Symmetrising operations, symmetric bilinear forms and central simple algebras

Seán McGarraghy

University College Dublin

LewisFest, July 2009

Outline

Outline

Outline

1. Exterior powers of forms and the Witt-Grothendieck ring
    - Preliminaries
    - Exterior powers
    - The Witt-Grothendieck ring as a λ-ring
2. Schur powers of forms
    - Immanants, idempotents and symmetrisers
    - Schur polynomials
3. Central simple algebras with involution
    - λ-powers of central simple algebras and extensions
    - Current and future work

Outline

1. Exterior powers of forms and the Witt-Grothendieck ring
   - Preliminaries
   - Exterior powers
   - The Witt-Grothendieck ring as a λ-ring
2. Schur powers of forms
   - Immanants, idempotents and symmetrisers
   - Schur polynomials
3. Central simple algebras with involution
   - λ-powers of central simple algebras and extensions
   - Current and future work

Preliminaries

Question

Exterior and other symmetrising powers of quadratic forms arise from actions of the symmetric group; can we define something similar for central simple algebras with involution?

Usual disclaimer: Let K be a field of characteristic $\neq 2$. All vector spaces will be finite-dimensional over K.

Notation

Let $\widehat{W}(K)^+$ denote the commutative cancellation semi-ring of isometry classes of symmetric bilinear forms under $\perp$ and $\otimes$, and let $\widehat{W}(K)$ be its Grothendieck completion, the **Witt-Grothendieck ring** of K. Then $W(K)$ is the quotient of $\widehat{W}(K)$ by the ideal generated by hyperbolic spaces.

Preliminaries

Question

Exterior and other symmetrising powers of quadratic forms arise from actions of the symmetric group; can we define something similar for central simple algebras with involution?

Usual disclaimer: Let K be a field of characteristic $\neq 2$. All vector spaces will be finite-dimensional over K.

Notation

Let $\widehat{W}(K)^+$ denote the commutative cancellation semi-ring of isometry classes of symmetric bilinear forms under $\perp$ and $\otimes$, and let $\widehat{W}(K)$ be its Grothendieck completion, the **Witt-Grothendieck ring** of K. Then $W(K)$ is the quotient of $\widehat{W}(K)$ by the ideal generated by hyperbolic spaces.

Preliminaries

Question

Exterior and other symmetrising powers of quadratic forms arise from actions of the symmetric group; can we define something similar for central simple algebras with involution?

Usual disclaimer: Let K be a field of characteristic $\neq 2$. All vector spaces will be finite-dimensional over K.

Notation

Let $\widehat{W}(K)^+$ denote the commutative cancellation semi-ring of isometry classes of symmetric bilinear forms under $\perp$ and $\otimes$, and let $\widehat{W}(K)$ be its Grothendieck completion, the **Witt-Grothendieck ring** of K. Then $W(K)$ is the quotient of $\widehat{W}(K)$ by the ideal generated by hyperbolic spaces.

Outline

Exterior powers

If V is a K-vector space with $\dim V = n$, then $\dim \bigwedge^k V = \binom{n}{k}$ (taken to be 0 for all $k > n$). Also

$$\bigwedge^k (V \oplus W) = \bigoplus_{i+j=k} \bigwedge^i V \otimes \bigwedge^j W.$$

What about bilinear forms?

Definition

Let $\varphi : V \times V \longrightarrow K$ be a bilinear form and let $k \in \mathbb{Z}$, $1 \leq k \leq n$. Define the k-fold **exterior power** $\bigwedge^k \varphi : \bigwedge^k V \times \bigwedge^k V \longrightarrow K$ by

$$\bigwedge^k \varphi(x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k) = \det\left(\varphi(x_i, y_j)\right)_{1 \leq i,j \leq k}.$$

We define $\bigwedge^0 \varphi := \langle 1 \rangle$, the identity form of dimension 1.
For $k > n$ we define $\bigwedge^k \varphi$ to be the zero form, since $\bigwedge^k V = 0$.

Exterior powers

If V is a K-vector space with $\dim V = n$, then $\dim \bigwedge^k V = \binom{n}{k}$ (taken to be 0 for all $k > n$). Also

$$\bigwedge^k (V \oplus W) = \bigoplus_{i+j=k} \bigwedge^i V \otimes \bigwedge^j W.$$

What about bilinear forms?

Definition

Let $\varphi : V \times V \longrightarrow K$ be a bilinear form and let $k \in \mathbb{Z}$, $1 \leq k \leq n$. Define the k-fold **exterior power** $\bigwedge^k \varphi : \bigwedge^k V \times \bigwedge^k V \longrightarrow K$ by

$$\bigwedge^k \varphi(x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k) = \det \left(\varphi(x_i, y_j)\right)_{1 \leq i,j \leq k}.$$

We define $\bigwedge^0 \varphi := \langle 1 \rangle$, the identity form of dimension 1.
For $k > n$ we define $\bigwedge^k \varphi$ to be the zero form, since $\bigwedge^k V = 0$.

Then $\bigwedge^k \varphi$ is a bilinear form, and is symmetric if φ is symmetric.

We have a similar definition for hermitian forms. We can define exterior powers of a quadratic form via its associated polar.

The exterior power construction also works for certain rings in which 2 is invertible, e.g., local rings.

Theorem

$\bigwedge^k$ is a (covariant) functor on the category of bilinear forms, and respects symmetry and non-singularity of forms.

Thus, if $\varphi \simeq \psi$, then $\bigwedge^k \varphi \simeq \bigwedge^k \psi$; so $\bigwedge^k$ is well-defined on $\widehat{W}(K)$.

However, φ hyperbolic $\not\Rightarrow$ $\bigwedge^k \varphi$ hyperbolic, as $\binom{2m}{k}$ may be odd.

Thus $\bigwedge^k$ is *not* well-defined on $W(K)$.

Then $\bigwedge^k \varphi$ is a bilinear form, and is symmetric if φ is symmetric.

We have a similar definition for hermitian forms. We can define exterior powers of a quadratic form via its associated polar.

The exterior power construction also works for certain rings in which 2 is invertible, e.g., local rings.

Theorem

$\bigwedge^k$ is a (covariant) functor on the category of bilinear forms, and respects symmetry and non-singularity of forms.

Thus, if $\varphi \simeq \psi$, then $\bigwedge^k \varphi \simeq \bigwedge^k \psi$; so $\bigwedge^k$ is well-defined on $\widehat{W}(K)$.

However, φ hyperbolic $\not\Rightarrow$ $\bigwedge^k \varphi$ hyperbolic, as $\binom{2m}{k}$ may be odd.

Thus $\bigwedge^k$ is *not* well-defined on $W(K)$.

Then $\bigwedge^k \varphi$ is a bilinear form, and is symmetric if φ is symmetric.

We have a similar definition for hermitian forms. We can define exterior powers of a quadratic form via its associated polar.

The exterior power construction also works for certain rings in which 2 is invertible, e.g., local rings.

Theorem

$\bigwedge^k$ is a (covariant) functor on the category of bilinear forms, and respects symmetry and non-singularity of forms.

Thus, if $\varphi \simeq \psi$, then $\bigwedge^k \varphi \simeq \bigwedge^k \psi$; so $\bigwedge^k$ is well-defined on $\widehat{W}(K)$.

However, φ hyperbolic $\not\Rightarrow$ $\bigwedge^k \varphi$ hyperbolic, as $\binom{2m}{k}$ may be odd.

Thus $\bigwedge^k$ is *not* well-defined on $W(K)$.

Then $\bigwedge^k \varphi$ is a bilinear form, and is symmetric if φ is symmetric.

We have a similar definition for hermitian forms. We can define exterior powers of a quadratic form via its associated polar.

The exterior power construction also works for certain rings in which 2 is invertible, e.g., local rings.

Theorem

$\bigwedge^k$ is a (covariant) functor on the category of bilinear forms, and respects symmetry and non-singularity of forms.

Thus, if $\varphi \simeq \psi$, then $\bigwedge^k \varphi \simeq \bigwedge^k \psi$; so $\bigwedge^k$ is well-defined on $\widehat{W}(K)$.

However, φ hyperbolic $\not\Rightarrow$ $\bigwedge^k \varphi$ hyperbolic, as $\binom{2m}{k}$ may be odd.

Thus $\bigwedge^k$ is *not* well-defined on $W(K)$.

Outline

λ-rings

Definition

A λ-**ring** is a commutative ring R with 1, and with unary operations $\lambda^n : R \longrightarrow R$, for $n = 0, 1, 2, \ldots$ such that for all $x, y \in R$:

(i) $\lambda^0(x) = 1$;

(ii) $\lambda^1(x) = x$;

(iii) $\lambda^n(x + y) = \sum_{i=0}^{n} \lambda^i(x)\lambda^{n-i}(y)$.

Definition

Equivalently: for $x \in R$, consider the formal power series in the indeterminate t defined by $\lambda_t(x) = \lambda^0(x) + \lambda^1(x)t + \lambda^2(x)t^2 + \cdots$. Then the conditions are (i), (ii) and

(iii′) $\lambda_t(x + y) = \lambda_t(x)\lambda_t(y)$.

We say that $x \in R$ is of **finite degree** n if $\lambda_t(x)$ is a polynomial of degree n.

For $x = x_1 + \cdots + x_n$, each x_i of degree 1, $\lambda^k(x)$ acts like the **elementary symmetric polynomial**

$$E_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}.$$

Definition

Equivalently: for $x \in R$, consider the formal power series in the indeterminate t defined by $\lambda_t(x) = \lambda^0(x) + \lambda^1(x)t + \lambda^2(x)t^2 + \cdots$. Then the conditions are (i), (ii) and

(iii′) $\lambda_t(x+y) = \lambda_t(x)\lambda_t(y)$.

We say that $x \in R$ is of **finite degree** n if $\lambda_t(x)$ is a polynomial of degree n.

For $x = x_1 + \cdots + x_n$, each x_i of degree 1, $\lambda^k(x)$ acts like the **elementary symmetric polynomial**

$$E_k = \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}.$$

As with vector spaces:

Theorem

For $\varphi, \psi \in \widehat{W}(K)$, we have $\bigwedge^k(\varphi \perp \psi) = \perp_{i+j=k} \bigwedge^i \varphi \otimes \bigwedge^j \psi$.

Since $\bigwedge^0\varphi$ is by definition the identity form, we have that $\widehat{W}(K)$ is a λ-ring with the exterior powers $\bigwedge^k$ acting as the λ-operations.

The degree of $\varphi \in \widehat{W}(K)^+$ is $\dim \varphi$ since $\bigwedge^k \varphi = 0$ for all $k > \dim \varphi$.

Since every form is a sum of 1-forms (each of λ-ring degree 1):

$$\bigwedge^k \langle a_1, \ldots, a_n \rangle = \perp_{1 \le i_1 < \cdots < i_k \le n} \langle a_{i_1} \cdots a_{i_k} \rangle = E_k(\langle a_1 \rangle, \ldots, \langle a_n \rangle).$$

As with vector spaces:

Theorem

For $\varphi, \psi \in \widehat{W}(K)$, we have $\bigwedge^k(\varphi \perp \psi) = \perp_{i+j=k} \bigwedge^i \varphi \otimes \bigwedge^j \psi$.

Since $\bigwedge^0 \varphi$ is by definition the identity form, we have that $\widehat{W}(K)$ is a λ-ring with the exterior powers $\bigwedge^k$ acting as the λ-operations.

The degree of $\varphi \in \widehat{W}(K)^+$ is $\dim \varphi$ since $\bigwedge^k \varphi = 0$ for all $k > \dim \varphi$.

Since every form is a sum of 1-forms (each of λ-ring degree 1):

$$\bigwedge^k \langle a_1, \ldots, a_n \rangle = \perp_{1 \le i_1 < \cdots < i_k \le n} \langle a_{i_1} \cdots a_{i_k} \rangle = E_k(\langle a_1 \rangle, \ldots, \langle a_n \rangle).$$

As with vector spaces:

Theorem

For $\varphi, \psi \in \widehat{W}(K)$, we have $\bigwedge^k(\varphi \perp \psi) = \bigperp_{i+j=k} \bigwedge^i \varphi \otimes \bigwedge^j \psi$.

Since $\bigwedge^0 \varphi$ is by definition the identity form, we have that $\widehat{W}(K)$ is a λ-ring with the exterior powers $\bigwedge^k$ acting as the λ-operations.

The degree of $\varphi \in \widehat{W}(K)^+$ is $\dim \varphi$ since $\bigwedge^k \varphi = 0$ for all $k > \dim \varphi$.

Since every form is a sum of 1-forms (each of λ-ring degree 1):

$$\bigwedge^k \langle a_1, \ldots, a_n \rangle = \bigperp_{1 \le i_1 < \cdots < i_k \le n} \langle a_{i_1} \cdots a_{i_k} \rangle = E_k(\langle a_1 \rangle, \ldots, \langle a_n \rangle).$$

As with vector spaces:

Theorem

For $\varphi, \psi \in \widehat{W}(K)$, we have $\bigwedge^k(\varphi \perp \psi) = \perp_{i+j=k} \bigwedge^i \varphi \otimes \bigwedge^j \psi$.

Since $\bigwedge^0 \varphi$ is by definition the identity form, we have that $\widehat{W}(K)$ is a λ-ring with the exterior powers $\bigwedge^k$ acting as the λ-operations.

The degree of $\varphi \in \widehat{W}(K)^+$ is $\dim \varphi$ since $\bigwedge^k \varphi = 0$ for all $k > \dim \varphi$.

Since every form is a sum of 1-forms (each of λ-ring degree 1):

$$\bigwedge^k \langle a_1, \dots, a_n \rangle = \perp_{1 \le i_1 < \cdots < i_k \le n} \langle a_{i_1} \cdots a_{i_k} \rangle = E_k(\langle a_1 \rangle, \dots, \langle a_n \rangle).$$

Power Sums

The (symmetric) power sums P_j may be defined in terms of the elementary symmetric polynomials E_k via Newton's formulas:

$$P_j = P_{j-1}E_1 - P_{j-2}E_2 + \cdots + (-1)^{j-2}P_1E_{j-1} + j(-1)^{j-1}E_j.$$

Solving for E_k by Cramer's rule gives

$$k! \times E_k = \det \begin{pmatrix} P_1 & 1 & & & & \\ P_2 & P_1 & 2 & & 0 & \\ P_3 & P_2 & P_1 & 3 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ P_{k-1} & \ddots & \ddots & P_2 & P_1 & k-1 \\ P_k & P_{k-1} & \dots & P_3 & P_2 & P_1 \end{pmatrix}.$$

Power Sums

The (symmetric) power sums P_j may be defined in terms of the elementary symmetric polynomials E_k via Newton's formulas:

$$P_j = P_{j-1}E_1 - P_{j-2}E_2 + \cdots + (-1)^{j-2}P_1E_{j-1} + j(-1)^{j-1}E_j.$$

Solving for E_k by Cramer's rule gives

$$k! \times E_k = \det \begin{pmatrix} P_1 & 1 & & & & \\ P_2 & P_1 & 2 & & 0 & \\ P_3 & P_2 & P_1 & 3 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ P_{k-1} & \ddots & \ddots & P_2 & P_1 & k-1 \\ P_k & P_{k-1} & \dots & P_3 & P_2 & P_1 \end{pmatrix}.$$

Definition

Let R be a λ-ring, let $x \in R$ and let k be a positive integer. We define the k^{th} **Adams operation**, Ψ^k, by

$$\Psi^k(x) := P_k(x).$$

There is a very simple characterisation of Adams operations in the Witt-Grothendieck λ-ring:

Proposition

Let $n, k \in \mathbb{N}$ and let φ be an n-dimensional form. Then, in $\widehat{W}(K)$,

$$\Psi^k(\varphi) = \begin{cases} n \times \langle 1 \rangle, & \text{for } k \text{ even;} \\ \varphi, & \text{for } k \text{ odd.} \end{cases}$$

Definition

Let R be a λ-ring, let $x \in R$ and let k be a positive integer. We define the k^{th} **Adams operation**, Ψ^k, by

$$\Psi^k(x) := P_k(x).$$

There is a very simple characterisation of Adams operations in the Witt-Grothendieck λ-ring:

Proposition

Let $n, k \in \mathbb{N}$ and let φ be an n-dimensional form. Then, in $\widehat{W}(K)$,

$$\Psi^k(\varphi) = \begin{cases} n \times \langle 1 \rangle, & \text{for } k \text{ even;} \\ \varphi, & \text{for } k \text{ odd.} \end{cases}$$

Thus the earlier Cramer's rule determinant becomes

$$k! \times \bigwedge^k(\varphi) = \det \begin{pmatrix} \varphi & 1 & & & & \\ n & \varphi & 2 & & 0 & \\ \varphi & n & \varphi & 3 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & n & \varphi & k-1 \\ & & \dots & \varphi & n & \varphi \end{pmatrix} \quad (*).$$

Let p_n be the Lewis polynomial

$$p_n(t) := (t-n)(t-n+2)\cdots(t+n) \in \mathbb{Z}[t].$$

Thus the earlier Cramer's rule determinant becomes

$$k! \times \bigwedge^k(\varphi) = \det \begin{pmatrix} \varphi & 1 & & & & \\ n & \varphi & 2 & & 0 & \\ \varphi & n & \varphi & 3 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & n & \varphi & k-1 \\ & & \dots & \varphi & n & \varphi \end{pmatrix} \quad (*).$$

Let p_n be the Lewis polynomial

$$p_n(t) := (t-n)(t-n+2)\cdots(t+n) \in \mathbb{Z}[t].$$

Using elementary row and column operations on $(*)$ and setting $k = n + 1$ gives

Theorem

Let (V, φ) be a symmetric bilinear space of dimension n. Then

$$(n+1)! \times \bigwedge^{n+1}\varphi = p_n(\varphi) \text{ in } \widehat{W}(K).$$

Since $\binom{n}{n+1} = 0$, we have recovered David Lewis's result:

Theorem

Let (V, φ) be a symmetric bilinear space of dimension n. Then the polynomial p_n annihilates the Witt class of φ in $W(K)$.

Using elementary row and column operations on $(*)$ and setting $k = n+1$ gives

Theorem

Let (V, φ) be a symmetric bilinear space of dimension n. Then

$$(n+1)! \times \bigwedge^{n+1} \varphi = p_n(\varphi) \text{ in } \widehat{W}(K).$$

Since $\binom{n}{n+1} = 0$, we have recovered David Lewis's result:

Theorem

Let (V, φ) be a symmetric bilinear space of dimension n. Then the polynomial p_n annihilates the Witt class of φ in $W(K)$.

Outline

Partitions

Let K be of characteristic 0. We'll see three main extensions of the exterior powers, the first two based on symmetrising projection operators in the group algebra $K[S_k]$.

Definition

Let k be a positive integer. A **partition** of k is a sequence of non-negative integers

$$\pi = [\pi_1, \pi_2, \pi_3, \ldots]$$

with $\pi_1 \geq \pi_2 \geq \pi_3 \geq \ldots$ and $\sum_i \pi_i = k$. We write $\pi \vdash k$.

Can we extend the idea of k-fold exterior power — which is associated to the partition $[1^k] = [1, 1, \ldots, 1]$ of k — to other partitions of k?

Partitions

Let K be of characteristic 0. We'll see three main extensions of the exterior powers, the first two based on symmetrising projection operators in the group algebra $K[S_k]$.

Definition

Let k be a positive integer. A **partition** of k is a sequence of non-negative integers

$$\pi = [\pi_1, \pi_2, \pi_3, \ldots]$$

with $\pi_1 \geq \pi_2 \geq \pi_3 \geq \ldots$ and $\sum_i \pi_i = k$. We write $\pi \vdash k$.

Can we extend the idea of k-fold exterior power — which is associated to the partition $[1^k] = [1, 1, \ldots, 1]$ of k — to other partitions of k?

Partitions

Let K be of characteristic 0. We'll see three main extensions of the exterior powers, the first two based on symmetrising projection operators in the group algebra $K[S_k]$.

Definition

Let k be a positive integer. A **partition** of k is a sequence of non-negative integers

$$\pi = [\pi_1, \pi_2, \pi_3, \ldots]$$

with $\pi_1 \geq \pi_2 \geq \pi_3 \geq \ldots$ and $\sum_i \pi_i = k$. We write $\pi \vdash k$.

Can we extend the idea of k-fold exterior power — which is associated to the partition $[1^k] = [1, 1, \ldots, 1]$ of k — to other partitions of k?

Idempotents: Young Symmetrisers

As each conjugacy class in S_k is determined by its cycle structure, each $\pi \vdash k$ gives an irreducible character, χ_π, of S_k. One defines a primitive idempotent, the **Young symmetriser** c_π, in $K[S_k]$. (We need to fix a tableau: a labelling of the Young diagram of π.)

Weyl construction: $\mathbf{GL}(V)$ acts diagonally on $V^{\otimes k}$ on the left. Also S_k acts on $V^{\otimes k}$ (on the right) by permuting factors,

$$(v_1 \otimes \cdots \otimes v_k) \cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}.$$

Thus c_π may be viewed as an endomorphism of $V^{\otimes k}$.

We denote the image of $V^{\otimes k}$ under c_π by $\mathbb{S}^\pi V := \mathrm{Im}(c_\pi|_{V^{\otimes k}})$. It is an irreducible representation of $\mathbf{GL}(V)$.

$\mathbb{S}^\pi$ is a functor from the category of K-vector spaces to itself.

Idempotents: Young Symmetrisers

As each conjugacy class in S_k is determined by its cycle structure, each $\pi \vdash k$ gives an irreducible character, χ_π, of S_k. One defines a primitive idempotent, the **Young symmetriser** c_π, in $K[S_k]$.
(We need to fix a tableau: a labelling of the Young diagram of π.)

Weyl construction: $\mathbf{GL}(V)$ acts diagonally on $V^{\otimes k}$ on the left.
Also S_k acts on $V^{\otimes k}$ (on the right) by permuting factors,

$$(v_1 \otimes \cdots \otimes v_k) \cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}.$$

Thus c_π may be viewed as an endomorphism of $V^{\otimes k}$.

We denote the image of $V^{\otimes k}$ under c_π by $\mathbb{S}^\pi V := \mathrm{Im}(c_\pi|_{V^{\otimes k}})$.
It is an irreducible representation of $\mathbf{GL}(V)$.

$\mathbb{S}^\pi$ is a functor from the category of K-vector spaces to itself.

Idempotents: Young Symmetrisers

As each conjugacy class in S_k is determined by its cycle structure, each $\pi \vdash k$ gives an irreducible character, χ_π, of S_k. One defines a primitive idempotent, the **Young symmetriser** c_π, in $K[S_k]$. (We need to fix a tableau: a labelling of the Young diagram of π.)

Weyl construction: $\mathbf{GL}(V)$ acts diagonally on $V^{\otimes k}$ on the left. Also S_k acts on $V^{\otimes k}$ (on the right) by permuting factors,

$$(v_1 \otimes \cdots \otimes v_k) \cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}.$$

Thus c_π may be viewed as an endomorphism of $V^{\otimes k}$.

We denote the image of $V^{\otimes k}$ under c_π by $\mathbb{S}^\pi V := \mathrm{Im}(c_\pi|_{V^{\otimes k}})$. It is an irreducible representation of $\mathbf{GL}(V)$.

$\mathbb{S}^\pi$ is a functor from the category of K-vector spaces to itself.

Idempotents: Young Symmetrisers

As each conjugacy class in S_k is determined by its cycle structure, each $\pi \vdash k$ gives an irreducible character, χ_π, of S_k. One defines a primitive idempotent, the **Young symmetriser** c_π, in $K[S_k]$. (We need to fix a tableau: a labelling of the Young diagram of π.)

Weyl construction: $\mathbf{GL}(V)$ acts diagonally on $V^{\otimes k}$ on the left. Also S_k acts on $V^{\otimes k}$ (on the right) by permuting factors,

$$(v_1 \otimes \cdots \otimes v_k) \cdot \sigma = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(k)}.$$

Thus c_π may be viewed as an endomorphism of $V^{\otimes k}$.

We denote the image of $V^{\otimes k}$ under c_π by $\mathbb{S}^\pi V := \mathrm{Im}(c_\pi|_{V^{\otimes k}})$. It is an irreducible representation of $\mathbf{GL}(V)$.

$\mathbb{S}^\pi$ is a functor from the category of K-vector spaces to itself.

Immanants

Definition

Let $A = (a_{ij})$ be a $k \times k$ matrix. Let χ be a (complex) character of the symmetric group S_k. Then the **immanant of A associated to χ** is

$$d_\chi(A) := \sum_{\sigma \in S_k} \chi(\sigma) a_{1\sigma(1)} \cdots a_{k\sigma(k)}.$$

(Recall: every complex character of S_k is integer-valued.)

The determinant is the immanant associated to the sign character; the permanent is the immanant associated to the trivial character.

Immanants

Definition

Let $A = (a_{ij})$ be a $k \times k$ matrix. Let χ be a (complex) character of the symmetric group S_k. Then the **immanant of A associated to χ** is

$$d_\chi(A) := \sum_{\sigma \in S_k} \chi(\sigma) a_{1\sigma(1)} \cdots a_{k\sigma(k)}.$$

(Recall: every complex character of S_k is integer-valued.)

The determinant is the immanant associated to the sign character; the permanent is the immanant associated to the trivial character.

Central idempotents

We also have a central idempotent,

$$\eta_\pi = \frac{\deg(\chi_\pi)}{k!} \sum_{\sigma \in S_k} \chi_\pi(\sigma)\sigma \in K[S_k]$$

and we write

$$\mathbb{P}^\pi V := \operatorname{Im}(\eta_\pi|_{V^{\otimes k}}) = \operatorname{Span}\{v_1 * \cdots * v_k : v_1, \ldots, v_k \in V\}.$$

We can define $\mathbb{P}^\pi \varphi : \mathbb{P}^\pi V \times \mathbb{P}^\pi V \longrightarrow K$ by the immanant

$$\mathbb{P}^\pi \varphi(x_1 * \cdots * x_k, y_1 * \cdots * y_k) = d_{\chi_\pi}(\varphi(x_i, y_j)).$$

Proposition

The functors $\mathbb{P}^\pi$ and $\mathbb{S}^\pi$ are well-defined on isometry classes of forms and in $\widehat{W}(K)$.

Central idempotents

We also have a central idempotent,

$$\eta_\pi = \frac{\deg(\chi_\pi)}{k!} \sum_{\sigma \in S_k} \chi_\pi(\sigma)\sigma \in K[S_k]$$

and we write

$$\mathbb{P}^\pi V := \operatorname{Im}(\eta_\pi|_{V^{\otimes k}}) = \operatorname{Span}\{v_1 * \cdots * v_k : v_1, \ldots, v_k \in V\}.$$

We can define $\mathbb{P}^\pi \varphi : \mathbb{P}^\pi V \times \mathbb{P}^\pi V \longrightarrow K$ by the immanant

$$\mathbb{P}^\pi \varphi(x_1 * \cdots * x_k, y_1 * \cdots * y_k) = d_{\chi_\pi}(\varphi(x_i, y_j)).$$

Proposition

The functors $\mathbb{P}^\pi$ and $\mathbb{S}^\pi$ are well-defined on isometry classes of forms and in $\widehat{W}(K)$.

Central idempotents

We also have a central idempotent,

$$\eta_\pi = \frac{\deg(\chi_\pi)}{k!} \sum_{\sigma \in S_k} \chi_\pi(\sigma)\sigma \in K[S_k]$$

and we write

$$\mathbb{P}^\pi V := \operatorname{Im}(\eta_\pi|_{V^{\otimes k}}) = \operatorname{Span}\{v_1 * \cdots * v_k : v_1, \ldots, v_k \in V\}.$$

We can define $\mathbb{P}^\pi \varphi : \mathbb{P}^\pi V \times \mathbb{P}^\pi V \longrightarrow K$ by the immanant

$$\mathbb{P}^\pi \varphi(x_1 * \cdots * x_k, y_1 * \cdots * y_k) = d_{\chi_\pi}(\varphi(x_i, y_j)).$$

Proposition

The functors $\mathbb{P}^\pi$ and $\mathbb{S}^\pi$ are well-defined on isometry classes of forms and in $\widehat{W}(K)$.

Outline

The **Schur polynomial** associated to $\pi \vdash k$ in n indeterminates $X_1, \ldots, X_n$ is the symmetric homogeneous polynomial of degree k

$$s_\pi = s_\pi(X_1, \ldots, X_n) = \frac{\det(X_j^{\pi_i + n - i})}{\det(X_j^{n-i})}.$$

It has non-negative integer coefficients: the **Kostka numbers**.

Note: $\dim \mathbb{S}^\pi V = s_\pi(1, \ldots, 1)$.

The polynomial Schur power of a diagonalised s.b.f. $\varphi = \langle a_1, \ldots, a_n \rangle : V \times V \longrightarrow K$ (or its class in $\widehat{W}(K)^+$) is

$$\mathbb{T}^\pi \varphi : \mathbb{S}^\pi V \times \mathbb{S}^\pi V \longrightarrow K \quad \text{defined by} \quad \mathbb{T}^\pi \varphi = s_\pi(\langle a_1 \rangle, \ldots, \langle a_n \rangle).$$

This is well-defined on $\widehat{W}(K)$ since every symmetric function is a polynomial in the elementary symmetric polynomials.

The **Schur polynomial** associated to $\pi \vdash k$ in n indeterminates $X_1, \ldots, X_n$ is the symmetric homogeneous polynomial of degree k

$$s_\pi = s_\pi(X_1, \ldots, X_n) = \frac{\det(X_j^{\pi_i + n - i})}{\det(X_j^{n-i})}.$$

It has non-negative integer coefficients: the **Kostka numbers**.

Note: $\dim \mathbb{S}^\pi V = s_\pi(1, \ldots, 1)$.

The polynomial Schur power of a diagonalised s.b.f. $\varphi = \langle a_1, \ldots, a_n \rangle : V \times V \longrightarrow K$ (or its class in $\widehat{W}(K)^+$) is

$$\mathbb{T}^\pi \varphi : \mathbb{S}^\pi V \times \mathbb{S}^\pi V \longrightarrow K \quad \text{defined by} \quad \mathbb{T}^\pi \varphi = s_\pi(\langle a_1 \rangle, \ldots, \langle a_n \rangle).$$

This is well-defined on $\widehat{W}(K)$ since every symmetric function is a polynomial in the elementary symmetric polynomials.

The **Schur polynomial** associated to $\pi \vdash k$ in n indeterminates $X_1, \ldots, X_n$ is the symmetric homogeneous polynomial of degree k

$$s_\pi = s_\pi(X_1, \ldots, X_n) = \frac{\det(X_j^{\pi_i + n - i})}{\det(X_j^{n-i})}.$$

It has non-negative integer coefficients: the **Kostka numbers**.

Note: $\dim \mathbb{S}^\pi V = s_\pi(1, \ldots, 1)$.

The polynomial Schur power of a diagonalised s.b.f. $\varphi = \langle a_1, \ldots, a_n \rangle : V \times V \longrightarrow K$ (or its class in $\widehat{W}(K)^+$) is

$$\mathbb{T}^\pi \varphi : \mathbb{S}^\pi V \times \mathbb{S}^\pi V \longrightarrow K \quad \text{defined by} \quad \mathbb{T}^\pi \varphi = s_\pi(\langle a_1 \rangle, \ldots, \langle a_n \rangle).$$

This is well-defined on $\widehat{W}(K)$ since every symmetric function is a polynomial in the elementary symmetric polynomials.

The **Schur polynomial** associated to $\pi \vdash k$ in n indeterminates $X_1, \ldots, X_n$ is the symmetric homogeneous polynomial of degree k

$$s_\pi = s_\pi(X_1, \ldots, X_n) = \frac{\det(X_j^{\pi_i+n-i})}{\det(X_j^{n-i})}.$$

It has non-negative integer coefficients: the **Kostka numbers**.

Note: $\dim \mathbb{S}^\pi V = s_\pi(1, \ldots, 1)$.

The polynomial Schur power of a diagonalised s.b.f. $\varphi = \langle a_1, \ldots, a_n \rangle : V \times V \longrightarrow K$ (or its class in $\widehat{W}(K)^+$) is

$$\mathbb{T}^\pi \varphi : \mathbb{S}^\pi V \times \mathbb{S}^\pi V \longrightarrow K \quad \text{defined by} \quad \mathbb{T}^\pi \varphi = s_\pi(\langle a_1 \rangle, \ldots, \langle a_n \rangle).$$

This is well-defined on $\widehat{W}(K)$ since every symmetric function is a polynomial in the elementary symmetric polynomials.

The **Schur polynomial** associated to $\pi \vdash k$ in n indeterminates $X_1, \ldots, X_n$ is the symmetric homogeneous polynomial of degree k

$$s_\pi = s_\pi(X_1, \ldots, X_n) = \frac{\det(X_j^{\pi_i + n - i})}{\det(X_j^{n-i})}.$$

It has non-negative integer coefficients: the **Kostka numbers**.

Note: $\dim \mathbb{S}^\pi V = s_\pi(1, \ldots, 1)$.

The polynomial Schur power of a diagonalised s.b.f. $\varphi = \langle a_1, \ldots, a_n \rangle : V \times V \longrightarrow K$ (or its class in $\widehat{W}(K)^+$) is

$$\mathbb{T}^\pi \varphi : \mathbb{S}^\pi V \times \mathbb{S}^\pi V \longrightarrow K \quad \text{defined by} \quad \mathbb{T}^\pi \varphi = s_\pi(\langle a_1 \rangle, \ldots, \langle a_n \rangle).$$

This is well-defined on $\widehat{W}(K)$ since every symmetric function is a polynomial in the elementary symmetric polynomials.

Giambelli's formula $s_\pi = \det(E_{\mu_i - i + j})$ gives

$$\mathbb{T}^\pi \varphi = \det(\textstyle\bigwedge^{\mu_i - i + j} \varphi)$$

where $\mu = [\mu_1, \ldots, \mu_\ell]$ is the conjugate partition of π.

One can also show that $k! \times \mathbb{T}^\pi \varphi =$

$$d_{\chi_\pi} \begin{pmatrix} \Psi^1(\varphi) & 1 & & & & \\ \Psi^2(\varphi) & \Psi^1(\varphi) & 2 & & & 0 \\ \Psi^3(\varphi) & \Psi^2(\varphi) & \Psi^1(\varphi) & 3 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ \Psi^{k-1}(\varphi) & \Psi^{k-2}(\varphi) & \ddots & \ddots & \Psi^1(\varphi) & k-1 \\ \Psi^k(\varphi) & \Psi^{k-1}(\varphi) & \Psi^{k-2}(\varphi) & \ldots & \Psi^2(\varphi) & \Psi^1(\varphi) \end{pmatrix}$$

and use this to derive annihilating polynomials in $W(K)$ as before.

Giambelli's formula $s_\pi = \det(E_{\mu_i - i + j})$ gives

$$\mathbb{T}^\pi \varphi = \det(\textstyle\bigwedge^{\mu_i - i + j} \varphi)$$

where $\mu = [\mu_1, \ldots, \mu_\ell]$ is the conjugate partition of π.

One can also show that $k! \times \mathbb{T}^\pi \varphi =$

$$d_{\chi_\pi} \begin{pmatrix} \Psi^1(\varphi) & 1 & & & & \\ \Psi^2(\varphi) & \Psi^1(\varphi) & 2 & & & 0 \\ \Psi^3(\varphi) & \Psi^2(\varphi) & \Psi^1(\varphi) & 3 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \\ \Psi^{k-1}(\varphi) & \Psi^{k-2}(\varphi) & \ddots & \ddots & \Psi^1(\varphi) & k-1 \\ \Psi^k(\varphi) & \Psi^{k-1}(\varphi) & \Psi^{k-2}(\varphi) & \ldots & \Psi^2(\varphi) & \Psi^1(\varphi) \end{pmatrix}$$

and use this to derive annihilating polynomials in $W(K)$ as before.

Outline

1. Exterior powers of forms and the Witt-Grothendieck ring
   - Preliminaries
   - Exterior powers
   - The Witt-Grothendieck ring as a λ-ring
2. Schur powers of forms
   - Immanants, idempotents and symmetrisers
   - Schur polynomials
3. Central simple algebras with involution
   - λ-powers of central simple algebras and extensions
   - Current and future work

Question

How far can definitions on vector spaces/elements of $\widehat{W}(K)$ be carried over to

- *central simple K-algebras with involution;*
- *other K-algebras with involution (or just antiautomorphism)?*

In [KMRT, 1998], λ-powers of central simple algebras with involution are defined using an S_k action (c.f. Weyl construction).

The **Goldman element** is the unique element $\alpha \in A \otimes_K A$ such that $\mathrm{Sand}(\alpha) = \mathrm{Trd}_A$, the reduced trace.

α plays the role of a transposition in this sense: if $A = \mathrm{End}_K V$, then, identifying $A \otimes_K A = \mathrm{End}_K(V \otimes_K V)$, α is defined by

$$\alpha(v_1 \otimes v_2) = v_2 \otimes v_1 \quad \text{for all } v_1, v_2 \in V.$$

Question

How far can definitions on vector spaces/elements of $\widehat{W}(K)$ be carried over to

- *central simple K-algebras with involution;*
- *other K-algebras with involution (or just antiautomorphism)?*

In [KMRT, 1998], λ-powers of central simple algebras with involution are defined using an S_k action (c.f. Weyl construction).

The **Goldman element** is the unique element $\alpha \in A \otimes_K A$ such that $\text{Sand}(\alpha) = \text{Trd}_A$, the reduced trace.

α plays the role of a transposition in this sense: if $A = \text{End}_K V$, then, identifying $A \otimes_K A = \text{End}_K(V \otimes_K V)$, α is defined by

$$\alpha(v_1 \otimes v_2) = v_2 \otimes v_1 \quad \text{for all } v_1, v_2 \in V.$$

Question

How far can definitions on vector spaces/elements of $\widehat{W}(K)$ be carried over to

- *central simple K-algebras with involution;*
- *other K-algebras with involution (or just antiautomorphism)?*

In [KMRT, 1998], λ-powers of central simple algebras with involution are defined using an S_k action (c.f. Weyl construction).

The **Goldman element** is the unique element $\alpha \in A \otimes_K A$ such that $\text{Sand}(\alpha) = \text{Trd}_A$, the reduced trace.

α plays the role of a transposition in this sense: if $A = \text{End}_K V$, then, identifying $A \otimes_K A = \text{End}_K(V \otimes_K V)$, α is defined by

$$\alpha(v_1 \otimes v_2) = v_2 \otimes v_1 \quad \text{for all } v_1, v_2 \in V.$$

Question

How far can definitions on vector spaces/elements of $\widehat{W}(K)$ be carried over to

- *central simple K-algebras with involution;*
- *other K-algebras with involution (or just antiautomorphism)?*

In [KMRT, 1998], λ-powers of central simple algebras with involution are defined using an S_k action (c.f. Weyl construction).

The **Goldman element** is the unique element $\alpha \in A \otimes_K A$ such that $\mathrm{Sand}(\alpha) = \mathrm{Trd}_A$, the reduced trace.

α plays the role of a transposition in this sense: if $A = \mathrm{End}_K V$, then, identifying $A \otimes_K A = \mathrm{End}_K(V \otimes_K V)$, α is defined by

$$\alpha(v_1 \otimes v_2) = v_2 \otimes v_1 \quad \text{for all } v_1, v_2 \in V.$$

We proceed as in [KMRT, 10.A]:

Proposition

Let $k \geq 1$. There is a natural homomorphism $\alpha_k : S_k \longrightarrow (A^{\otimes k})^\times$ such that in case $A = \mathrm{End}_K V$, and identifying $A^{\otimes k} = \mathrm{End}_K(V^{\otimes k})$,

$$\alpha_k(\sigma)(v_1 \otimes \cdots \otimes v_k) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}$$

for all $\sigma \in S_k$ and $v_1, \ldots, v_k \in V$.

Let $\pi \vdash k$. We define

$$\zeta_\pi = \sum_{\sigma \in S_k} \chi_\pi(\sigma)\alpha_k(\sigma) \in A^{\otimes k}$$

Given a tableau on π, we can also define an element e_π of $A^{\otimes k}$ analogous to a Young symmetriser.

We proceed as in [KMRT, 10.A]:

Proposition

Let $k \geq 1$. There is a natural homomorphism $\alpha_k : S_k \longrightarrow (A^{\otimes k})^\times$ such that in case $A = \mathrm{End}_K V$, and identifying $A^{\otimes k} = \mathrm{End}_K(V^{\otimes k})$,

$$\alpha_k(\sigma)(v_1 \otimes \cdots \otimes v_k) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}$$

for all $\sigma \in S_k$ and $v_1, \ldots, v_k \in V$.

Let $\pi \vdash k$. We define

$$\zeta_\pi = \sum_{\sigma \in S_k} \chi_\pi(\sigma)\alpha_k(\sigma) \in A^{\otimes k}$$

Given a tableau on π, we can also define an element e_π of $A^{\otimes k}$ analogous to a Young symmetriser.

We proceed as in [KMRT, 10.A]:

Proposition

Let $k \geq 1$. There is a natural homomorphism $\alpha_k : S_k \longrightarrow (A^{\otimes k})^\times$ such that in case $A = \mathrm{End}_K V$, and identifying $A^{\otimes k} = \mathrm{End}_K(V^{\otimes k})$,

$$\alpha_k(\sigma)(v_1 \otimes \cdots \otimes v_k) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(k)}$$

for all $\sigma \in S_k$ and $v_1, \ldots, v_k \in V$.

Let $\pi \vdash k$. We define

$$\zeta_\pi = \sum_{\sigma \in S_k} \chi_\pi(\sigma)\alpha_k(\sigma) \in A^{\otimes k}$$

Given a tableau on π, we can also define an element e_π of $A^{\otimes k}$ analogous to a Young symmetriser.

Lemma

If A is split, there are natural isomorphisms of $A^{\otimes k}$-modules:

$$A^{\otimes k} e_\pi = \mathrm{Hom}_K(\mathbb{S}^\pi V, V^{\otimes k}), \quad A^{\otimes k} \zeta_\pi = \mathrm{Hom}_K(\mathbb{P}^\pi V, V^{\otimes k})$$

with reduced dimensions
$\mathrm{rdim}(A^{\otimes k} e_\pi) = s_\pi(1, \dots, 1)$, $\mathrm{rdim}(A^{\otimes k} \zeta_\pi) = \deg(\chi_\pi) s_\pi(1, \dots, 1)$.

Definition

Let $\pi \vdash k$. Define $\mathbb{S}^\pi A = \mathrm{End}_{A^{\otimes k}}(A^{\otimes k} e_\pi)$, $\mathbb{P}^\pi A = \mathrm{End}_{A^{\otimes k}}(A^{\otimes k} \zeta_\pi)$

As in [KMRT], each of these is a central simple K-algebra, Brauer-equivalent to $A^{\otimes k}$, with degree as given by the reduced dimensions in the lemma. There are natural isomorphisms $\mathbb{S}^\pi \mathrm{End}_K V = \mathrm{End}_K \mathbb{S}^\pi V$ and $\mathbb{P}^\pi \mathrm{End}_K V = \mathrm{End}_K \mathbb{P}^\pi V$.

Lemma

If A is split, there are natural isomorphisms of $A^{\otimes k}$-modules:

$$A^{\otimes k} e_\pi = \mathrm{Hom}_K(\mathbb{S}^\pi V, V^{\otimes k}), \quad A^{\otimes k} \zeta_\pi = \mathrm{Hom}_K(\mathbb{P}^\pi V, V^{\otimes k})$$

with reduced dimensions
$\mathrm{rdim}(A^{\otimes k} e_\pi) = s_\pi(1, \dots, 1)$, $\mathrm{rdim}(A^{\otimes k} \zeta_\pi) = \deg(\chi_\pi) s_\pi(1, \dots, 1)$.

Definition

Let $\pi \vdash k$. Define $\mathbb{S}^\pi A = \mathrm{End}_{A^{\otimes k}}(A^{\otimes k} e_\pi)$, $\mathbb{P}^\pi A = \mathrm{End}_{A^{\otimes k}}(A^{\otimes k} \zeta_\pi)$

As in [KMRT], each of these is a central simple K-algebra, Brauer-equivalent to $A^{\otimes k}$, with degree as given by the reduced dimensions in the lemma. There are natural isomorphisms $\mathbb{S}^\pi \mathrm{End}_K V = \mathrm{End}_K \mathbb{S}^\pi V$ and $\mathbb{P}^\pi \mathrm{End}_K V = \mathrm{End}_K \mathbb{P}^\pi V$.

Lemma

If A is split, there are natural isomorphisms of $A^{\otimes k}$-modules:

$$A^{\otimes k} e_\pi = \mathrm{Hom}_K(\mathbb{S}^\pi V, V^{\otimes k}), \quad A^{\otimes k} \zeta_\pi = \mathrm{Hom}_K(\mathbb{P}^\pi V, V^{\otimes k})$$

with reduced dimensions
$\mathrm{rdim}(A^{\otimes k} e_\pi) = s_\pi(1, \ldots, 1)$, $\mathrm{rdim}(A^{\otimes k} \zeta_\pi) = \deg(\chi_\pi) s_\pi(1, \ldots, 1)$.

Definition

Let $\pi \vdash k$. Define $\mathbb{S}^\pi A = \mathrm{End}_{A^{\otimes k}}(A^{\otimes k} e_\pi)$, $\mathbb{P}^\pi A = \mathrm{End}_{A^{\otimes k}}(A^{\otimes k} \zeta_\pi)$

As in [KMRT], each of these is a central simple K-algebra, Brauer-equivalent to $A^{\otimes k}$, with degree as given by the reduced dimensions in the lemma. There are natural isomorphisms $\mathbb{S}^\pi \mathrm{End}_K V = \mathrm{End}_K \mathbb{S}^\pi V$ and $\mathbb{P}^\pi \mathrm{End}_K V = \mathrm{End}_K \mathbb{P}^\pi V$.

We can use Schur powers of hermitian forms to get induced (adjoint) involutions on Schur powers of central simple algebras.

The adjoint involution to $h^{\otimes k}$ is $\tau_{h^{\otimes k}} = (\tau_h)^{\otimes k}$.

Every $f \in \mathbb{P}^\pi A = \mathrm{End}_{A^{\otimes k}}(A^{\otimes k}\zeta_\pi)$ is right multiplication by $u\zeta_\pi$, for some $u \in A^{\otimes k}$.

Define $\mathbb{P}^\pi \tau(f)$ to be right multiplication by $\tau^{\otimes k}(u)\zeta_\pi$.

Proposition

If $A = \mathrm{End}_K V$ and $\tau = \tau_h$ is the adjoint involution w.r.t. a non-singular hermitian form h on V, then under the isomorphism $\mathbb{P}^\pi \mathrm{End}_K V = \mathrm{End}_K \mathbb{P}^\pi V$, the involution $\mathbb{P}^\pi \tau$ is the adjoint involution w.r.t. the hermitian form $\mathbb{P}^\pi h$ on $\mathbb{P}^\pi V$.

We can use Schur powers of hermitian forms to get induced (adjoint) involutions on Schur powers of central simple algebras.

The adjoint involution to $h^{\otimes k}$ is $\tau_{h^{\otimes k}} = (\tau_h)^{\otimes k}$.

Every $f \in \mathbb{P}^\pi A = \mathrm{End}_{A^{\otimes k}}(A^{\otimes k}\zeta_\pi)$ is right multiplication by $u\zeta_\pi$, for some $u \in A^{\otimes k}$.

Define $\mathbb{P}^\pi\tau(f)$ to be right multiplication by $\tau^{\otimes k}(u)\zeta_\pi$.

Proposition

If $A = \mathrm{End}_K V$ and $\tau = \tau_h$ is the adjoint involution w.r.t. a non-singular hermitian form h on V, then under the isomorphism $\mathbb{P}^\pi \mathrm{End}_K V = \mathrm{End}_K \mathbb{P}^\pi V$, the involution $\mathbb{P}^\pi\tau$ is the adjoint involution w.r.t. the hermitian form $\mathbb{P}^\pi h$ on $\mathbb{P}^\pi V$.

We can use Schur powers of hermitian forms to get induced (adjoint) involutions on Schur powers of central simple algebras.

The adjoint involution to $h^{\otimes k}$ is $\tau_{h^{\otimes k}} = (\tau_h)^{\otimes k}$.

Every $f \in \mathbb{P}^\pi A = \operatorname{End}_{A^{\otimes k}}(A^{\otimes k}\zeta_\pi)$ is right multiplication by $u\zeta_\pi$, for some $u \in A^{\otimes k}$.

Define $\mathbb{P}^\pi\tau(f)$ to be right multiplication by $\tau^{\otimes k}(u)\zeta_\pi$.

Proposition

If $A = \operatorname{End}_K V$ and $\tau = \tau_h$ is the adjoint involution w.r.t. a non-singular hermitian form h on V, then under the isomorphism $\mathbb{P}^\pi \operatorname{End}_K V = \operatorname{End}_K \mathbb{P}^\pi V$, the involution $\mathbb{P}^\pi\tau$ is the adjoint involution w.r.t. the hermitian form $\mathbb{P}^\pi h$ on $\mathbb{P}^\pi V$.

Outline

Work in progress

Question

Can we apply Schur polynomials to

- *central simple K-algebras with involution;*
- *other K-algebras with involution (or just antiautomorphism)?*

- There is an associative commutative orthogonal sum $\perp$ of Morita-equivalent central simple K-algebras with involution [Dejaiffe, 1998] which commutes with taking degrees
- $\perp$ can be extended to Morita-equivalent K-algebras with antiautomorphism [Cortella and Lewis, 2009]
- $\lambda^k A$ (& $\mathbb{P}^\pi A, \dots$) is Brauer-equivalent to $A^{\otimes k}$ [KMRT, 10.4]
- For A a central simple K-algebra with involution, of exponent 2 (e.g., involution of first kind), or split: $A^{\otimes k}$ and $A^{\otimes \ell}$ are Brauer-equivalent if $k \equiv \ell \pmod 2$

Work in progress

Question

Can we apply Schur polynomials to

- *central simple K-algebras with involution;*
- *other K-algebras with involution (or just antiautomorphism)?*

- There is an associative commutative orthogonal sum $\perp$ of Morita-equivalent central simple K-algebras with involution [Dejaiffe, 1998] which commutes with taking degrees
- $\perp$ can be extended to Morita-equivalent K-algebras with antiautomorphism [Cortella and Lewis, 2009]
- $\lambda^k A$ (& $\mathbb{P}^\pi A, \ldots$) is Brauer-equivalent to $A^{\otimes k}$ [KMRT, 10.4]
- For A a central simple K-algebra with involution, of exponent 2 (e.g., involution of first kind), or split: $A^{\otimes k}$ and $A^{\otimes \ell}$ are Brauer-equivalent if $k \equiv \ell \pmod 2$

Work in progress

Question

Can we apply Schur polynomials to

- *central simple K-algebras with involution;*
- *other K-algebras with involution (or just antiautomorphism)?*

- There is an associative commutative orthogonal sum $\perp$ of Morita-equivalent central simple K-algebras with involution [Dejaiffe, 1998] which commutes with taking degrees
- $\perp$ can be extended to Morita-equivalent K-algebras with antiautomorphism [Cortella and Lewis, 2009]
- $\lambda^k A$ (& $\mathbb{P}^\pi A, \ldots$) is Brauer-equivalent to $A^{\otimes k}$ [KMRT, 10.4]
- For A a central simple K-algebra with involution, of exponent 2 (e.g., involution of first kind), or split: $A^{\otimes k}$ and $A^{\otimes \ell}$ are Brauer-equivalent if $k \equiv \ell \pmod 2$

Work in progress

Question

Can we apply Schur polynomials to

- *central simple K-algebras with involution;*
- *other K-algebras with involution (or just antiautomorphism)?*

- There is an associative commutative orthogonal sum $\perp$ of Morita-equivalent central simple K-algebras with involution [Dejaiffe, 1998] which commutes with taking degrees
- $\perp$ can be extended to Morita-equivalent K-algebras with antiautomorphism [Cortella and Lewis, 2009]
- $\lambda^k A$ (& $\mathbb{P}^\pi A, \ldots$) is Brauer-equivalent to $A^{\otimes k}$ [KMRT, 10.4]
- For A a central simple K-algebra with involution, of exponent 2 (e.g., involution of first kind), or split: $A^{\otimes k}$ and $A^{\otimes \ell}$ are Brauer-equivalent if $k \equiv \ell \pmod 2$

Work in progress

Question

Can we apply Schur polynomials to

- *central simple K-algebras with involution;*
- *other K-algebras with involution (or just antiautomorphism)?*

- There is an associative commutative orthogonal sum $\perp$ of Morita-equivalent central simple K-algebras with involution [Dejaiffe, 1998] which commutes with taking degrees
- $\perp$ can be extended to Morita-equivalent K-algebras with antiautomorphism [Cortella and Lewis, 2009]
- $\lambda^k A$ (& $\mathbb{P}^\pi A, \ldots$) is Brauer-equivalent to $A^{\otimes k}$ [KMRT, 10.4]
- For A a central simple K-algebra with involution, of exponent 2 (e.g., involution of first kind), or split: $A^{\otimes k}$ and $A^{\otimes \ell}$ are Brauer-equivalent if $k \equiv \ell \pmod 2$

Definition

A **partial abelian monoid** (PAM, for short) is a nonempty set M together with a partially defined binary relation $\oplus$ with domain $P \subseteq M \times M$ satisfying the following conditions.

P1 (Commutativity). If aPb then bPa and $a \oplus b = b \oplus a$.

P2 (Associativity). If aPb and $(a \oplus b)Pc$ then bPc and $aP(b \oplus c)$, and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

P3 (Neutral element). There is a (necessarily unique) element $0 \in M$ such that for all $a \in M$ we have $0Pa$ and $a \oplus 0 = a$.

Then the set $Y(K)^+$ of isomorphism classes of central simple K-algebras with involution under $\perp$ is a PAM, where only Morita-equivalent ones are related.

Definition

A **partial abelian monoid** (PAM, for short) is a nonempty set M together with a partially defined binary relation $\oplus$ with domain $P \subseteq M \times M$ satisfying the following conditions.

P1 (Commutativity). If aPb then bPa and $a \oplus b = b \oplus a$.

P2 (Associativity). If aPb and $(a \oplus b)Pc$ then bPc and $aP(b \oplus c)$, and $(a \oplus b) \oplus c = a \oplus (b \oplus c)$.

P3 (Neutral element). There is a (necessarily unique) element $0 \in M$ such that for all $a \in M$ we have $0Pa$ and $a \oplus 0 = a$.

Then the set $Y(K)^+$ of isomorphism classes of central simple K-algebras with involution under $\perp$ is a PAM, where only Morita-equivalent ones are related.

Speculation

Idea: Consider Grothendieck completion $Y(K)$ of $Y(K)^+$ under $\perp$, restricting to where $\perp$ makes sense. Cancellation seems unlikely!

Note: Don't collapse hyperbolic involutions to one point as in [Lewis, 2000], since exterior powers not well-defined on Witt ring

Need to check:

- Tensor product $\otimes$ on elements of $Y(K)$ distributes over $\perp$: would make $Y(K)$ a partial ring
- Formula

$$\bigwedge^k((A,\sigma)\perp(B,\tau)) = \coprod_{i+j=k} \bigwedge^i(A,\sigma)\otimes\bigwedge^j(B,\tau)$$

holds: would make $Y(K)$ a partial λ-ring. (Evidence so far: the degrees add correctly)

Speculation

Idea: Consider Grothendieck completion $Y(K)$ of $Y(K)^+$ under $\perp$, restricting to where $\perp$ makes sense. Cancellation seems unlikely!

Note: Don't collapse hyperbolic involutions to one point as in [Lewis, 2000], since exterior powers not well-defined on Witt ring

Need to check:

- Tensor product $\otimes$ on elements of $Y(K)$ distributes over $\perp$: would make $Y(K)$ a partial ring
- Formula

$$\bigwedge^k((A,\sigma) \perp (B,\tau)) = \coprod_{i+j=k} \bigwedge^i(A,\sigma) \otimes \bigwedge^j(B,\tau)$$

holds: would make $Y(K)$ a partial λ-ring. (Evidence so far: the degrees add correctly)

Speculation

Idea: Consider Grothendieck completion $Y(K)$ of $Y(K)^+$ under $\perp$, restricting to where $\perp$ makes sense. Cancellation seems unlikely!

Note: Don't collapse hyperbolic involutions to one point as in [Lewis, 2000], since exterior powers not well-defined on Witt ring

Need to check:

- Tensor product $\otimes$ on elements of $Y(K)$ distributes over $\perp$: would make $Y(K)$ a partial ring
- Formula

$$\bigwedge^k((A,\sigma) \perp (B,\tau)) = \coprod_{i+j=k} \bigwedge^i(A,\sigma) \otimes \bigwedge^j(B,\tau)$$

holds: would make $Y(K)$ a partial λ-ring. (Evidence so far: the degrees add correctly)

Speculation

Idea: Consider Grothendieck completion $Y(K)$ of $Y(K)^+$ under $\perp$, restricting to where $\perp$ makes sense. Cancellation seems unlikely!

Note: Don't collapse hyperbolic involutions to one point as in [Lewis, 2000], since exterior powers not well-defined on Witt ring

Need to check:

- Tensor product $\otimes$ on elements of $Y(K)$ distributes over $\perp$: would make $Y(K)$ a partial ring
- Formula

$$\bigwedge^k((A,\sigma)\perp(B,\tau)) = \coprod_{i+j=k} \bigwedge^i(A,\sigma)\otimes\bigwedge^j(B,\tau)$$

  holds: would make $Y(K)$ a partial λ-ring. (Evidence so far: the degrees add correctly)

Possible outcomes

Then polynomials in one indeterminate which are either even or odd (or homogeneous) would make sense in $Y(K)$, as all the summands will be Morita equivalent.

Thus determinantal formulas (e.g., Giambelli) could be used to define polynomial Schur powers on $Y(K)$.

If there are reasonably straightforward expressions for Adams operations, we may even be able to define annihilating polynomials in $Y(K)$, since these polynomials are necessarily either even or odd (they must respect the signature).

Possible outcomes

Then polynomials in one indeterminate which are either even or odd (or homogeneous) would make sense in $Y(K)$, as all the summands will be Morita equivalent.

Thus determinantal formulas (e.g., Giambelli) could be used to define polynomial Schur powers on $Y(K)$.

If there are reasonably straightforward expressions for Adams operations, we may even be able to define annihilating polynomials in $Y(K)$, since these polynomials are necessarily either even or odd (they must respect the signature).

Possible outcomes

Then polynomials in one indeterminate which are either even or odd (or homogeneous) would make sense in $Y(K)$, as all the summands will be Morita equivalent.

Thus determinantal formulas (e.g., Giambelli) could be used to define polynomial Schur powers on $Y(K)$.

If there are reasonably straightforward expressions for Adams operations, we may even be able to define annihilating polynomials in $Y(K)$, since these polynomials are necessarily either even or odd (they must respect the signature).