EXACT BRAIDS AND OCTAGONS

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Quadratic Forms

and their Applications

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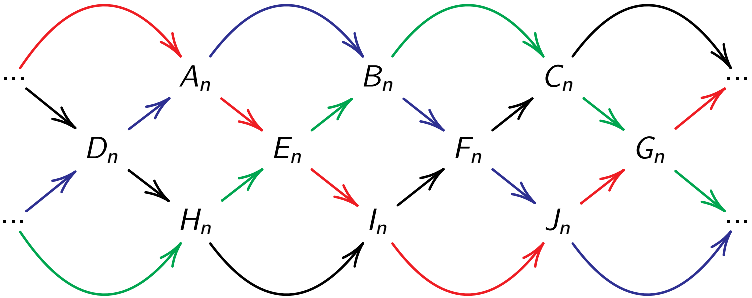
Exact braids

- An **exact braid** is a commutative diagram of 4 exact sequences



- The 4 exact sequences are

$$\cdots \longrightarrow A_n \longrightarrow E_n \longrightarrow I_n \longrightarrow J_n \longrightarrow G_n \longrightarrow \cdots$$

$$\cdots \longrightarrow D_n \longrightarrow A_n \longrightarrow B_n \longrightarrow F_n \longrightarrow J_n \longrightarrow \cdots$$

$$\cdots \longrightarrow D_n \longrightarrow H_n \longrightarrow I_n \longrightarrow F_n \longrightarrow C_n \longrightarrow \cdots$$

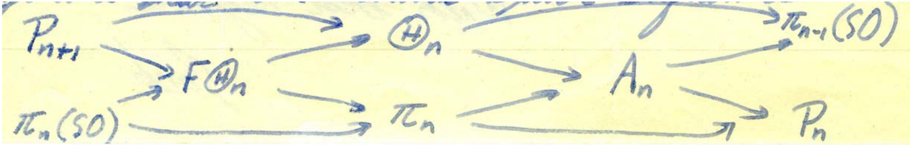
$$\cdots \longrightarrow H_n \longrightarrow E_n \longrightarrow B_n \longrightarrow C_n \longrightarrow G_n \longrightarrow \cdots$$

Brief history of exact braids

- Eilenberg and Steenrod (1952) Axiomatic treatment of Mayer-Vietoris exact sequences, with commutative diagrams.
- Kervaire-Milnor (1963), Levine (1965/1984). Application of braids to the classification of exotic spheres.
- Wall (1966) *On the exactness of interlocking sequences*. General theory: exactness of three sequences implies exactness of fourth. Applications in homology theory, simplifying the Eilenberg-Steenrod treatment of triples and the Mayer-Vietoris sequence.
- 1966 – ... Many applications in the surgery theory of high-dimensional manifolds (Wall, R., Hambleton-Taylor-Williams, Harsiladze ...)
- Hardie and Kamps (1985) Homotopy theory application.
- Iversen (1986) Triangulated category application.
- 1983 – ... Many applications in quadratic form theory of equivariant forms and Clifford algebras, via the exact octagons of Lewis et al.

The first exact braid

- In a letter from Milnor to Kervaire, 29 June, 1961:



  with $\Theta_n = \pi_n(PL/O)$ the group of n-dimensional exotic spheres, $F\Theta_n = \pi_n(PL)$ the group of framed n-dimensional exotic spheres, $P_n = L_n(\mathbb{Z}) = \pi_n(G/PL)$ the simply-connected surgery obstruction group, $\pi_n = \Omega_n^{fr} = \pi_n(G)$ the stable homotopy groups of spheres = the framed cobordism group, $A_n = \pi_n(G/O)$ the almost framed cobordism group, and $\pi_n(SO) \to \pi_n$ the J-homomorphism.
- Exact braids are sometimes called **Kervaire diagrams**.

Homotopy and homology groups

- The **homotopy groups** of a space X are the groups of homotopy classes of maps $S^n \to X$

$$\pi_n(X) \;=\; [S^n, X] \;\; (n \geqslant 1) \; .$$

- The **relative homotopy groups** $\pi_n(X, Y)$ of a map of spaces $Y \to X$ are the homotopy classes of commutative squares

$$\begin{array}{ccc} S^{n-1} & \longrightarrow & Y \\ \downarrow & & \downarrow \\ D^n & \longrightarrow & X \end{array}$$

  with an exact sequence

$$\cdots \to \pi_n(Y) \to \pi_n(X) \to \pi_n(X, Y) \to \pi_{n-1}(Y) \to \dots \; .$$

- Similarly for **homology** $H_*(X)$, $H_*(X, Y)$.

Fibre squares

- A commutative square of spaces and maps

$$\begin{array}{ccc} Y & \longrightarrow & X^+ \\ \big\downarrow & & \big\downarrow \\ X^- & \longrightarrow & X \end{array}$$

  is a **fibre square** if the natural maps of relative homotopy groups

$$\pi_*(X^+, Y) \to \pi_*(X, X^-)$$

  are isomorphisms, or equivalently if the natural maps

$$\pi_*(X^-, Y) \to \pi_*(X, X^+)$$

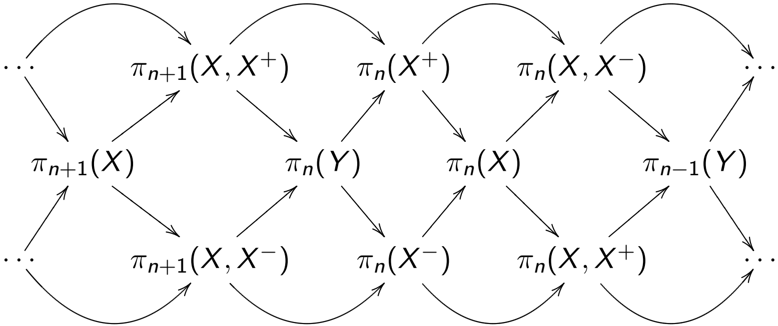
  are isomorphisms.

The exact braid of homotopy groups of a fibre square

- **Proposition** The homotopy groups of a fibre square

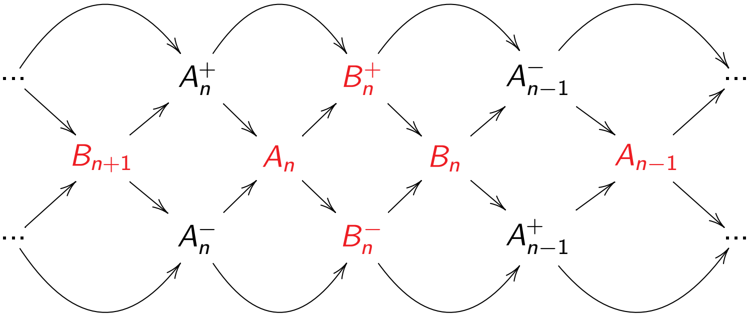
$$\begin{array}{ccc} Y & \longrightarrow & X^{+} \\ \downarrow & & \downarrow \\ X^{-} & \longrightarrow & X \end{array}$$

fit into an exact braid



The Mayer-Vietoris sequence of an exact braid

- **Proposition** An exact braid



determines an exact sequence

$$\cdots \longrightarrow B_{n+1} \longrightarrow A_n \longrightarrow B_n^+ \oplus B_n^- \longrightarrow B_n \longrightarrow A_{n-1} \longrightarrow \cdots$$

- Exactness proved by diagram chasing.

The Mayer-Vietoris exact sequence of a union

- Let X be a topological space with a decomposition

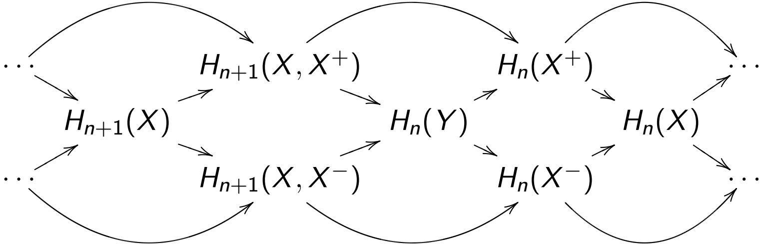
$$X \ = \ X^+ \cup_Y X^-$$

with $X^+, X^-, Y \subseteq X$ closed subspaces, $Y = X^+ \cap X^-$.

- **Proposition** The excision isomorphisms

$$H_*(X^+, Y) \ \cong \ H_*(X, X^-) \ , \ H_*(X^-, Y) \ \cong \ H_*(X, X^+)$$

determine an exact braid of homology sequences

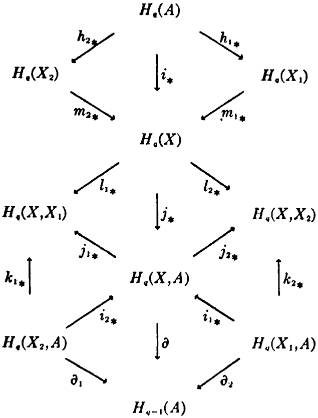


and hence the Mayer-Vietoris exact sequence

$$\cdots \to H_{n+1}(X) \to H_n(Y) \to H_n(X^+) \oplus H_n(X^-) \to H_n(X) \to \cdots$$

Almost an exact braid

- From Eilenberg and Steenrod, *Foundations of algebraic topology* (1952)

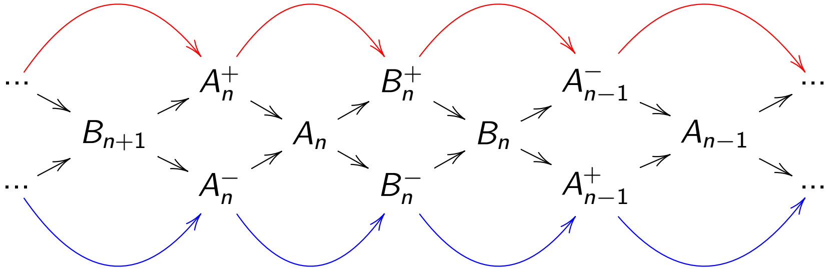


DEFINITION 15.2. The *Mayer-Vietoris sequence of a proper triad* $(X;X_1,X_2)$ *with* $X = X_1 \cup X_2$ and $A = X_1 \cap X_2$ is the lower sequence

$$\cdots \leftarrow H_{q-1}(A) \overset{\Delta}{\leftarrow} H_q(X) \overset{\phi}{\leftarrow} H_q(X_1) + H_q(X_2) \overset{\psi}{\leftarrow} H_q(A) \leftarrow \cdots$$

The homology isomorphisms

- **Proposition** The top and bottom rows of an exact braid



  are chain complexes with isomorphic homology

$$\frac{\ker(B_n^+ \to A_{n-1}^-)}{\operatorname{im}(A_n^+ \to B_n^+)} \cong \frac{\ker(B_n^- \to A_{n-1}^+)}{\operatorname{im}(A_n^- \to B_n^-)} .$$

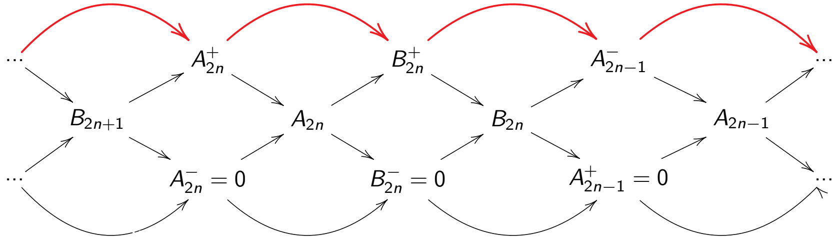
- The elements $b^+ \in \ker(B_n^+ \to A_{n-1}^-)$, $b^- \in \ker(B_n^- \to A_{n-1}^+)$ match up if and only if they have the same image in B_n.

4-periodicity

- An exact braid is **4-periodic** if

$$X_n \;=\; X_{n+4} \text{ for } X \in \{A, B, A^+, B^+, A^-, B^-\}\ .$$

- **Proposition** For a 4-periodic exact braid with bottom row 0

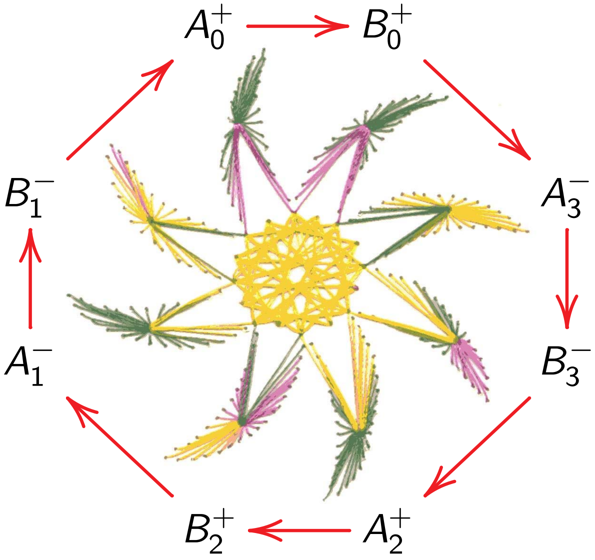


the top row is an exact sequence

$$\cdots \longrightarrow A^+_{2n} \longrightarrow B^+_{2n} \longrightarrow A^-_{2n-1} \longrightarrow B^-_{2n-1} \longrightarrow A^-_{2n-2} \longrightarrow \cdots$$

defining . . .

The exact octagon of a 4-periodic exact braid with bottom row 0



The coat of arms of the Isle of Man

The surgery exact braid

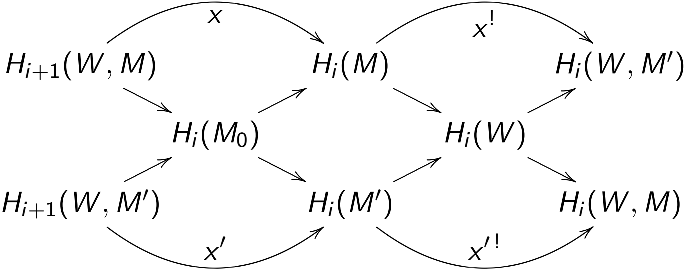
- Given an m-dimensional manifold M and $x : S^n \times D^{m-n} \subset M$ define the m-dimensional manifold M' obtained from M by **surgery**

$$M' \;=\; M_0 \cup D^{n+1} \times S^{m-n-1} \text{ with } M_0 \;=\; \text{cl.}(M \backslash S^n \times D^{m-n}) \;.$$

- The homology groups of the **trace cobordism**

$$(W; M, M') \;=\; (M \times I \cup D^{n+1} \times D^{m-n}; M, M')$$

fit into an exact braid



with $H_{n+1}(W, M) = \mathbb{Z}$, $H_{m-n}(W, M') = \mathbb{Z}$, $= 0$ otherwise.

Algebraic L-theory via forms and automorphisms

- Wall (1970) defined the 4-periodic **algebraic L-groups**

$$L_n(A) \ = \ L_{n+4}(A)$$

  of a ring with involution A. Applications to surgery theory of n-dimensional manifolds with $n \geqslant 5$.
- $L_{2k}(A)$ is the Witt group of nonsingular $(-)^k$-quadratic forms on f.g. free A-modules.
- $L_{2k+1}(A)$ is the commutator quotient of the stable unitary group of automorphisms of the hyperbolic $(-)^k$-quadratic forms on f.g. free A-modules.
- If X is an n-dimensional space with Poincaré duality and a normal vector bundle there is an obstruction in $L_n(\mathbb{Z}[\pi_1(X)])$ to X being homotopy equivalent to an n-dimensional manifold.
- If $f : M \to X$ is a normal homotopy equivalence of n-dimensional manifolds there is an obstruction in $L_{n+1}(\mathbb{Z}[\pi_1(X)])$ to f being homotopic to a diffeomorphism.

Algebraic L-theory via Poincaré chain complexes

- (R., 1980) Expression of $L_n(A)$ as the cobordism group of n-dimensional f.g. free A-module chain complexes
$$C \;:\; C_n \to C_{n-1} \to \cdots \to C_1 \to C_0$$
with an n-dimensional quadratic Poincaré duality
$$H^{n-*}(C) \;\cong\; H_*(C) \;.$$
- Quadratic Poincaré complexes C, C' are cobordant if there exists an $(n+1)$-dimensional f.g. free A-module chain complex D with chain maps $C \to D$, $C' \to D$ and an $(n+1)$-dimensional quadratic Poincaré-Lefschetz duality
$$H^{n+1-*}(D, C) \;\cong\; H_*(D, C') \;.$$
- The 4-periodicity isomorphisms are defined by double suspension
$$L_n(A) \to L_{n+4}(A) \;;\; C \mapsto S^2C$$
with $(S^2C)_r = C_{r-2}$.

Induction in L-theory

- A morphism of rings with involution $f : A \to B$ determines an **induction** functor of additive categories with duality involution

$$f_! \ : \ \{\text{f.g. free } A\text{-modules}\} \to \{\text{f.g. free } B\text{-modules}\} \ ; \ M \mapsto B \otimes_A M$$

- (R., 1980) The relative L-group $L_n(f_!)$ in the exact sequence

$$\cdots \longrightarrow L_n(A) \xrightarrow{f_!} L_n(B) \longrightarrow L_n(f_!) \longrightarrow L_{n-1}(A) \longrightarrow \cdots$$

  is the cobordism group of pairs (D, C) with C an $(n-1)$-dimensional quadratic Poincaré complex over A and D a null-cobordism of $f_! C$ over B

$$L_n(f_!) \to L_{n-1}(A) \ ; \ (D, C) \mapsto C \ .$$

Restriction in L-theory

- A morphism of rings with involution $f : A \to B$ with B f.g. free as an A-module determines the **restriction** functor

$$f^! \ : \ \{\text{f.g. free } B\text{-modules}\} \to \{\text{f.g. free } A\text{-modules}\} \ ; \ N \mapsto N$$

- (R., 1980) The relative L-group $L_n(f^!)$ in the exact sequence

$$\cdots \longrightarrow L_n(B) \xrightarrow{f^!} L_n(A) \longrightarrow L_n(f^!) \longrightarrow L_{n-1}(B) \longrightarrow \cdots$$

  is the cobordism group of pairs (D, C) with C an $(n-1)$-dimensional quadratic Poincaré complex over B and D a null-cobordism of $f^!C$ over A

$$L_n(f^!) \to L_{n-1}(B) \ ; \ (D, C) \mapsto C \ .$$

Quadratic extensions of a ring with involution

- Given a ring A and a non-square central unit $a \in A^\bullet$ let

$$A[\sqrt{a}] \;=\; A[t]/(t^2 - a)$$

be the quadratic extension of A adjoining the square roots of a.

- Given an involution $A \to A; x \mapsto \overline{x}$ with $\overline{a} = a$ let $A[\sqrt{a}]^+$, $A[\sqrt{a}]^-$ denote the ring $A[\sqrt{a}]$ with the involution on A extended by

$$A[\sqrt{a}]^+ \to A[\sqrt{a}]^+ \; ; \; x + y\sqrt{a} \mapsto x + y\sqrt{a} \; ,$$
$$A[\sqrt{a}]^- \to A[\sqrt{a}]^- \; ; \; x + y\sqrt{a} \mapsto x - y\sqrt{a} \; .$$

with the inclusions denoted by

$$i^+ \; : \; A \to A[\sqrt{a}]^+ \; , \; i^- \; : \; A \to A[\sqrt{a}]^- \; .$$

The Witt groups of a quadratic extension of a field

- **Proposition** (Jacobson 1940, Milnor and Husemoller 1973)
  Let K be a field with the identity involution, of characteristic $\neq 2$, and let $J = K[\sqrt{a}]$ be a quadratic extension for some non-square $a \in K^\bullet$.
  The Witt groups of J^+, J^-, K are related by an exact sequence

$$0 \longrightarrow L_0(J^-) \xrightarrow{(i^-)^!} L_0(K) \xrightarrow{i_!^+} L_0(J^+)$$

  with $i^+ : K \to J^+$, $i^- : K \to J^-$ the inclusions
- $(i^-)^!$ is a special case of the Scharlau transfer for the Witt groups of finite algebraic extensions of fields.
- There is also a version for characteristic 2.

$\mathbb{R}$ and $\mathbb{C}$

- **Example** For $K = \mathbb{R}$ with the identity involution and $a = -1$

$$K[\sqrt{a}]^+ = \mathbb{C} \text{ with identity involution },$$
$$K[\sqrt{a}]^- = \mathbb{C} \text{ with complex conjugation }.$$

- The signatures and mod 2 rank define isomorphisms

$$\text{signature}/2 : L_0(\mathbb{C}^-) \cong \mathbb{Z} ,$$
$$\text{signature} : L_0(\mathbb{R}) \cong \mathbb{Z} ,$$
$$\text{mod 2 rank} : L_0(\mathbb{C}^+) \cong \mathbb{Z}_2$$

- The Witt groups are related by the exact sequence

$$0 \longrightarrow L_0(\mathbb{C}^-) = \mathbb{Z} \xrightarrow{2} L_0(\mathbb{R}) = \mathbb{Z} \longrightarrow L_0(\mathbb{C}^+) = \mathbb{Z}_2 \longrightarrow 0$$

L-theory excision of quadratic extensions

- Browder and Livesay (1967), Wall (1970), Lopez de Medrano (1971) and Hambleton (1982) worked on the surgery obstruction theory for splitting homotopy equivalences of manifolds along codimension 1 submanifolds with nontrivial normal bundle, such as $\mathbb{RP}^n \subset \mathbb{RP}^{n+1}$, giving codimension 1 isomorphisms of relative L-groups for group rings.
- **Proposition** (R. 1987) For any ring with involution A the relative L-groups of induction and restriction of the inclusions

$$i^+ \ : \ A \to A[\sqrt{a}]^+ \ , \ i^- \ : \ A \to A[\sqrt{a}]^-$$

are related by isomorphisms

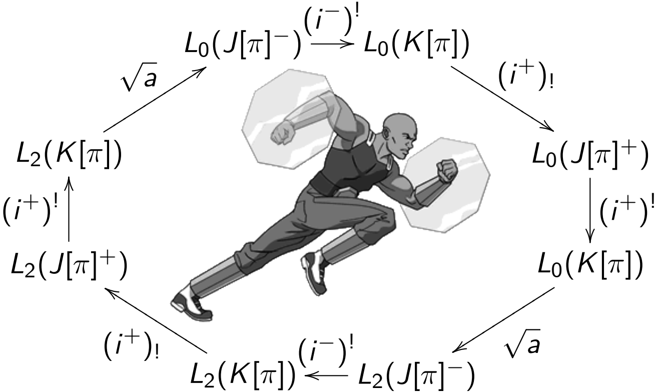
$$L_n(i^+_!) \xrightarrow{\cong} L_{n+1}(i^-_!) \ ; \ (D,C) \mapsto (SD,SC) \ ,$$

$$L_n((i^-)^!) \xrightarrow{\cong} L_{n+1}((i^+)^!) \ ; \ (D,C) \mapsto (SD,SC)$$

with $SC_r = C_{r-1}$.

Some results of David Lewis

- (1977) The computation of $L_{2*}(\mathbb{R}[\pi])$ for a finite group π in terms of the multisignature.
- (1983/5) The extensions of the Milnor-Husemoller exact sequence to exact octagons of Witt groups of $J[\pi]^+, J[\pi]^-, K[\pi]$ for a finite group π, and to Clifford algebras

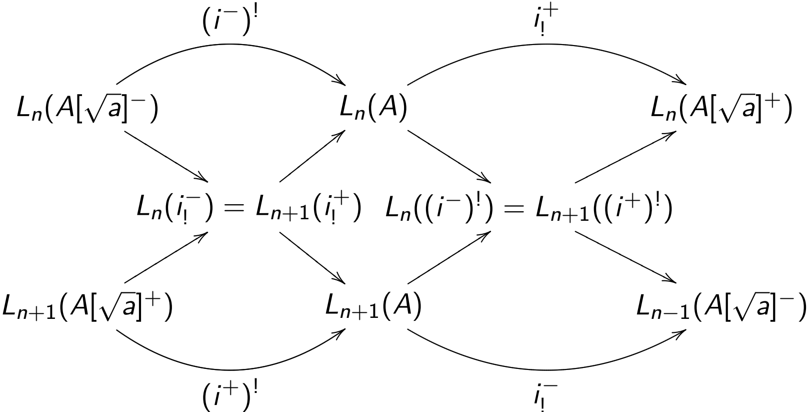


The L-theory exact braid of a quadratic extension

- **Proposition** (Hambleton-Taylor-Williams, R. 1984, 1987, 1992)
  The isomorphisms

$$L_*(i_!^+) \;\cong\; L_{*+1}(i_!^-)\ ,\ L_*((i^-)^!) \;\cong\; L_{*+1}((i^+)^!)$$

  determine an exact braid of L-groups



The L-theory exact octagon of a quadratic extension

- **Proposition** (R. 1978) $L_{2*+1}(A) = 0$ for a semisimple A.
- **Proposition** (Warshauer 1982, Lewis 1983/5, Hambleton, Taylor and Williams 1984, R. 1992, Grenier-Boley and Mahmoudi 2005) If A and $A[\sqrt{a}]$ are semisimple there is defined an exact octagon of Witt groups

$$L_0(A[\sqrt{a}]^-) \xrightarrow{(i^-)^!} L_0(A) \xrightarrow{(i^+)_!} L_0(A[\sqrt{a}]^+) \xrightarrow{(i^+)^!} L_0(A) \xrightarrow{\sqrt{a}} L_2(A[\sqrt{a}]^-) \xrightarrow{(i^-)^!} L_2(A) \xrightarrow{(i^+)_!} L_2(A[\sqrt{a}]^+) \xrightarrow{(i^+)^!} L_2(A) \xrightarrow{\sqrt{a}} L_0(A[\sqrt{a}]^-)$$

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