B Equivalence Relations

B.1 The terms set and member of a set are accepted as undefined.

B.2 We take as known the notation and standard theorems of elementary set theory. For example, \( x \in A, A \subset B \), \( A = \{ x : x \text{ has property } P \} \), and so on.

B.3 We normally use upper-case Roman letters to denote sets and lower-case Roman letters to denote elements of sets. We use upper-case script letters to denote sets of sets. For example, we could write \( a \in B \in \mathcal{C}, \{ a \} \subset B \), and \( \{ B \} \subset \mathcal{C} \).

B.4 We also take as known the usual sets of numbers:

\[
\mathbb{N} \subset \mathbb{Z}^+ \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}
\]

where \( \mathbb{N} = \{1, 2, 3, 4, \ldots\} \), \( \mathbb{Z}^+ = \{0, 1, 2, 3, \ldots\} \), and \( \mathbb{Z}, \mathbb{Q}, \mathbb{R} \) denote the integers, the rational numbers, and the real numbers, respectively.

We also take as known the arithmetical operations and the order properties of the real numbers.

B.5 A set with precisely two distinct members \( \{ x, y \} \), say, is called a doubleton. An ordered pair \( (x, y) \) is a doubleton where the order in which its members are presented is significant. Thus \( \{ x, y \} = \{ y, x \} \) but \( (x, y) \neq (y, x) \) unless \( x = y \). We can define \( (x, y) \) in purely set-theoretic terms by

\[
(x, y) = \{ x, y, \{ x \} \}.
\]

B.6 Definition
The Cartesian product \( A \times B \) of the sets \( A \) and \( B \) is defined by

\[
A \times B = \{ (x, y) : x \in A \text{ and } y \in B \}.
\]

B.7 Definition
Suppose that \( A \) is a non-empty set. A binary relation or simply a relation on \( A \) is a subset \( R \subset A \times A \).

B.8 Suppose that \( R \) is a binary relation on \( A \). Suppose that \( x, y \in A \). Then we say that \( x \text{ is related to } y \) (with respect to \( R \)) iff \( (x, y) \in R \).

[“iff” is an abbreviated form of “if and only if”.

B.9 Example
Let \( A \) be a non-empty set. Then two examples of a binary relation on \( A \) are \( \emptyset \)—no two elements of \( A \) are related—and \( A \times A \)—any two elements of \( A \) are related.

B.10 Suppose that \( A = \{2, 3, \ldots, 8\} \) and that the binary relation \( R \subset A \times A \) is defined by

\[
R = \{ (2, 2), (2, 4), (2, 6), (2, 8), (3, 3), (3, 6), (4, 4), (4, 8), (5, 5), (6, 6), (7, 7), (8, 8) \}.
\]

Then \( (x, y) \in R \) iff \( x \mid y \).

B.11 Suppose that \( R \subset A \times A \) is a binary relation on \( A \). We can then define an operator \( \sim \) (known as tilde or twiddles) by the formula

\[
\forall x, y \in A, x \sim y \iff (x, y) \in R
\]

and then use \( \sim \) instead of \( R \) to denote the binary relation on \( A \). This is how mathematicians usually describe binary relations in practice. For example, the binary relation \( R \) of the previous example could also be described as
“the relation | on the set \{2,3,\ldots,8\}”.

**B.12** As usual, if \(x \sim y\) means “\(x\) is related to \(y\)” then \(x \not\sim y\) means “\(x\) is not related to \(y\)”.

**B.13 Definition**
Suppose that \(\sim\) is a binary relation on \(A\).

(i) We say that \(\sim\) is **reflexive** iff for all \(x \in A\), \(x \sim x\).

(ii) We say that \(\sim\) is **symmetric** iff for all \(x,y \in A\), \(x \sim y\) implies that \(y \sim x\).

(iii) We say that \(\sim\) is **transitive** iff for all \(x,y,z \in A\), \(x \sim y\) and \(y \sim z\) implies that \(x \sim z\).

**B.14 Example**
In each of the following examples \(\sim\) denotes a binary relation on \(\mathbb{N}\).

(i) \(x \sim y\) iff \(x \leq y\).

Clearly, \(\sim\) is reflexive and transitive but not symmetric.

(ii) \(x \sim y\) iff \(|x - y| \leq 3\).

Clearly, \(\sim\) is reflexive and symmetric. But \(\sim\) is not transitive: for example, \(1 \sim 4\) and \(4 \sim 6\) but \(1 \not\sim 6\).

(iii) \(x \sim y\) iff \(x\) and \(y\) are both even numbers.

It is easy to see that \(\sim\) is symmetric and transitive. But \(\sim\) is not reflexive: for example, \(1 \not\sim 1\).

**B.15** The examples above show that none of the three properties, reflexivity, symmetry, or transitivity, can be inferred from the other two.

**B.16 Definition**
A binary relation \(\sim\) on a non-empty set \(A\) is said to be an **equivalence relation** on \(A\) iff it is reflexive, symmetric, and transitive.

**B.17** Clearly, **equality** is an equivalence relation on any non-empty set.

**B.18 Example**
Let \(T\) be the set of all triangles in the real plane. Then similarity and congruence, as defined in Euclidean geometry, are both equivalence relations on \(T\).

**B.19 Lemma**
Suppose that

- \((M,d)\) is a metric space.
- \(y \in M\).
- \(f\) and \(g : [0,1] \to M\) are both continuous on \([0,1] \subset \mathbb{E}_1\).
- \(f(1) = g(0) = y\).
- \(h : [0,1] \to M\) is defined by

\[
h : [0,1] \to M : t \to \begin{cases} f(2t) & \text{if } t \leq 1/2 \\ g(2t - 1) & \text{if } t \geq 1/2. \end{cases}
\]

Then \(h\) is continuous at \(1/2\).

**Proof**
Let \(\varepsilon > 0\).

Since \(f\) is continuous at 1, \(\exists \delta_1 > 0\) such that

\[
\forall u \in [0,1], |u - 1| < \delta_1 \implies d(f(u),y) < \varepsilon. \tag{1}
\]
Since $g$ is continuous at $0$, $\exists \delta_2 > 0$ such that
\[
\forall u \in [0, 1], |u| < \delta_2 \implies d(g(u), y) < \varepsilon. \tag{2}
\]

Let
\[
\delta = \frac{1}{2} \min \{ \delta_1, \delta_2 \}.
\]
Suppose that $t \in [0, 1]$ satisfies $|t - 1/2| < \delta$.
If $t \leq 1/2$ then
\[
|t - 1/2| < \delta \implies |2t - 1| < 2\delta \\
\implies |2t - 1| < \delta_1 \\
\implies d(f(2t), y) < \varepsilon \\
\iff d(h(t), y) < \varepsilon.
\]
If $t \geq 1/2$ then
\[
|t - 1/2| < \delta \implies |2t - 1| < 2\delta \\
\implies |2t - 1| < \delta_2 \\
\implies d(g(2t - 1), y) < \varepsilon \\
\iff d(h(t), y) < \varepsilon.
\]
Therefore $h$ is continuous at $1/2$. □