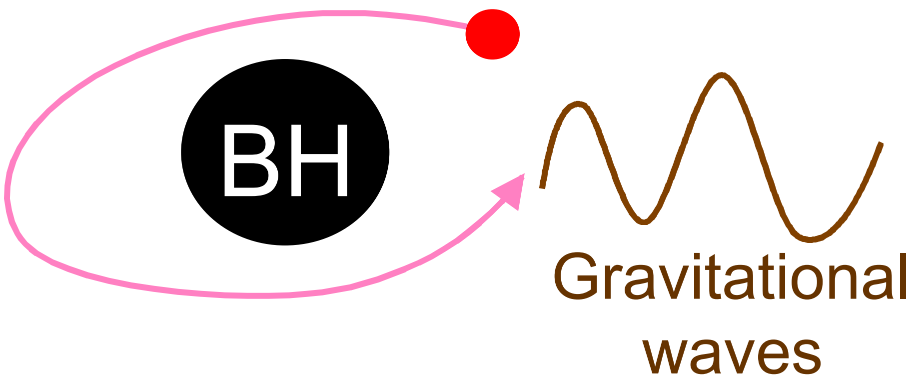
Adiabatic evolution of the constants of motion in resonance (I)



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Extrime Mass Ratio Inspiral(EMRI)

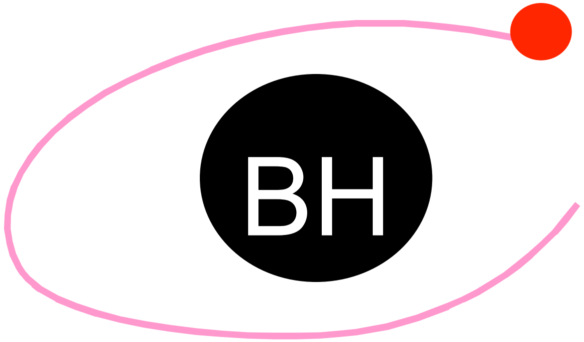
- Inspiral of 1 ~ 100M_{sol} BH of NS into the super massive BH at galactic center (typically $10^6 M_{sol}$)

- Very relativistic wave form can be calculated using BH perturbation

- Many cycles before the coalescence ~ $O(M/\mu)$ allow us to determine the orbit precisely.

- Clean system

  The best place to test GR.



Leading order wave form

Energy balance argument is sufficient.

$$\boxed{\frac{dE_{GW}}{dt} = \frac{dE_{orbit}}{dt}}$$

Here E is energy divided by mass.

Wave form $\equiv \dfrac{df}{dt}$ for quasi-circular orbits, for example.

$$\frac{df}{dt} = \frac{dE_{orbit}}{dt} \bigg/ \frac{dE_{orbit}}{df}$$

leading order

$$\frac{dE_{orbit}}{dt} = 0 + O(\mu) + O(\mu^2)$$

$$\frac{dE_{orbit}}{df} = (\text{geodesic}) + O(\mu) + O(\mu^2)$$

Evolution of general orbits

If we know four velocity u^μ at each time accurately,

⇒ we can solve the orbital evolution.

On Kerr background there are four "constants of motion"

constant in case of no radiation reaction

Normalization of four velocity: $-1 = u^\mu u_\mu$

Energy: $E = -u^\mu \xi^{(t)}_\mu$ ← Killing vector for time translation sym.

Angular momentum: $L_z = u^\mu \xi^{(\phi)}_\mu$ ← Killing vector for rotational sym.

Carter constant: $Q = u^\mu u^\nu K_{\mu\nu}$ ← Killing tensor

Quadratic and un-related to Killing vector

One-to-one correspondence

$$E, L_z, Q \Leftrightarrow u^\mu$$

Secular evolution of E, L_z, Q is necessary.

$$\frac{dE_{orbit}}{dt} = 0 + O(\mu) + O(\mu^2)$$

$$\frac{dE_{orbit}}{df} = (\text{geodesic}) + O(\mu) + O(\mu^2)$$

The issue of radiation reaction to Carter constant

- $E, L_z \Leftrightarrow$ Killing vector
  Conserved current for the field corresponding to Killing vector exists.

$$E_{GW} = \int d\Sigma^{\mu} t^{(GW)}_{\mu\nu} \xi^{\nu}$$

$$\dot{E} = -\dot{E}_{GW}$$ As a sum conservation law holds.

However, $Q \not\Leftrightarrow$ Killing vector

- We need to directly evaluate the self-force acting on the particle.

§3 Adiabatic approximation for Q

which is different from energy balance argument.

- $T << \tau_{RR}$

  T: orbital period

  τ_{RR} : timescale of radiation reaction

Approximation procedure

- The trajectory of a particle is assumed to be given by a geodesic specified by E, L_z, Q.
- We evaluate the radiative field instead of the retarded field.

$$\boxed{h_{\mu\nu}^{(rad)} = \left[h_{\mu\nu}^{(ret)} - h_{\mu\nu}^{(adv)}\right]/2}$$

- Self-force is computed from the radiative field, and it determines the change rates of E, L_z, Q.

$$\boxed{\left\langle \frac{dQ}{d\tau} \right\rangle = \frac{1}{\mu} \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} d\tau \frac{\partial Q}{\partial u^\alpha} F^\alpha \left[h_{\mu\nu}^{(rad)}\right]}$$

Why does this approximation work?

- For E and L_z the results are equivalent to the balance argument. (shown by Gal'tsov '82)
- For Q, the estimate using the radiative field is shown to give the correct long time average. (shown by Mino '03)
  - Key point: Under the transformation
    $$\boxed{(t, r, \theta, \phi) \to (-t, r, \theta, -\phi)}$$
    a geodesic is transformed back into itself.

- Radiative field is free from divergence at the location of the particle and easy to evaluate.
  - Divergent part is common for both retarded and advanced fields.

Outstanding property of Kerr geodesic

Introducing a new time parameter λ by $\quad d\lambda \equiv \dfrac{d\tau}{r^2 + a^2\cos^2\theta}$

$$\Rightarrow \quad \left(\frac{dr}{d\lambda}\right)^2 = R(r) \qquad \left(\frac{d\theta}{d\lambda}\right)^2 = \Theta(\theta)$$

r- and θ-oscillations can be solved independently.

$$\frac{dt}{d\lambda} = (r\text{-dependent part}) + (\theta\text{-dependent part})$$

$$\boxed{\frac{d\phi}{d\lambda} = \text{similar}}$$

$$t(\lambda) = \underline{t^{(r)}} + \underline{t^{(\theta)}} + \left\langle \frac{dt}{d\lambda} \right\rangle \lambda$$

Periodic functions with frequencies Ω_r, Ω_θ

- Only discrete Fourier components arise in an orbit

$$\omega = \omega_m^{n_r, n_\theta} = \langle dt/d\lambda \rangle^{-1} \left(m \langle d\phi/d\lambda \rangle + \underline{n_r}\Omega_r + \underline{n_\theta}\Omega_\theta \right)$$

Final expression for *dQ/dt* in adiabatic approximation

The resulting formula is so simple.

(Sago, TT, Hikida, Nakano PTP 115 (2005) 873)

$$\left\langle \frac{dQ}{dt} \right\rangle = 2\langle f(r)\rangle \left\langle \frac{dE}{dt} \right\rangle - 2\langle g(r)\rangle \left\langle \frac{dL_z}{dt} \right\rangle + 2 \sum_{l,m,\omega=\omega_{l,m}^{n_r,n_\theta}} \frac{n_r \Omega_r}{\omega} \left| A_{l,m,\omega} \right|^2$$

$$\boxed{A_{l,m,\omega} = \int (\text{mode function})_{l,m,\omega} \times (\text{source term}) d^4 x}$$

amplitude of the partial wave

This expression is as easy to evaluate as *dE/dt* and *dL/dt*.

$$\left\langle \frac{dE}{dt} \right\rangle \approx - \sum_{l,m,\omega} \left| A_{l,m,\omega} \right|^2 \qquad \left\langle \frac{dL}{dt} \right\rangle \approx - \sum_{l,m,\omega} \frac{m}{\omega} \left| A_{l,m,\omega} \right|^2$$

Analytic formula up to 2.5PN order:

Ganz, Hikida, Nakano, Sago, TT, PTP 117 (2007) 1041

Numerical calculation: Fujita, Hikida, Tagoshi, PTP 121 (2009) 843

Resonant orbit

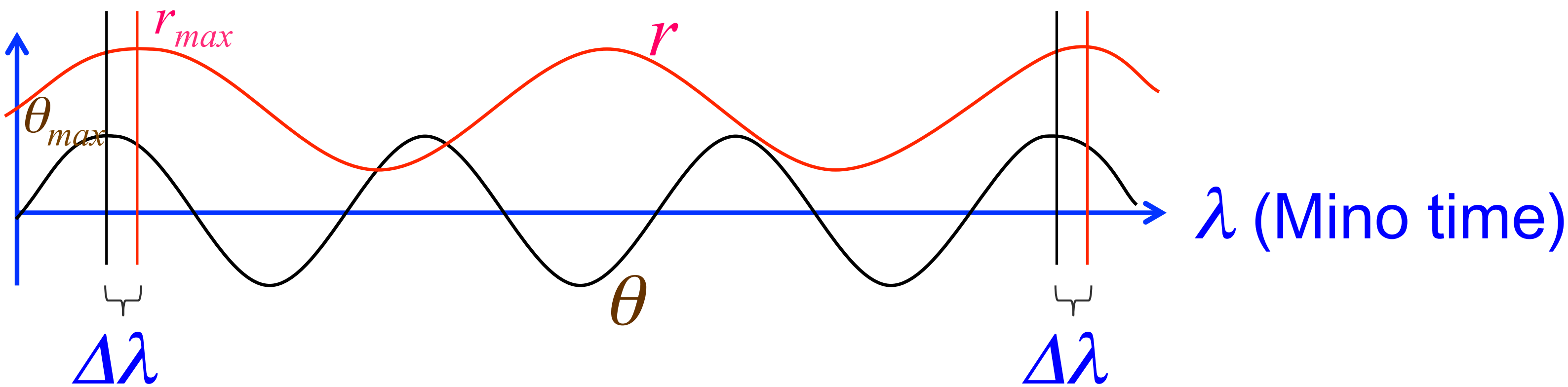
- Key point: Under Mino's transformation

$$\boxed{(t,r,\theta,\phi)\rightarrow(-t,r,\theta,-\phi)}$$

a geodesic is transformed back into the same geodesic.

However, for resonant case: $j_\theta\Omega_r = j_r\Omega_\theta$ with integer j_r & j_θ

$\Delta\lambda$ (separation from θ_{max} to r_{max}) has physical meaning.



Under Mino's transformation, a resonant geodesic with $\Delta\lambda$ transforms into a resonant geodesics with -$\Delta\lambda$.

dQ/dt at resonance

$$G^{(ret)}(x,x') = G^{(rad)}(x,x') + G^{(sym)}(x,x')$$

For the radiative part (retarded-advaneced)/2, a formula similar to the non-resonant case can be obtained:

$$\left\langle \frac{dQ}{dt} \right\rangle = 2\langle f(r)\rangle \left\langle \frac{dE}{dt} \right\rangle - 2\langle g(r)\rangle \left\langle \frac{dL}{dt} \right\rangle + 2\sum_{l,m,n_r,n_\theta} \frac{n_r \Omega_r}{\omega} \left| A_{l,m,n_r,n_\theta} \right|^2$$

$$2\sum_{l,m,N} \frac{\Omega_r}{\omega} A_{l,m,N} \overline{B_{l,m,N}}$$

(Flanagan, Hughes, Ruangsri, 1208.3906)

$$A_{l,m,N} = \sum_{n_r\Omega_r + n_\theta\Omega_\theta = N\Omega} A_{l,m,n_r,n_\theta} \qquad B_{l,m,N} = \sum_{n_r\Omega_r + n_\theta\Omega_\theta = N\Omega} n_r A_{l,m,n_r,n_\theta} \qquad \Omega \equiv \frac{\Omega_r}{j_r} = \frac{\Omega_r}{j_\theta}$$

Sum for the same frequency is to be taken first.

We recently developed a method to evaluate the symmetric part contribution.

→ The next Soichiro's talk

Impact of the resonance on the phase evolution

$$\frac{d\Delta\lambda}{dt} \approx \frac{\Delta\Omega}{\Omega} \qquad \frac{d\Delta\Omega}{dt} \approx \frac{\mu}{M}\Omega^2$$

$\Rightarrow$ $\Delta t_{res} = O\left(\sqrt{M/\mu}\,\Omega^{-1}\right)$: duration staying around resonance

$\Rightarrow$ $\Delta\Omega_{res} = O\left(\sqrt{\mu/M}\,\Omega\right)$: frequency shift caused by passing resonance

$\Rightarrow$ $\Delta\varphi_{res} = \underline{O\left(\sqrt{M/\mu}\right)}$: overall phase error due to resonance

$\neq O((\mu/M)^0)$

If $\dfrac{d\Delta\Omega}{dt} = 0$ for $\Delta\lambda = \Delta\lambda_c$,

Oscillation period is much shorter than the radiation reaction time

$$\left\{\begin{aligned} \frac{d\left(\Delta\lambda - \Delta\lambda_c\right)}{dt} &= O\left(\frac{\Delta\Omega}{\Omega}\right) \\ \frac{d\Delta\Omega}{dt} &= O\left(\frac{\mu}{M}\Omega\left(\Delta\lambda - \Delta\lambda_c\right)\right) \end{aligned}\right. \Rightarrow \frac{d^2\left(\Delta\lambda - \Delta\lambda_c\right)}{dt^2} \approx \boxed{\beta\frac{\mu}{M}}\Omega^2\left(\Delta\lambda - \Delta\lambda_c\right)$$

If β stays negative, resonance may persist for a long time.

Conclusion

Introduction to the next talk. Sorry for containing nothing new.

Adiabatic radiation reaction for the Carter constant is as easy to compute as those for energy and angular momentum.

leading order

second order

$$\frac{dE_{orbit}}{dt} = 0 + \boxed{O(\mu)} + O(\mu^2)$$

$$\frac{dE_{orbit}}{df} = (\text{geodesic}) + \boxed{O(\mu)} + O(\mu^2)$$

Hence, the leading order waveform whose phase is correct at $O(M/\mu)$ has already been ready to compute.

The orbital evolution may cross resonance, which induces $O((M/\mu)^{1/2})$ correction to the phase.

For the change rate of the Carter constant, we need to evaluate the symmetric part in the resonance case.