**Proof** Let $F = M \setminus U = \{ x \in M \mid x \notin U \}$.

(i) We shall prove that if $U$ is open then $F$ is closed by proving that if $F$ is not closed then $U$ is not open.

Suppose that $F$ is not closed. This implies that there is a sequence $(a_n) \subset F$ that converges to $a$ where $a \notin F$, that is, where $a \in U$. Since $(a_n)$ converges to $a$, Lemma 5.25 implies that every open ball centred at $a$ contains terms of the sequence $(a_n)$, that is, contains points of $F$. Therefore there is no open ball centred at $a$ that is a subset of $U$. Therefore $U$ is not open.

(ii) We shall prove that if $F$ is closed then $U$ is open by proving that if $U$ is not open then $F$ is not closed.

Suppose that $U$ is not open. This implies that there exists $u \in U$ such that no open ball centred at $u$ is a subset of $U$, that is, that every open ball centred at $u$ contains at least one point of $F$. Therefore, for all $n \in \mathbb{N}$, there exists $a_n \in F$ such that $a_n \in B(u; 1/n)$. Therefore,

$$0 < d(a_n, u) < \frac{1}{n}.$$

Since $\lim_{n} (1/n) = 0$, the “sandwich rule” or “squeezing rule” for sequences of real numbers implies that

$$\lim_{n} (d(a_n, u)) = 0.$$

Therefore, by Lemma 5.29,

$$\lim_{n} (a_n) = u$$

(1)

Since $(a_n) \subset F$ and $u \notin F$, equation (1) implies that $F$ is not closed. □
5.31 Definition
Suppose that $A \subset M$.

(i) The interior of $A$, $\text{Int}(A)$, is the set of all the interior points of $A$.

(ii) A point $x \in M$ is said to be an exterior point of $A$ if and only if there is an open ball $B$ centred at $x$ such that $B \cap A = \emptyset$.

(iii) The exterior of $A$, $\text{Ext}(A)$, is the set of all the exterior points of $A$.

5.32 Theorem
For all $A \subset M$,

(i) $\text{Int}(A) \subset A$;

(ii) $\text{Ext}(A) \subset M \setminus A$;

(iii) $\text{Int}(M \setminus A) = \text{Ext}(A)$.

(iv) $\text{Ext}(M \setminus A) = \text{Int}(A)$;

(v) $A$ is open if and only if $\text{Int}(A) = A$. □

5.33 Example
In the above diagram $A$ is a subset of $\mathbb{E}^2$, $a$ is an interior point of $A$, and $b$ is an exterior point of $A$.

5.34 Definition
Suppose that $A \subset M$ and that $x \in M$. Then $x$ is a boundary point of $A$ if and only if every open ball centred at $x$ contains at least one point that belongs to $A$ and at least one point that does not belong to $A$.

5.35 Definition
Suppose that $A \subset M$. The boundary of $A$, $\text{Bd}(A)$, is the set of all the boundary points of $A$.

5.36 Some authors call the boundary of a set, as defined above, the frontier of that set and give boundary a different meaning.

5.37 Lemma
For all $A \subset M$, $\text{Bd}(A) = \text{Bd}(M \setminus A)$. □
5.38 Example
In the above diagram $c$ and $d$ are both boundary points of $A$. Notice that $c \in A$ and $d \notin A$.

5.39 Theorem
Suppose that $(M,d)$ is a metric space and that $A$ is a subset of $M$. Then $\{ \text{Int}(A), \text{Ext}(A), \text{Bd}(A) \}$ is a partition of $M$.

Proof
Suppose that $x \in M$. Then one and only one of the following two statements is true:

(i) there is an open ball centred at $x$ that is a subset of $A$;
(ii) every open ball centred at $x$ contains a point of $M \setminus A$.

Given that (ii) is true, one and only one of the following two statements is true.

(iia) there is an open ball centred at $x$ that is a subset of $M \setminus A$;
(iib) every open ball centred at $x$ contains a point of $M \setminus A$ and a point of $A$.

Therefore one and only one of the following three statements is true:

$x \in \text{Int}(A), x \in \text{Ext}(A), x \in \text{Bd}(A)$. □

5.40 Corollary
For all $A \subset M$, $A$ is open if and only if $A \cap \text{Bd}(A) = \emptyset$.

Proof
Suppose that $A \subset M$. Then $A$ is open $\iff A = \text{Int}(A)$

$\iff A \cap [\text{Bd}(A) \cup \text{Ext}(A)] = \emptyset$

$\iff [A \cap \text{Bd}(A)] \cup [A \cap \text{Ext}(A)] = \emptyset$

$\iff A \cap \text{Bd}(A) = \emptyset$. □

5.41 Corollary
For all $A \subset M$, $A$ is closed if and only if $\text{Bd}(A) \subset A$.

Proof
Suppose that $A \subset M$. Then $A$ is closed $\iff M \setminus A$ is open

$\iff [M \setminus A] \cap \text{Bd}(M \setminus A) = \emptyset$

$\iff [M \setminus A] \cap \text{Bd}(A) = \emptyset$. □
5.42 Definition
Suppose that $A \subset M$. The closure of $A$, $\text{Cl}(A)$, is the union of $A$ and its boundary, that is, $\text{Cl}(A) = A \cup \text{Bd}(A)$.

5.43 $A \subset M$ is open if and only if it contains none of its boundary points and closed if and only if it contains all of them.

5.44 Corollary
For all $A \subset M$, $A$ is closed if and only if $A = \text{Cl}(A)$. □

5.45 Example
Suppose that $(H, d)$ is a metric subspace of $\mathbb{R}^2$ where

$$H = \{ (x, y) \in \mathbb{R}^2 | x \geq 0 \}.$$  

Suppose that

$$V = \{ x \in H | d(x, 0) < 1 \}.$$  

Then, in the subspace $H$, $V = B(0, 1)$.

If $V$ is treated a subset of $\mathbb{R}^2$ then $V$ is not open since, for example, $(0, 0)$ clearly belongs to both $V$ and the boundary of $V$. If, however, $V$ is treated as a subset of $H$ then $V$ is open since it is an open ball.

5.46 Theorem
Suppose that $(S, d)$ is a metric subspace of $(M, d)$ and that $A$ is a subset of $S$. Then $A$ is open in $S$ if and only if there is an open subset $U$ of $M$ such that $A = U \cap S$. 
Proof
(i) Suppose that $A = U \cap S$ where $U$ is open in $(M, d)$. Let $a \in A$. Since $U$ is open in $M$, $a \in \text{Int}(U)$, that is, there exists $\rho > 0$ such that

$$B(a; \rho) = \{x \in M \mid d(x, a) < \rho\} \subset U.$$ 

Let $B'(a; \rho) = B(a; \rho) \cap S$. Then

$$B'(a; \rho) \subset U \cap S = A. \quad (1)$$

But $B'(a; \rho) = \{x \in S \mid d(x, a) < \rho\}$ is an open ball in $S$. Therefore (1) implies that $a \in \text{Int}(A)$ (in $S$). Therefore $A$ is open in $S$.

(ii) Now suppose that $A$ is open in $S$ and that $a \in A$. Since $a \in \text{Int}(A)$, there exists $\rho_a > 0$ such that, in $S$,

$$B_a = B(a; \rho_a) = \{x \in S \mid d(x, a) < \rho_a\} \subset A.$$ 

Now $B_a = S \cap B'_a$ where, in $M$,

$$B'_a = B(a; \rho_a) = \{x \in M \mid d(x, a) < \rho_a\}.$$ 

Since $B'_a$ is an open ball in $M$, $B'_a$ is open in $M$. Let

$$U = \bigcup \{B'_a \mid a \in A\}.$$ 

Then, by Theorem 5.28 (ii), $U$ is an open subset of $M$.

For all $a \in A$, $a \in B_a = B'_a \cap S \subset U \cap S$. Therefore

$$A \subset U \cap S. \quad (2)$$

5.47 Corollary

Suppose that $(S, d)$ is a metric subspace of $(M, d)$ and that $A$ is a subset of $S$. Then $A$ is closed in $S$ if and only if there is an open subset $U$ of $M$ such that $A = U \cap S$.

Proof Exercise. Use Theorem 5.30. \qed

5.54

For all $a \in A$, $A \supset B_a = B'_a \cap S$. Therefore

$$A \supset \bigcup \{B'_a \cap S \mid a \in A\} = \bigcup \{B'_a \mid a \in A\} \cap S = U \cap S. \quad (3)$$

Statements (2) and (3) imply that $A = U \cap S$. \qed