Joining Forces: Combining the gravitational and scalar self-force

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Motivation

- There are numerous interesting situations involving extreme mass ratio systems in non-vacuum environments:
  - Self force as an enforcer of cosmic censorship in the overcharging scenario. (PZ *et al.*, 2012) [1211.3889]
  - Scalar fields in Kerr – stability of floating orbits (V. Cardoso *et al.*, 2011) [1109.6021]
  - Alternative theories of gravity (S. Gralla, 2013) [1303.0269]

General quadratic actions in non-diagonal bases

- The quadratic action for a set of arbitrary fields $\psi_\ell(x)$ coupled linearly to an external current $J_\ell(x)$ is given by

$$S[\psi] = -\frac{1}{2}\int_x \int_{x'} \sqrt{-g} \sum_{\ell\ell'} \mathscr{O}_{\ell\ell'}(x,x')\psi_\ell(x)\psi_{\ell'}(x') + \int_x \sqrt{-g} \sum_\ell \psi_\ell(x) J_\ell(x)$$

where

$$\mathscr{O}_{\ell\ell'}(x,x') = -\frac{1}{\sqrt{-g}} \frac{\delta^2 S[\psi]}{\delta\psi_\ell(x)\delta\psi_{\ell'}(x')}$$

- The Green's function $\Delta_{\ell\ell'}(x,x')$ of $\mathscr{O}_{\ell\ell'}(x,x')$ is defined as

$$\int_{x_2} \sqrt{-g} \sum_{\ell_2} \mathscr{O}_{\ell_1\ell_2}(x_1,x_2)\Delta_{\ell_2\ell_3}(x_2,x_3) = \delta_{\ell_1\ell_3}\delta^4(x_1-x_3)$$

and the equations of motion read

$$\psi_\ell = \int_{x'} \sum_{\ell'} \Delta_{\ell\ell'}(x,x') J_{\ell'}(x').$$

Scalar gravity action

- We consider an effective point particle possessing scalar charge q and mass m that is coupled to the metric and a real scalar field φ.

$$S = \int_x \sqrt{-\mathsf{g}}\left(\tfrac{1}{\kappa}\mathsf{R} - \tfrac{1}{2}\mathsf{g}^{\alpha\beta}\nabla_\alpha\varphi\nabla_\beta\varphi\right) + q\int \mathrm{d}\lambda\,\varphi - m\int \mathrm{d}\lambda + \cdots .$$

The mass term can be removed via the shift $\varphi \to \varphi + m/q$.

Perturbations

- By neglecting the microphysics of the small body, we have tacitly assumed that its internal structure is irrelevant at the scales of interest. We are most interested in the extreme mass ratio limit

$$\epsilon \sim \frac{m}{M} \sim \frac{q}{M} \ll 1.$$

- The scalar and gravitational field are decomposed into background and perturbations of the background fields

$$\varphi = \Phi + f, \qquad \mathrm{g}_{\alpha\beta} = g_{\alpha\beta} + h_{\alpha\beta},$$

where Φ and $g_{\alpha\beta}$ are assumed solutions to the background Einstein Klein Gordon field equations

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{\kappa}{2}\left(\Phi_{;\alpha}\Phi_{;\beta} - \frac{1}{2} g_{\alpha\beta} g^{\mu\nu}\Phi_{;\mu}\Phi_{;\nu}\right),$$

$$g^{\alpha\beta}\Phi_{;\alpha\beta} = 0.$$

Bulk action and interaction

- The second order variation of the bulk action in the Lorenz gauge

$$\delta^2 S = S_{f^2} + S_{h^2} + S_{mix}$$

introduces an interaction term which couples the scalar and gravitational fields

$$S_{mix} = -\int_x \sqrt{-g}\,\Phi^{;\alpha\beta}\left(h_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}h\right)f.$$

Effective Action and dissipation

- The effective action is

$$S_{\text{eff}}(z) = S(h_{\mu\nu}(z), z),$$

where $h_{\mu\nu}$ is a solution of the field equations evaluated at z.

- Dissipation in the action formalism arises from letting a degree of freedom leave the system
- In field theory the path integral is dominated by the saddle point at the classical value

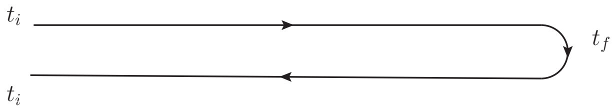
$$Z = \int \mathcal{D}h \; e^{iS(h,z)} = e^{iS(h_{cl},z)} + O(\hbar).$$

- The machinery for computing $e^{iS_{\text{eff}}}$ is provided by QFT almost free of charge.

In-in formalism

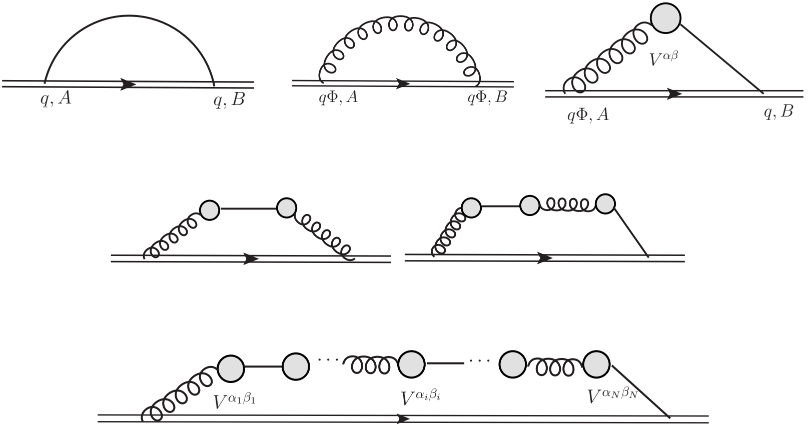
- We adopt the in-in path integral formalism where the fields are time ordered with respect to a closed time path (CTP) beginning and ending in the remote past. This is to insure that the future state of the system is determined causally from initial conditions.
- Vacuum transition amplitudes are given by

$$\begin{aligned}\langle \mathsf{IN}, \mathsf{VAC} \mid \mathsf{IN}, \mathsf{VAC} \rangle &= \int \mathcal{D}\phi_* \langle \mathsf{in}, t_i \mid \phi_*(t_f) \rangle \langle \phi_*(t_f) \mid \mathsf{in}, t_i \rangle \\ &= \int \mathcal{D}\phi_* \int \mathcal{D}\phi_1 \, e^{i \int_i^f L(\phi_1) dt} \int \mathcal{D}\phi_2 \, e^{-i \int_i^f L(\phi_2) dt} \\ &\equiv \int_{\mathsf{CTP}} \mathcal{D}\phi \, e^{iS_1 - iS_2}.\end{aligned}$$



First order mixed self-force diagrams

At first order, the self-force is obtained from the effective action given by the diagrams



Divergences

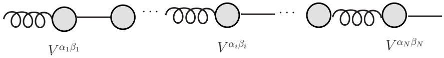
- To calculate the singular part of the effective action we investigate the local behaviour of the propagator using a near-point expansion centred on the world line.

- General covariance and dimensional analysis dictate that propagators will take the schematic form

$$D(z,z') \sim \int \mathrm{d}^d k \frac{e^{ik\cdot(z-z')}}{k^2}\left(\underbrace{1}_{\text{power div.}} \oplus \underbrace{\frac{R}{k^2}}_{\text{log div. and/or constant}} \oplus \underbrace{\frac{R^2}{k^4} \oplus \cdots}_{\text{convergent}}\right)$$

Divergences of mixed diagrams

- The form of the divergent part of the mixed propagator, which corresponds to propagation in flat spacetime, is found to be



$$\sim \int \mathrm{d}^d k \frac{e^{ik\cdot z}}{k^{2(N+1)}}$$

$$\sim \Lambda^{d-2(N+1)}.$$

- In $d = 4$, all mixing diagrams will be finite for $N > 1$. We therefore only consider the marginal case of one insertion.
- Each successive insertion of a mixing vertex is suppressed by a small ratio $\sim 1/\Lambda$.

Near point propagators

- The near point expansion of the scalar propagator, obtained to $O(\partial^2)$ inclusive, reads

$$D(z,0) = \int \frac{\mathrm{d}^d k}{(2\pi)^d} \frac{e^{iz\cdot k}}{k^2} \left[1 + \frac{1}{3}\frac{R}{k^2} - \frac{2}{3}\frac{R_{ab}k^a k^b}{k^4} + O(k^{-3}) \right].$$

- The expression for the near point gravitational propagator displays similar structure with respect to divergences

$$D_{abcd}(z,0) = \int \frac{\mathrm{d}^d q}{(2\pi)^d} \frac{e^{iz\cdot q}}{q^2} \Bigg[P_{abcd} - \frac{1}{6\big(d-2\big)q^4} \Bigg(4q^i q^k \Big(\big(d-2\big) P_{abcd} R_{ik}$$

$$+ 2\,(2-d)\,\left(\eta_{a(c}\, R_{\,d)kbi} + \eta_{b(c}\, R_{\,d)kai}\right) + 2\,(R_{aibk}\eta_{cd} + R_{cidk}\eta_{ab})\Big)$$

$$- q^2 \Big(\big(d-2\big) P_{abcd} R + 6\big((2-d) R_{acbd} + (d-2) R_{adbc} + R_{cd}\eta_{ab} + R_{ab}\eta_{cd}\big) \Big) \Bigg) \Bigg]$$

Near point propagator integrals

- In dimensional regularization the three integrals comprising the divergent part of the retarded propagator read

$$\int_{C_{ret}} \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{e^{ip\cdot z}}{p^2} \xrightarrow{d=4-\epsilon} -\frac{\theta_+(r^2)}{2\pi r^2}\epsilon$$

$$\int_{C_{ret}} \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{e^{ip\cdot z}}{p^4} \xrightarrow{d=4-\epsilon} \frac{\theta_+(r^2)}{8\pi}$$

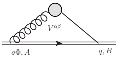
$$\int_{C_{ret}} \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{e^{ip\cdot z}p_a p_b}{p^6} \xrightarrow{d=4-\epsilon} -\frac{1}{64\pi}\partial_a\partial_b\left(\theta_+(r^2)r^2\right)$$

where

$$r^2 = -\eta_{ab}z^a z^b.$$

First mixing contribution

$$-i\,\frac{\delta}{\delta z^\mu}\Bigg($$



$$\Bigg) = \frac{q^2}{2}\int_{-\infty}^{\infty} \mathrm{d}\tau' \int_x \sqrt{-g}\,\Phi^{;\alpha\beta}\left(I_\mu^{(1)} + I_\mu^{(2)} + I_\mu^{(3)}\right)$$

where

$$I_\mu^{(1)} = \int_x \sqrt{-g}\,\Phi^{;\alpha\beta} w_\mu{}^\lambda \left(\nabla_\lambda D(z,x)\right) D_{\alpha\beta\gamma'\delta'}(x,z')\dot{z}^{\gamma'}\dot{z}^{\delta'} \sim w_\mu{}^\lambda \nabla_\lambda \theta_+(-\sigma(z,z'))$$

$$I_\mu^{(2)} = -4\int_x \sqrt{-g}\,\Phi^{;\alpha\beta}\Phi(z) w_\mu{}^{\gamma\delta\lambda}\left(\nabla_\lambda D_{\alpha\beta\gamma\delta}(x,z)\right)\ D(x,z') \sim w_\mu{}^\lambda \nabla_\lambda \theta_+(-\sigma(z,z'))$$

$$I_\mu^{(3)} = -2\int_x \sqrt{-g}\,\Phi^{;\alpha\beta} D_{\rho\sigma\alpha\beta}(z,x) D(x,z') w_\mu{}^{\rho\sigma\lambda}\nabla_\lambda \Phi(z) \sim \theta_+(-\sigma(z,z'))\nabla_\mu \Phi(z)$$

Equation of Motion

After regularizing the divergent part of the diagram we find that the finite self-force is given by

$$m(\tau)a_\mu = q w_\mu{}^\nu \nabla_\nu \Phi + f_\mu^{\mathsf{Loc}} + q^2 \int_{-\infty}^{\tau_-} \mathrm{d}\tau' w_\mu{}^\nu \nabla_\nu D^{ret}(z,z')$$

$$+ \frac{1}{2}\int_{-\infty}^{\tau_-} \mathrm{d}\tau' m(\tau') w_\mu{}^{\alpha\beta\lambda} \nabla_\lambda \left(m(\tau) D^{ret}_{\alpha\beta\gamma'\delta'}(z,z') \right) \dot{z}^{\gamma'} \dot{z}^{\delta'}$$

$$+ \frac{q}{4\pi}\int_{-\infty}^{\tau_-} \mathrm{d}\tau' \Phi_{;\alpha\beta}(z') w_\mu{}^{\alpha\beta\lambda} \nabla_\lambda m(\tau) + \cdots$$

where

$$4\pi f_\mu^{\mathsf{Loc}} = w_\mu{}^\nu \left[q^2 \left(\frac{1}{3}\dot{a_\nu} + \frac{1}{6} R_{\nu\alpha} u^\alpha \right) - \frac{2}{3} m^2(\tau) R_{\nu\alpha} u^\alpha + c m^2(\tau) \dot{a}_\nu \right]$$

$$\frac{dm}{d\tau} = -\frac{1}{4\pi}\frac{q^2}{12} R \qquad\qquad + \text{nonlocal}$$

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$$+ \frac{1}{2}\int_{-\infty}^{\tau_-} \mathrm{d}\tau' m(\tau') w_\mu{}^{\alpha\beta\lambda} \nabla_\lambda \left(m(\tau) D^{ret}_{\alpha\beta\gamma'\delta'}(z,z') \right) \dot{z}^{\gamma'} \dot{z}^{\delta'}$$

$$+ \frac{q}{4\pi}\int_{-\infty}^{\tau_-} \mathrm{d}\tau' \Phi_{;\alpha\beta}(z') w_\mu{}^{\alpha\beta\lambda} \nabla_\lambda m(\tau) + \cdots$$

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$$\frac{dm}{d\tau} = -\frac{1}{4\pi}\frac{q^2}{12} R + \frac{1}{4\pi} q m(\tau) u^\alpha u^\beta \Phi_{;\alpha\beta} + \text{nonlocal}$$

Summary, conclusion, and future work

- A formulation of the coupled self-force problem is motivated by open theoretical problems.
- The motion of a scalar charge is modified due to the coupling between gravitational and scalar fields when a background scalar field is present.
- A local expansion of interaction between the fields reveals that the coupling of the fields only contributes to the finite part.
- The coupling generates new local terms.
- Future work will focus on producing a DW-type decomposition of the fields using the Hadamard form for practical computations.