The mathematics of David W. Lewis

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David's publications

Items reviewed in *Mathematical Reviews* (as of July 2009):

- 57 papers (1977–2007)
- 1 book: "Matrix Theory" World Scientific Pub., 1991
- 1 volume of conference proceedings: "Quadratic forms and their applications (Dublin, 1999)" Amer. Math. Soc., 2000

Superficial remarks

- Substantial number of surveys.
- Journals: the local moorings: 7 papers in the *Bulletin of the Irish Mathematical Society*, 4 papers in the *Proceedings of the Royal Irish Academy*.
- Collaborators: the Belgian connection: Dejaiffe, De Wannemacker, Tignol, Unger, Van Geel.

Outline

The objects

Quadratic forms . . .

. . . with a noncommutative twist

David's mathematics

Hermitian forms: 1977–1985

Levels of skew fields: 1985–1990

Ring-theoretic results on Witt rings: 1987–1992

Quadratic forms over function fields of conics: 1994–1995

Involutions on central simple algebras: 1999–. . .

The fundamental objects

Quadratic forms + noncommutative twist

Quadratic form = homogeneous polynomial of degree 2

$$x_1^2 - 2x_2x_3 \simeq y_1^2 + y_2^2 - y_3^2$$

$$a_1x_1^2 + \cdots + a_nx_n^2 = \langle a_1, \ldots, a_n \rangle$$

Geometric viewpoint

V vector space over F (char $\neq 2$)

Quadratic form = map $q \colon V \to F$ such that

$$b(x, y) = q(x + y) - q(x) - q(y)$$

is a bilinear pairing $V \times V \to F$.

The Witt ring

Ernst Witt (1937):
$\langle a_1, \ldots, a_n \rangle \perp \langle b_1, \ldots, b_m \rangle = \langle a_1, \ldots, a_n, b_1, \ldots, b_m \rangle$
$\langle a_1, \ldots, a_n \rangle \otimes \langle b_1, \ldots, b_m \rangle = \langle a_1b_1, \ldots, a_ib_j, \ldots, a_nb_m \rangle$

Hyperbolic quadratic form = $\langle 1, -1, \ldots, 1, -1 \rangle$

Theorem (Witt cancellation)

$\langle a_1, \ldots, a_n \rangle \simeq \langle b_1, \ldots, b_n \rangle$ *iff*

$\langle a_1, \ldots, a_n \rangle \perp \langle -1 \rangle \langle b_1, \ldots, b_n \rangle$ *is hyperbolic.*

Definitions

q_1, q_2 are *Witt-equivalent* if $q_1 \perp \langle -1 \rangle q_2$ is hyperbolic

Witt ring of F:

$$WF := \{\text{Witt-equivalence classes of quadratic forms over } F\}$$

Examples of Witt groups

Example

$W(\mathbb{C}) = \mathbb{Z}/2\mathbb{Z}, \quad q \mapsto \dim q \bmod 2.$

Proof.

$\langle a_1, a_2 \rangle \simeq \langle 1, -1 \rangle = 0.$ □

Example

$W(\mathbb{R}) = \mathbb{Z}, \quad q \mapsto \operatorname{sgn} q = \#\{a_i > 0\} - \#\{a_i < 0\}.$

Proof.

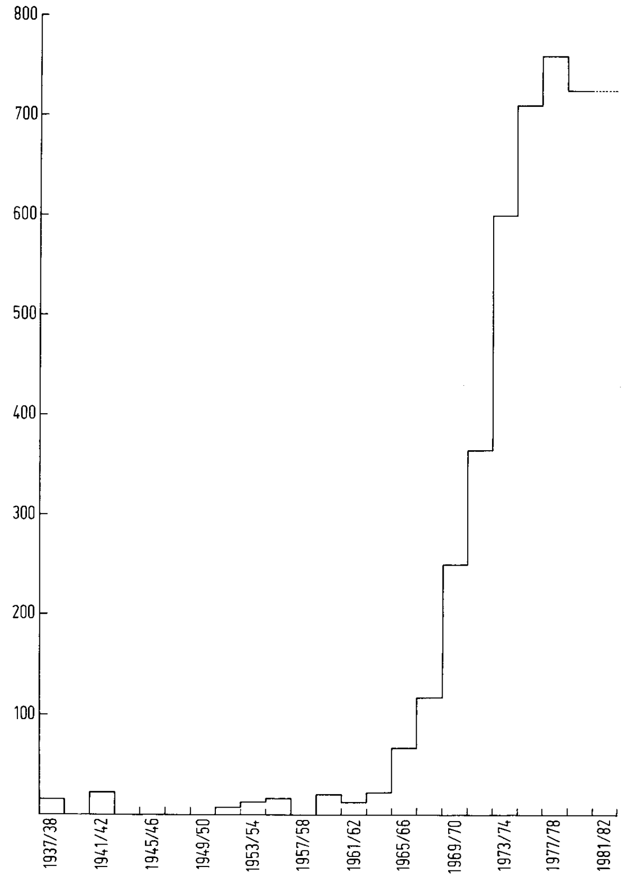
Sylvester's law of inertia. □

Example

$W(\mathbb{Q}) \simeq \mathbb{Z} \oplus \left(\bigoplus_p W(\mathbb{F}_p)\right),$

$$W(\mathbb{F}_p) \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } p = 2, \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & \text{if } p \equiv 1 \bmod 4, \\ \mathbb{Z}/4\mathbb{Z} & \text{if } p \equiv 3 \bmod 4. \end{cases}$$

p. 417 in *W. Scharlau, Quadratic and Hermitian Forms, Springer, 1985:*



The algebraic theory of quadratic forms:
Number of pages published in research articles

Multiplicative forms

$$(x_1^2 + x_2^2)(y_1^2 + y_2^2) = (x_1 y_1 - x_2 y_2)^2 + (x_1 y_2 + x_2 y_1)^2$$

(Diophantus, 3d century)

$$\begin{aligned}(x_1^2 + x_2^2 + x_3^2 + x_4^2)(y_1^2 + y_2^2 + y_3^2 + y_4^2) = \\ (x_1 y_1 - x_2 y_2 - x_3 y_3 - x_4 y_4)^2 \\ +(x_1 y_2 + x_2 y_1 + x_3 y_4 - x_4 y_3)^2 \\ +(x_1 y_3 + x_3 y_1 + x_4 y_2 - x_2 y_4)^2 \\ +(x_1 y_4 + x_4 y_1 + x_2 y_3 - x_3 y_2)^2\end{aligned}$$

(Euler, 1748)

Multiplicative forms

$$\begin{aligned}(x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2) \\ (y_1^2 + y_2^2 + y_3^2 + y_4^2 + y_5^2 + y_6^2 + y_7^2 + y_8^2) = \\ (x_1 y_1 - x_2 y_2 - x_3 y_3 - x_4 y_4 - x_5 y_5 - x_6 y_6 - x_7 y_7 - x_8 y_8)^2 \\ +(x_1 y_2 + x_2 y_1 + x_3 y_4 - x_4 y_3 + x_5 y_6 - x_6 y_5 - x_7 y_8 + x_8 y_7)^2 \\ +(x_1 y_3 - x_2 y_4 + x_3 y_1 + x_4 y_2 + x_5 y_7 + x_6 y_8 - x_7 y_5 - x_8 y_6)^2 \\ +(x_1 y_4 + x_2 y_3 - x_3 y_2 + x_4 y_1 + x_5 y_8 - x_6 y_7 + x_7 y_6 - x_8 y_5)^2 \\ +(x_1 y_5 - x_2 y_6 - x_3 y_7 - x_4 y_8 + x_5 y_1 + x_6 y_2 + x_7 y_3 + x_8 y_4)^2 \\ +(x_1 y_6 + x_2 y_5 - x_3 y_8 + x_4 y_7 - x_5 y_2 + x_6 y_1 - x_7 y_4 + x_8 y_3)^2 \\ +(x_1 y_7 + x_2 y_8 + x_3 y_5 - x_4 y_6 - x_5 y_3 + x_6 y_4 + x_7 y_1 - x_8 y_2)^2 \\ +(x_1 y_8 - x_2 y_7 + x_3 y_6 + x_4 y_5 - x_5 y_4 - x_6 y_3 + x_7 y_2 + x_8 y_1)^2\end{aligned}$$

(Graves, 1843)

Multiplicative forms

Theorem (Hurwitz, 1898)

If there exist bilinear polynomials $f_1(x,y), \dots, f_n(x,y)$ *such that*

$$(x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2) = f_1(x,y)^2 + \cdots + f_n(x,y)^2,$$

then $n = 1$, 2, 4 *or* 8.

Theorem (Pfister, 1965)

For every $n = 2^k$ *there exist rational functions* $f_1(x,y), \dots,$ $f_n(x,y)$ *such that*

$$(x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2) = f_1(x,y)^2 + \cdots + f_n(x,y)^2.$$

The level of a field

Definition

$s(F) = \inf\{n \mid -1 = x_1^2 + \cdots + x_n^2\}$.

Examples

$s(\mathbb{R}) = \infty$, $s(\mathbb{C}) = 1$, $s\big(\mathbb{Q}(\sqrt{-2})\big) = 2$, ...

Theorem (Pfister)

The level of a field is a power of 2 *(or* ∞*).*

Proof.

Say $x_1^2 + \cdots + x_{12}^2 = -1$; then

$$x_1^2 + \cdots + x_8^2 = -(1 + x_9^2 + x_{10}^2 + x_{11}^2 + x_{12}^2)$$

Let $z = 1 + x_9^2 + x_{10}^2 + x_{11}^2 + x_{12}^2$; then

$y_1^2 + \cdots + y_8^2 = (x_1^2 + \cdots + x_8^2)(1 + x_9^2 + x_{10}^2 + x_{11}^2 + x_{12}^2) = -z^2$.

Hence $\left(\frac{y_1}{z}\right)^2 + \cdots + \left(\frac{y_8}{z}\right)^2 = -1$. □

The noncommutative twist

V right vector space over a skew field D
$b\colon V \times V \to D$ bilinear: $\quad b(x\alpha, y\beta) = b(x,y)\alpha\beta = b(x,y)\beta\alpha.$
Each pairing b induces

$$\begin{aligned} \hat{b}\colon V &\to V^* = \mathrm{Hom}_D(V, D), \qquad \textit{left } D\text{-vector space.} \\ x &\mapsto b(x, \bullet) \end{aligned}$$

Definitions

$\overline{}\colon D \to D$ is an *involution* if for α, $\beta \in D$,

$$\overline{\alpha + \beta} = \overline{\alpha} + \overline{\beta}, \quad \overline{\alpha\beta} = \overline{\beta}\,\overline{\alpha}, \quad \overline{\overline{\alpha}} = \alpha.$$

$b\colon V \times V \to D$ is *hermitian* if for α, $\beta \in D$ and x, $y \in V$,

$$b(x\alpha, y\beta) = \overline{\alpha} b(x,y)\beta, \quad b(y,x) = \overline{b(x,y)}.$$

$W(D, \overline{}) =$ Witt group of hermitian forms for $\overline{}$.

David's Hermitian period: 1977–1985

in the wake of his thesis with C.T.C. Wall

Highlight: Exact octagons of Witt groups

Basic observation (Milnor–Husemoller):
if $h\colon V \times V \to F(\sqrt{a})$ is a hermitian form, then

$$h(x,x) = \overline{h(x,x)} \in F \quad \text{for all } x \in V,$$

so $x \mapsto h(x,x)$ is a quadratic form on V over F.

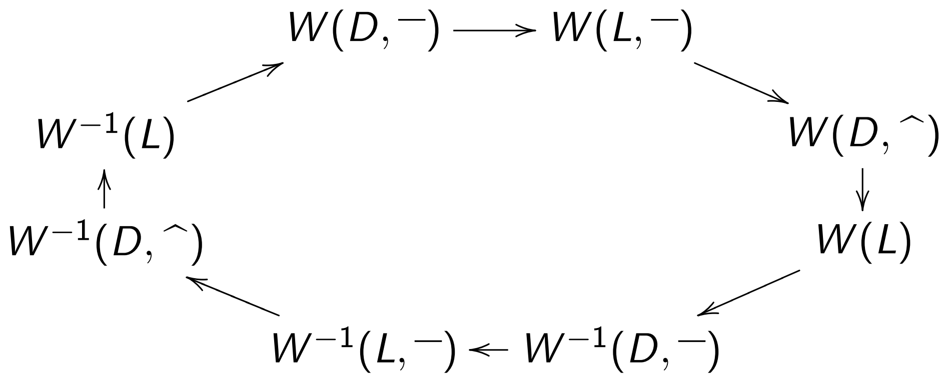
Exact sequence: $\quad 0 \to W(F(\sqrt{a}), \overline{}) \to W(F) \to W\big(F(\sqrt{a})\big).$

David:

$$0 \to W(\sqrt{a}), \overline{}) \to W(F) \to W\big(F(\sqrt{a})\big) \to W(F) \to$$
$$\to W(F(\sqrt{a}), \overline{}) \to 0 = W^{-1}(F)$$

The exact octagon

For D a quaternion algebra with quadratic subfield L,



- Variants: Witt groups of equivariant forms, of Clifford algebras
- Related to Clifford algebra periodicity
- Deep relation between various types of hermitian forms and general quadratic extensions, used by Bayer–Parimala in their proof of Serre's "Conjecture II" for classical groups

Levels of skew fields: 1985–1990

Idea: extend results on sums of squares to skew fields

Problem: $x^2y^2 \neq (xy)^2 = xyxy$

For D finite-dimensional over its center F, there is a trace map $\mathrm{Tr}\colon D \to F$, hence a quadratic form

$$q_D\colon D \to F, \qquad q_D(x) = \mathrm{Tr}(x^2).$$

Theorem (Solution of a conjecture of Leep– Shapiro– Wadsworth)

-1 is a sum of squares in D iff q_D is weakly isotropic, i.e. $n \times q_D$ is isotropic for some $n \geq 1$.

Levels of skew fields

Definitions

$s(D) = \inf\{n \mid -1 = x_1^2 + \cdots + x_n^2\}$
$s(D, ^{-}) = \inf\{n \mid -1 = x_1\overline{x_1} + \cdots + x_n\overline{x_n}\}$

Theorem

For every $k \geq 0$, there exist quaternion division algebras D, D' with $s(D) = 2^k$ and $s(D') = 2^k + 1$.
For a quaternion division algebra D, $s(D, ^{-})$ is a power of 2 (or ∞).

- Raises interesting questions in relation with trace forms
- Spurred important research activity by Bauwens, Denert, Hoffmann, Koprowski, Krüskemper, Leep, Serhir, Van Geel, Vast, Wadsworth, ...

Ring-theoretic results on Witt rings: 1987–1992

Theorem

For every quadratic form q of dimension n over a field F,

$$q(q^2 - 2^2)(q^2 - 4^2)\cdots(q^2 - n^2) = 0 \text{ in } W(F) \quad \text{if } n \text{ is even,}$$
$$(q^2 - 1^2)(q^2 - 3^2)\cdots(q^2 - n^2) = 0 \text{ in } W(F) \quad \text{if } n \text{ is odd.}$$

Corollary

New proofs of structure theorems for Witt rings:

- *no odd torsion, no odd-dimensional zero divisors, no nontrivial idempotents;*
- *if $W(F)$ contains torsion elements, every even-dimensional form is a zero-divisor;*
- *and much more.*

Annihilating polynomials

Theorem (Conner, 1987)

If q is the trace quadratic form of a separable field extension of degree n, then

$$q(q-2)(q-4)\cdots(q-n) = 0 \text{ in } W(F) \text{ if } n \text{ is even,}$$
$$(q-1)(q-3)\cdots(q-n) = 0 \text{ in } W(F) \text{ if } n \text{ is odd.}$$

Improvement (taking into account the Galois group): Beaulieu–Palfrey (1997)

Further improvement: Lewis–McGarraghy (2000) (using the Burnside ring of a finite group viewed as Grothendieck ring of a category of étale algebras).

Related work: Hurrelbrink, Sładek, Epkenhans, Ongenae–Van Geel, De Wannemacker, . . .

Quadratic forms over function fields of conics: 1994–1995

Two joint papers with Van Geel and Hoffmann–Van Geel

Theme: Which quadratic forms become isotropic over $F(x, y)$ where $ax^2 + by^2 = 1$?

Obvious answer: those that contain a multiple of $\langle a, b, -1\rangle$.

But there are many more:

- Classification in terms of "splitting sequences" (based on work of Rost)
- Characterization of the 5-dimensional forms that become isotropic over $F(x, y)$

Involutions on central simple algebras: (1993) 1999–...

Every hermitian pairing $h\colon V \times V \to D$ induces an *adjoint* involution $\mathrm{ad}_h = {}^*\colon \mathrm{End}_D V \to \mathrm{End}_D V$ such that

$$h(f(x), y) = h(x, f^*(y)) \text{ for } x, y \in V,\ f \in \mathrm{End}_D V$$

hermitian or skew-hermitian forms on V up to a scalar factor	$\longleftrightarrow$	involutions on $\mathrm{End}\, V$

Theorem (Weil, 1960)

Every classical simple linear algebraic group of adjoint type is a group of automorphisms of a central simple algebra with involution.

1990's:

- Schofield–Van den Bergh, Merkurjev: "index reduction formulas" point to a bridge between quadratic forms and linear algebraic groups: involutions as "virtual" quadratic forms
- Knus–Parimala–Sridharan rediscover the discriminant and Clifford algebra of involutions (Jacobson, Tits)

Classification results

A central simple algebra over a field F

Definition (Lewis–Tignol, 1993)

For $\sigma\colon A \to A$ involution of the first kind ($\sigma|_F = \mathrm{Id}_F$), P ordering on F,

$$\mathrm{sgn}_P\, \sigma = \sqrt{\mathrm{sgn}_P\, T_\sigma}$$

where $T_\sigma(x) = \mathrm{Tr}(\sigma(x)x)$ for $x \in A$.

For σ of the second kind: Quéguiner (1995)

Example

$\mathrm{sgn}_P\, \mathrm{ad}_q = |\mathrm{sgn}_P\, q|$

Theorem (Lewis–Tignol, 1999)

If $\mathrm{cd}\, F \leq 2$*, involutions on central simple* F*-algebras are classified by their "classical" invariants.*
If $\mathrm{cd}\, F(\sqrt{-1}) \leq 2$*: classification by classical invariants and signatures, provided* F *is ED (e.g. number fields).*

Local-global principles

Local properties

- σ totally indefinite: $\mathrm{sgn}_P\,\sigma < \deg A$ for every ordering P
- σ totally hyperbolic: $\mathrm{sgn}_P\,\sigma = 0$ for every ordering P

Global properties

$\sigma \otimes t_n$ on $A \otimes M_n(F) = M_n(A)$ is $n \times \sigma$

- σ is weakly isotropic: $\sigma \otimes t_n$ isotropic for some n
- σ is weakly hyperbolic: $\sigma \otimes t_n$ hyperbolic for some n

Local–global principles

weakly isotropic $\iff$ totally indefinite: weak Hasse principle
weakly hyperbolic $\iff$ totally hyperbolic: Pfister's local-global principle

Local-global principles

Theorem (Lewis–Scheiderer–Unger)

The weak Hasse principle holds for involutions of the first kind iff F satisfies ED.

Note: The weak Hasse principle holds for quadratic forms iff F satisfies SAP. (Elman–Lam–Prestel)

Theorem (Lewis–Unger)

Pfister's local-global principle holds for involutions of any kind.

Local-global principles

Let A be a central simple algebra over a global field.

Theorem (Lewis–Unger–Van Geel)

Orthogonal involutions on A are conjugate iff they are conjugate at every prime $\mathfrak{p}$, provided their discriminant is not a square at any $\mathfrak{p}$ such that $A_{\mathfrak{p}}$ is not split.

- Originally expressed as a Hasse principle for similarity of skew-hermitian forms
- Corrects a statement made by Hijikata (1963)

To be continued . . .