Algebras with involution hyperbolic over the function field of a conic

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Question

Can we describe the algebras with involution $(A, \sigma)/k$ that become hyperbolic over $F = k(\langle 1, -a, -b \rangle)$?

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Example

- (A, σ) hyperbolic $/k$
- $(Q, \bar{\ })$, where $Q = (a, b)_k$
- $(A, \sigma) \supset (Q, \bar{\ })$ that is $(A, \sigma) = (Q, \bar{\ }) \otimes (B, \tau)$

Typical example

φ odd-dimensional quadratic form over k
such that φ is isotropic over $F = k(\langle 1, -a, -b\rangle)$.

$$(A, \sigma) = (\mathcal{C}_0(\varphi), \mathrm{can}) \text{ is hyperbolic over } F$$

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Example

- $\varphi = \langle 1, -a, -b\rangle \quad \rightsquigarrow \quad (A, \sigma) = (Q, \bar{\ })$
- $\varphi \supset \langle 1, -a, -b\rangle \quad \rightsquigarrow \quad (A, \sigma) \supset (Q, \bar{\ })$

Question

Can we describe the quadratic forms φ/k that become isotropic over $F = k(\langle 1, -a, -b\rangle)$?

Example

- isotropic forms
- $\langle\lambda\rangle\langle 1, -a, -b\rangle$
- $\varphi \supset \langle\lambda\rangle\langle 1, -a, -b\rangle$

Question

Can we describe the quadratic forms φ/k that become isotropic over $F = k(\langle 1, -a, -b\rangle)$?

Example

- isotropic forms
- $\langle \lambda \rangle \langle 1, -a, -b\rangle$
- $\varphi \supset \langle \lambda \rangle \langle 1, -a, -b\rangle$

Question (Lam, 70's)

If φ/k is anisotropic and isotropic over $F = k(\langle 1, -a, -b\rangle)$, does $\varphi \supset \langle \lambda \rangle \langle 1, -a, -b\rangle$

Bibliography

- M. Rost : On quadratic forms isotropic over the function field of a conic. Math Annalen (1990)

Bibliography

- M. Rost : On quadratic forms isotropic over the function field of a conic. Math Annalen (1990)
- D. Lewis and J. Van Geel : Quadratic forms isotropic over the function field of a conic. Indag. Mathem. (1994)
- D. Hoffmann, D. Lewis and J. Van Geel : Minimal forms for functions fields of conics. Proc. Symposia in Pure Math. (1995)
- D. Hoffmann and J. Van Geel : Minimal forms with respect to function fields of conics. Manuscripta Math. (1995)

Minimal Forms of dimension 5

Theorem (Hoffmann - Lewis - Van Geel)

Let φ be a 5-dimensional form over k

φ is F-minimal

$$\Longleftrightarrow$$

- $\varphi \subset \langle\langle a, b, \ell\rangle\rangle$ *for some* $\ell \in k^\times$
- $\mathcal{C}_0(\varphi) \sim_{\mathrm{Br}} (c, d)_k$ *with* $(c, d)_k \otimes (a, b)_k$ *is division*

Degree 4 symplectic case

Theorem (Q-T)

Let (A,σ) be a degree 4 algebra with symplectic involution.

Assume $\begin{cases} (A,\sigma) \text{ is non-hyperbolic,} \\ (A,\sigma)_F \text{ is hyperbolic.} \end{cases}$ *Then*

Either $(A,\sigma) = (Q, \bar{\ }) \otimes (H,\rho)$ *(1)*

Or $(A,\sigma) = (M_2(F), \mathrm{ad}_{\langle\langle \ell \rangle\rangle}) \otimes ((c,d)_k, \bar{\ })$ *(2)*
with $(a,b)_k \otimes (c,d)_k$ *division*
and $\langle\langle \ell, a, b \rangle\rangle \simeq \langle\langle \ell, c, d \rangle\rangle$.

Subform Theorem

Theorem (QT)

Assume $A = M_n(F)$ or $M_n(Q)$, and (A, σ) is anisotropic.

$$\text{Then } (A, \sigma) \text{ is hyperbolic over } F$$
$$\Leftrightarrow$$
$$(A, \sigma) \supset (Q, \bar{\ }).$$

Degree 8 orthogonal case

Let φ be the 7-dimensional minimal form constructed by Hoffmann and Van Geel
And $(A, \sigma) = (\mathcal{C}_0(\varphi), \mathrm{c}an)$.

1) A is division ; hence (A, σ) is anisotropic.

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2) (A, σ) is hyperbolic over F.

3) $A \supset Q$;

4) $(A, \sigma) \not\supset (Q, ^{-})$.