Higher product levels

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The commutative case

The n-th level of a field F was defined by J.-P. Joly [1970] as

$$s_n(F) = \min\{t \mid -1 \in \sum_{i=1}^{t} F^n\}.$$

The convention $\min \emptyset = \infty$ is used. If n is odd then $s_n(F) = 1$, hence the interesting case is $n = 2m$.

Joly proved that there exists a function $u \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that

$$s_{2m}(F) \leq u(s_2(F), 2m)$$

for every field F and every $m \in \mathbb{N}$. In particular,

> if $s_2(F)$ is finite then $s_{2m}(F)$ is finite for every $m \in \mathbb{N}$.

The commutative case

The construction of Joly's function u is based on Hilbert identities:

Write $L(r,n) := \binom{n+2r-1}{n-1}$. For every $r, n \in \mathbb{N}$ there exist $\lambda_i \in \mathbb{Q}^+$ and $a_{ij} \in \mathbb{Z}$ $(1 \le i \le L(r,n),\ 1 \le j \le n)$ such that

$$(x_1^2 + \ldots + x_n^2)^r = \sum_{i=1}^{L(r,n)} \lambda_i (a_{i1}x_1 + \ldots + a_{in}x_n)^{2r}.$$

They are used in the proof the Waring numbers $\mathrm{G}(n)$ are finite.

If char $F \neq 2$ and $t := \mathrm{s}_2(F) < \infty$ (say $-1 = a_1^2 + \cdots + a_t^2$), then every element $b \in F$ can be written as a sum of $t+1$ squares:

$$b = \left(\frac{1+b}{2}\right)^2 - \left(\frac{1-b}{2}\right)^2 = \left(\frac{1+b}{2}\right)^2 + (a_1^2 + \ldots + a_t^2)\left(\frac{1-b}{2}\right)^2.$$

The commutative case

It follows, that for any odd l we have

$$-1 = (-1)^l = (a_1^2 + \cdots + a_t^2)^l = \sum_{j_1=1}^{L(l,t)} \lambda_{j_1} b_{j_1}^{2l} =$$

$$= \sum_{j_1=1}^{L(l,t)} \lambda_{j_1} (c_{j_1,1}^2 + \cdots + c_{j_1,t+1}^2)^{2l} = \sum_{j_2=1}^{L(2l,t+1)} \sum_{j_1=1}^{L(l,t)} \lambda_{j_1 j_2} b_{j_1 j_2}^{4l} =$$

$$= \cdots = \sum_{j_k=1}^{L(2^{k-1}l,t+1)} \cdots \sum_{j_2=1}^{L(2l,t+1)} \sum_{j_1=1}^{L(l,t)} \lambda_{j_1 j_2 \ldots j_k} b_{j_1 j_2 \ldots j_k}^{2^k l}.$$

Therefore
$s_{2^k l}(F) \leq G(2^k l) L(l,t) L(2l,t+1) \ldots L(2^{k-1} l, t+1) =: u(t, 2^k l)$.

The definition in the noncommutative case

The problem in the noncommutative case is that a product of two n-th powers need not be an n-th power. Hence we can define higher levels of skew fields in several different ways. We choose the following definition:

Let D be a skew field and $n \in \mathbb{N}$. Write D^n for the set of all n-th powers of elements from D and $[D^\times, D^\times]$ for the set of all multiplicative commutators of nonzero elements from D. The set of all finite products of elements from $D^n \cup [D^\times, D^\times]$ will be denoted by $\Pi_n(D)$. The number

$$\mathrm{ps}_n(D) = \min\{k \mid \exists z_1, \dots, z_k \in \Pi_n(D) \colon -1 = \sum_{i=1}^{k} z_i\}$$

will be called **the n-th product level of D**.

The main result

The main result is:

- If $\mathrm{ps}_{2^k}(D)$ is finite for some k, then $\mathrm{ps}_{2^k l}(D)$ is finite for every l. Moreover, there exists a function $f : \mathbb{N}^3 \to \mathbb{N}$, such that $\mathrm{ps}_{2^k l}(D) \leq f(\mathrm{ps}_{2^k}(D), k, l)$.
- For every k there exists a skew field D such that $\mathrm{ps}_{2^k}(D)$ is finite but $\mathrm{ps}_{2^{k+1}}(D)$ is not. Moreover, we can replace 'finite' by 'equal to one'.

Orderings of higher exponent

How do we prove that $\mathrm{ps}_n(D) = \infty$? We contruct **an ordering of exponent** n **on** D, i.e. a normal subgroup P of $D^\times$ such that the factor group $D^\times/P$ is cyclic of order n, $-1 \notin P$ and $P + P \subseteq P$.

Clearly, the existence of an ordering of exponent n on D implies that $\mathrm{ps}_n(D) = \infty$. Namely, every ordering of exponent n contains all finite sums of elements from $\Pi_n(D)$ but it does not contain -1.

The opposite direction, i.e. that $\mathrm{ps}_n(D) = \infty$ implies the existence of an ordering of exponent n, is much more difficult to prove. This was done by V. Powers (by modifying E. Becker's original proof in the commutative case.)

$\mathrm{ps}_{2^k} < \infty \Rightarrow \mathrm{ps}_{2^k l} < \infty$, Proof 1

If P is an ordering of some exponent, then the set

$$\mathrm{A}(P) = \{a \in A \mid \exists r \in \mathbb{Q}^+ : r \pm a \in P\}$$

is a valuation ring and the set $\mathrm{Arch}(P) \cap \mathrm{A}(P)$ where

$$\mathrm{Arch}(P) = \{a \in A \mid \forall r \in \mathbb{Q}^+ : r + a \in P\}$$

is an ordering with exponent 2 on $\mathrm{A}(P)$.

It follows that every subgroup of $D^\times$ containing P is closed for addition, hence it is an ordering of some exponent.

In particular, if P has exponent $n = 2^k l$, then the multiplicative set $\{d \in D^\times \mid d^l \in P\}$ is an ordering of exponent 2^k. This is a nonconstructive proof of $\mathrm{ps}_{2^k}(D) < \infty \Rightarrow \mathrm{ps}_{2^k l}(D) < \infty$.

$\mathrm{ps}_{2^k} < \infty \Rightarrow \mathrm{ps}_{2^k l} < \infty$, Proof 2

Using the theory of orderings of higher exponents, it is possible to prove that for every odd number l, every number $k \geq 1$ and every $d \in D$ there exist $g_0, \ldots, g_{2^k-1} \in \Sigma_{i=1}^{a_{2^k l}} \mathbb{Q}(d)^{2^k l}$ such that

$$(1+d)^l = g_0 + g_1 d^l + \cdots + g_{2^k-1} d^{(2^k-1)l}.$$

By induction on t, one can obtain the following noncommutative analogue of Hilbert identities: If $s_1, \ldots, s_t \in \Pi_{2^k}(D)$ and k, l as above, then $(s_1 + \cdots + s_t)^l \in \Pi_{2^k l}(D)$.

If we do the induction carefully, we can obtain the bound

$$\mathrm{ps}_{2^k l}(D) \leq (2^k a_{2^k l})^{2\,\mathrm{ps}_{2^k}(D)}$$

where a_n is the upper bound for n-th Pythagoras numbers of $\mathbb{Q}(d)$, $d \in D$. (The existence of a_n was proved by E. Becker.)

$\mathrm{ps}_{2k} < \infty \not\Rightarrow \mathrm{ps}_{2k+1} < \infty$

Let k be a formally real field, m an integer and

$$R_m := k\langle X, Y\rangle / (YX + X^{2m+1}Y),$$

It is known that R_m is a left Ore domain, so it has a left skew-field of fractions D_m.

The relation implies that $-1 \in \Pi_m(D_m)$, hence $\mathrm{ps}_m(D_m) = 1$.

We can construct an ordering of exponent $2m$ on D_m, hence $\mathrm{ps}_{2m}(D_m) = \infty$. We provide the details of the construction.

Every element $f \in R_m$ can be written as $f = \sum_{i,j} c_{i,j} X^i Y^j$. We assume that monomials are ordered anti-lexicographically, i.e. $X^{i_1} Y^{j_1} \succ X^{i_2} Y^{j_2}$ if and only if $j_1 > j_2$ or $j_1 = j_2$ and $i_1 > i_2$.

If the leading term of f is $c_{p,q} X^p Y^q$, then we say that $f > 0$ iff ($p = 4m$ and $c_{p,q} > 0$) or ($p = 4m + 2$ and $c_{p,q} < 0$).

It turns out that the set

$$P = \{fg^{-1} \in D^\times \mid fg > 0\}$$

is an ordering on D of exponent $2m$.

Another higher product level

Let D be a skew field and $n \in \mathbb{N}$. Write $P_n(D)$ for the set of all finite products of n-th powers of elements from D. The number

$$\mathrm{ms}_n(D) = \min\{k \mid \exists z_1, \ldots, z_k \in P_n(D) \colon -1 = \sum_{i=1}^{k} z_i\}$$

can also be considered the n-th product level of D.

Is $\mathrm{ms}_n(D) = \mathrm{ps}_n(D)$ for every n and D? The answer is no!

Does $\mathrm{ms}_n(D) < \infty$ imply that $\mathrm{ms}_{kn}(D) < \infty$ for any $k > 1$? I don't know!

Is there a variant of orderings of exponent n corresponding to $\mathrm{ms}_n(D) = \infty$? Yes, replace the cyclic group by a nonabelian group of exponent n. Not much is known about such orderings!

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