Scalar self-force for highly eccentric orbits in Kerr spacetime

Jonathan Thornburg

in collaboration with

Barry Wardell
Outline

Goals, overall plan of the computation
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Brief review of effective-source (puncture-function) regularization
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Some details of the computation

- \( m \)-mode decomposition
- separate 2+1D time-domain evolution for each mode
- worldtube scheme
  - finite differencing across the worldtube boundary
  - moving the worldtube
- computing the effective source and puncture function
- finite differencing near the particle (where fields are only \( C^2 \))
- mesh refinement
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Sample results

Conclusions, Plans, Lessons Learned
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develop techniques for future work with gravitational field
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  • astrophysical EMRI s likely have inclined orbits, any $e$ up to $\sim 0.99$
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  - eLISA/NGO will eventually need parameter-estimation templates with phase error $\lesssim 0.01$ radians over $\sim 10^5$ orbits of inspiral
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- geodesic approximation (at least for now)
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Work in progress: some goals accomplished, some not yet!
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Effective-Source (also known as puncture-function) regularization
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- worldtube scheme
- worldtube moves in ($r, \theta$) to follow the particle around the orbit
- Cauchy evolution
- fixed mesh refinement; some (finer) grids follow the worldtube/particle
- (almost) causally-disconnected spatial boundaries
  (with mesh refinement this isn’t very expensive)
Effective-src (puncture-fn) regularization: Outline

The particle’s physical (retarded) field \( \varphi \) satisfies

\[
\square \varphi = \delta(x - x_{\text{particle}}(t))
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The particle’s physical (retarded) field $\varphi$ satisfies $\Box \varphi = \delta(x - x_{\text{particle}}(t))$

Detwiler and Whiting (2003) showed that $\varphi$ can be decomposed into a singular field $\varphi_{\text{singular}}$ which is spherically symmetric at the particle (and hence exerts no self-force), and a finite regular part $\varphi_{\text{regular}}$ which exerts the self-force
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Instead (Barack & Golbourn (2007), Vega & Detweiler (2008)) we construct a “puncture function” \( \varphi_p \) which closely approximates \( \varphi_{\text{singular}} \) near the particle, then numerically compute the (finite) “residual field” \( \varphi_r := \varphi - \varphi_p \) by solving

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If \(\varphi_p\) “closely-enough” approximates \(\varphi_{\text{singular}}\) near the particle, then the self-force is given by \(F^a = q (\nabla^a \varphi_r)\bigg|_{\text{particle}}\).
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Even though $\varphi_p \neq \varphi_{\text{singular}}$, then self-force is exact to $O(\mu)$.
The worldtube

Problems:

- \( \varphi_p \) and \( S_{\text{effective}} \) are only defined in a neighbourhood of the particle.
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• $\varphi_p$ and $S_{\text{effective}}$ are only defined in a neighbourhood of the particle
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Solution:
introduce finite worldtube containing the particle worldline

• define “numerical field” $\varphi_{\text{num}} = \begin{cases} \varphi_r & \text{inside the worldtube} \\ \varphi & \text{outside the worldtube} \end{cases}$

  (this has a jump discontinuity by $\pm \varphi_p$ across the worldtube boundary)
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- $S_{\text{effective}}$ is only needed inside the worldtube
- the self-force is given by $F^a = q \left( \nabla^a \varphi_{\text{num}} \right) \bigg|_{\text{particle}}$
- finite differencing must locally “adjust” (a copy of) $\varphi_{\text{num}}$ by $\mp \varphi_p$
  across the worldtube bndry to undo the jump discontinuity in $\varphi_{\text{num}}$
\textit{m}-mode decomposition

Instead of numerically solving $\Box \varphi_{\text{num}} = \begin{cases} S_{\text{effective}} & \text{inside the worldtube} \\ 0 & \text{outside the worldtube} \end{cases}$ in 3+1D, we Fourier-decompose and solve for each Fourier mode in 2+1D:
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- $\varphi_{\text{num}}(t, r, \theta, \varphi) = \sum_m e^{im\tilde{\phi}} \varphi_{\text{num}, m}(t, r, \theta)$

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where

$$S_{\text{effective}, m} = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{\text{effective}} e^{-im\tilde{\phi}} d\tilde{\phi}$$

[numerically solve this for each $m$ in 2+1D]
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- in practice, solve numerically for $0 \leq m \leq m_{\text{max}} \sim 20$;
  fit large-$m$ asymptotic series to estimate “tail sum” $\sum_{m=m_{\text{max}}+1}^{\infty}$
Initial data, boundary conditions

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- with mesh refinement, having very large domain is not expensive
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But for a non-circular orbit, the particle moves in \((r, \theta)\) during the orbit.
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Large eccentricity:
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- recall that our numerically-evolved field is
  \[
  \varphi_{\text{num}} := \begin{cases} 
  \varphi - \varphi_p & \text{inside the worldtube} \\
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  \]
  this means then if we move the worldtube,
  - a given \((r, \theta)\) may change from being inside the worldtube to being outside \(\Rightarrow\) must add \(\varphi_p\)
  - a given \((r, \theta)\) may change from being outside the worldtube to being inside \(\Rightarrow\) must subtract \(\varphi_p\)
- I was worried that this would be a source of numerical noise
  \(\Rightarrow\) not a problem in practice (modulo bugs!)
Current Status

Equatorial eccentric orbits:

- elliptic-integral puncture fn & effective src
- worldtube moves in \((r, \theta)\) to follow the particle around the orbit
- fixed mesh refinement with “hollow grids”; some (finer) grids follow the worldtube
- typical worldtube size particle \(\pm 5M\) in \(r_\ast\), particle \(\pm \pi/8\) (22.5\(^\circ\)) in \(\theta\)
- 4th order finite differencing in space & time
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Generic (inclined eccentric) orbits:

- our first attempt at an effective source had \(\sim 20\) million terms
  \(\Rightarrow\) impractical to compile machine-generated C code
- we are starting to explore various ideas to reduce the complexity,
  and are optimistic we can solve this
Self-force for $e = 0.4$ orbit

BH spin 0.6  orbit: $p=8M$, $e=0.4$

$10^3 \times r^3 F_r$

- Outwards
- Inwards
Self-force for $e = 0.8$ orbit (preliminary)

BH spin 0.6 orbit: $p=8M$, $e=0.8$

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outwards

inwards
Self-force for $e = 0.9$ orbit (very preliminary)

BH spin 0.6   orbit: $p=8M$, $e=0.9$

$10^3 r^3 F_r$

outwards

inwards

$5 10 20 50 100$

$-250 -200 -150 -100 -50 0 50 100 150 200 250 300$

July 16, 2013 13 / 14
Conclusions

Things that work well:

- puncture-function regularization
- worldtube
- $m$-mode decomposition and 2+1D evolution
  - gives moderate parallelism “for free”
  - allows different numerical parameters for different $m$
- moving worldtube (allows highly eccentric orbits)
- mesh refinement (moving with particle & worldtube)
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Things that don’t yet work well (a.k.a. directions for further research)

- evolved fields only $C^2 \Rightarrow$ hard to get higher-order finite-diff convergence
- inclined eccentric orbits $\Rightarrow$ effective src is too complicated to be usable

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