The virtues of similarity of orthogonal representations

July 24, 2009

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- K is a field
  - of characteristic 0 or
  - p relatively prime to $|G|$

so the group algebra KG is a separable algebra.

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$$\rho : G \to \mathrm{O}(V, b) \tag{1}$$

into the orthogonal group of b.

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Two orthogonal representations

$$\rho : G \to \mathrm{O}(V, b) \ \text{ and } \ \sigma : G \to \mathrm{O}(W, c)$$

are *equivalent* if there is an isometry $\varphi : (V, b) \to (W, c)$ which commutes with the action of G.

Notation: $\cong$ stands for equivalence, of forms or representations.

If $\alpha_0 \in \dot{K}$, then

$$\mathrm{O}(V, \alpha_0 b) = \mathrm{O}(V, b)$$

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Definition. The orthogonal representations

$$\rho : G \to \mathrm{O}(V, b) \quad \text{and} \quad \sigma : G \to \mathrm{O}(W, c)$$

are said to be *similar* if there is a similarity $\varphi : (V, b) \to (W, c)$ which commutes with the action of G. This means that there is some $\alpha_0 \in \dot{K}$ such that

$$c(\varphi v, \varphi v') = \alpha_0 b(v, v') \quad \text{for all } v, v' \in V.$$

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One immediately notices that a Witt-Grothendieck group cannot be defined in the obvious way, not even for symmetric bilinear forms since, e.g., if $K = \mathbb{Q}$, then $x^2 \simeq x^2$ and $y^2 \simeq 2y^2$, but $x^2 + y^2 \not\simeq x^2 + 2y^2$.

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But similarity is better suited to a multiplication of forms, in a sense to be defined.

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Let $\rho : G \to \mathrm{O}(V, b)$ be an orthogonal representation. Define

$$\hat{b}(v, v') = \sum_s b(v, \rho(s)v')s \in KG.$$

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Then

$$\hat{b} : V \times V \to (KG, ^{-})$$

is a nonsingular Hermitian form.

Theorem: *The category of orthogonal representations is isomorphic to the category of nonsingular Hermitian forms over* $(KG, ^{-})$ *via* $\rho \rightsquigarrow \hat{b}$.

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Theorem: *The category of orthogonal representations under similarity is isomorphic to the category of nonsingular Hermitian forms over* $(KG, ^{-})$ *under similarity via* $\rho \rightsquigarrow \hat{b}$.

Proof: One checks easily that $\varphi : V \to W$ is a KG-linear similarity between (V, b) and (W, c) if and only if φ is a similarity between $(V, \hat{b})$ and $(W, \hat{c})$. $\square$

Reduction to simple algebras.

The K-algebra KG is separable and so factors into a direct sum

$$KG = A_1 \oplus A_2 \oplus \cdots \oplus A_r$$

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Thus

$$(KG, ^-) = (B_1, ^-) \oplus (B_2, ^-) \oplus \cdots \oplus (B_t ^-)$$

where each B_i is either a simple component A_j such that $\bar{A}_j = A_j$ or is a direct sum $A_j \oplus \bar{A}_j$ of two of them.

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If V is a KG-module, there is a corresponding decomposition

$$V = B_1 V \oplus B_2 V \oplus \cdots B_t V$$

into "involution isotypic" components of V.

Since

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Thus

$$V = B_1V \perp B_2V \perp \cdots \perp B_tV$$

with respect to $\hat{b}$ (or b) and

$$\hat{b}|_{B_iV \times B_iV} : B_iV \times B_iV \to (B_i, \bar{\ })$$

is also a nonsingular Hermitian form.

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if and only if

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The hyperbolic factors ($B_i = A_j \oplus \bar{A}_j$ for some j) can be ignored since for them $(B_i V, \hat{b}|_{B_i V \times B_i V}) \simeq (B_i W, \hat{c}|_{B_i W \times B_i W})$ if and only if $B_i V$ and $B_i W$ have the same length.

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Thus the similarity problem is reduced to the special case when $V = BV$ for a simple algebra $(B, ^-)$ with involution.

Similarity classes

If $(A, {}^-)$ is an algebra with involution, denote by $\mathrm{Sim}(A, {}^-)$ the set of similarity classes of nonsingular Hermitian forms over $(A, {}^-)$.

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Theorem

Let $(A, ^-)$ and $(B, ^-)$ be isomorphic central simple K-algebras with K-involutions. Then there is a canonical bijection

$$\Phi_{(A,^-),(B,^-)} : \mathrm{Sim}(A, ^-) \to \mathrm{Sim}(B, ^-).$$

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It satisfies

$$\Phi_{(B,^{-}),(C,^{-})}\Phi_{(A,^{-}),(B,^{-})} = \Phi_{(A,^{-}),(C,^{-})},$$

$$\Phi_{(A,^{-}),(A,^{-})} = \mathrm{id}_{\mathrm{Sim}(A,^{-})}$$

and

$$\Phi_{(B,^{-}),(A,^{-})} = \Phi^{-1}_{(A,^{-}),(B,^{-})}.$$

Let $\varphi : (A, ^-) \to (B, ^-)$ be an isomorphism. Then if (V, h) is an Hermitian space over $(A, ^-)$, first make V into a B-module ${}^{\varphi}V$ by defining

$$b.v = (\varphi^{-1} b)v,$$

and then define ${}^{\varphi}h(u, v) = \varphi \circ h(u, v)$. The form ${}^{\varphi}h : {}^{\varphi}V \times {}^{\varphi}V \to (B, ^-)$ is a nonsingular Hermitian form, and $h \rightsquigarrow {}^{\varphi}h$ induces a map

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If $\psi : (A, \bar{\ }) \to (B, \bar{\ })$ is another isomorphism, the automorphism $\psi\varphi^{-1}$ of $(B, \bar{\ })$ is an inner automorphism $b \to cbc^{-1}$ for some $c \in B^\times$ satisfying $\bar{c}c \in \dot{K}$, say $\bar{c}c = \gamma$.

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$${}^{\psi}f(u, v) = \gamma^{-1}\,{}^{\varphi}f(c.u, c.v).$$

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$$ {}^\psi f(u, v) = \gamma^{-1} {}^\varphi f(c.u, c.v). $$

Thus ${}^\psi f \simeq {}^\varphi f$ and so $\Phi_{(A,^-),(B,^-)}$ is independent of the isomorphism $(A, ^-) \to (B, ^-)$ chosen, and it is bijective since the structure transfer via φ^{-1} results in the inverse map

$$ \mathrm{Sim}(B, ^-) \to \mathrm{Sim}(A, ^-). $$

The other properties are easy to verify. □

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$$\bar{a} = c_0^{-1} a^{t*} c_0$$

for some $c_0 \in A^\times$. It satisfies $c_0^{t*} = \varepsilon_0 c_0$ where $\varepsilon_0 = \pm 1$.

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A finitely generated A-module V is isomorphic to $D^{n \times m}$ for some m (the length of V), so any similarity class in $\mathrm{Sim}(\mathrm{M}(n, D), {}^-)$ contains a form $h : D^{n \times m} \times D^{n \times m} \to (A, {}^-)$.

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$$h(u, v) = u\underline{h}v^{t*} c_0$$

where $\underline{h} \in D^{m \times m}$ – it is the "Gram matrix" of h and also satisfies $\underline{h}^{t*} = \varepsilon_0 \underline{h}$.

Now suppose that $(B, \tilde{\ }) = (\mathrm{M}(p, D), \tilde{\ })$ is another involution algebra of the first kind and of the same type (symplectic or orthogonal) as (A^{-}), Now suppose that $(B, \tilde{\ }) = (\mathrm{M}(p, D), \tilde{\ })$ is another involution algebra of the first kind and of the same type (symplectic or orthogonal) as (A^{-}), so for some $d_0 \in \mathrm{GL}(p, D)$ satisfying $d_0^{t*} = \varepsilon_0 d_0$,

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Then the association

$$u\underline{h}v^{t*}c_0 \rightsquigarrow x\underline{h}y^{t*}d_0$$

induces a bijection

$$\mathrm{Sim}(A, {}^{-}) \to \mathrm{Sim}(B, \tilde{\ })$$

which is independent of the various choices made – $c_0, d_0, *$ and the form h in its similarity class, and so is canonical.

An aside

The ε_0-Hermitian form

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over $(D, \ ^*\ )$ is, up to a sign, the form over D which is "Morita equivalent" to h.

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Two Hermitian forms over $(A, \bar{\ })$ are equivalent, respectively similar, if and only if their Morita equivalent forms over $(D, \ ^*\ )$ are equivalent respectively similar.

The general canonical bijection

Suppose that $(A, ^{-})$ and $(B, \tilde{\ })$ are central simple algebras in the same "involutory Brauer class", i.e. A and B are in the same Brauer class, say as the central division algebra D, and that both are orthogonal or both are symplectic.

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Suppose that $(A, ^-) \cong (\mathrm{M}(n, D), ^-)$ and $(B, \tilde{\ }) \cong (\mathrm{M}(p, D), \tilde{\ })$. The canonical bijection

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The general canonical bijection has the expected properties:

- the composition of two of them is also one,
- $\mathrm{Sim}(A, \bar{\ }) \to \mathrm{Sim}(A, \bar{\ })$ is the identity, and
- $\mathrm{Sim}(B, \tilde{\ }) \to \mathrm{Sim}(A, \bar{\ })$ is the inverse of $\mathrm{Sim}(A, \bar{\ }) \to \mathrm{Sim}(B, \tilde{\ })$.

The product of forms

Let $(A, ^-)$ and $(B, \tilde{\ })$ be central simple algebras over K with K-involutions, and let $f : V \times V \to (A, ^-)$ and $g : W \times W \to (B, \tilde{\ })$ be nonsingular Hermitian forms.

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$$(f \otimes g)(v \otimes w, v' \otimes w') = f(v, v') \otimes g(w, w')$$

defines a nonsingular Hermitian form

$$f \otimes g : (V \otimes W) \times (V \otimes W) \to (A, \bar{\ }) \otimes (B, \tilde{\ }).$$

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which induces a product

$$(\mathrm{Sim}\,\mathsf{a}) \times (\mathrm{Sim}\,\mathsf{b}) \to \mathrm{Sim}\,\mathsf{ab} \quad (\mathsf{a}, \mathsf{b} \in \mathrm{Br}(K, \mathrm{id}))$$

by identification using the canonical bijections.

This can be restated by defining

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Suppose that f and g are nonsingular Hermitian forms over $(A, ^{-})$ and $(B, ^{\sim})$ respectively, and define

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and $\breve{g}$ similarly. They are nonsingular symmetric bilinear forms over K.

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$$\begin{aligned}\breve{f}(au, v) &= \mathrm{trd}_{A/K} f(au, v) = \mathrm{trd}_{A/K}(af(u, v)) \\ &= \mathrm{trd}_{A/K}(f(u, v)a) = \mathrm{trd}_{A/K} f(u, \bar{a}v) \\ &= \breve{f}(u, \bar{a}v),\end{aligned}$$

that is to say "$\breve{f}$ is $(A, {}^-)$-invariant."

The product has a nonsurprising interpretation in terms of invariant symmetric bilinear forms over K.
Suppose that f and g are nonsingular Hermitian forms over $(A, {}^{-})$ and $(B, {}^{\sim})$ respectively, and define

$$\check{f}(u, v) = \mathrm{trd}_{A/K} f(u, v)$$

and $\check{g}$ similarly. They are nonsingular symmetric bilinear forms over K. Moreover if $a \in A$,

$$\begin{aligned}\check{f}(au, v) &= \mathrm{trd}_{A/K} f(au, v) = \mathrm{trd}_{A/K}(af(u, v)) \\ &= \mathrm{trd}_{A/K}(f(u, v)a) = \mathrm{trd}_{A/K} f(u, \bar{a}v) \\ &= \check{f}(u, \bar{a}v),\end{aligned}$$

that is to say "$\check{f}$ is $(A, {}^{-})$-invariant."
Thus if $(A, {}^{-})$ is a summand of $(KG, {}^{-})$, $\check{f}$ is G-invariant and so gives rise to an orthogonal representation.

If $(B, \tilde{\ })$ is also a summand of $(KG, \bar{\ })$, then

$$\begin{aligned}(f \breve{\otimes} g)(v \otimes w, v' \otimes w') &= \mathrm{trd}_{A \otimes B/K}(f(v, v') \otimes g(w, w')) \\ &= (\mathrm{trd}_{A/K} f(v, v'))(\mathrm{trd}_{B/K} g(w, w'))\end{aligned}$$

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If $(B, \tilde{\ })$ is also a summand of $(KG, \bar{\ })$, then

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Thus if $(A, \bar{\ })$ and $(B, \tilde{\ })$ are summands of $(KG, \bar{\ })$, and f and g correspond to the orthogonal representations

$$\rho : G \to \mathrm{O}(V, \breve{f}) \quad \text{and} \quad \sigma : G \to \mathrm{O}(W, \breve{g})$$

respectively, then $f \otimes g$ corresponds to the orthogonal representation

$$\rho \otimes \sigma : G \to \mathrm{O}(V \otimes W, \breve{f}\breve{g}).$$

Remark

Let $f : V \times V \to (D, \,^*\,)$ and $g : W \times W \to (E, \,^\dagger\,)$ be sesquilinear forms over central simple division algebras over K. Then the product of f and g gives a product

$$\mathrm{Sim}(D, \,^*\,) \times \mathrm{Sim}(E, \,^\dagger\,) \to \mathrm{Sim}(F, \,^\ddagger\,)$$

where $[D, \mathrm{type}\,^*\,][E, \mathrm{type}\,^\dagger\,] = [F, \mathrm{type}\,^\ddagger\,]$ in $\mathrm{Br}(K, \mathrm{id})$.

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That is to say, one can define the product of Hermitian forms over central division K-algebras "up to a scalar".