Extraction of very high precision post-Newtonian parameters from self-force computations

Abhay Shah
John Friedman,
Bernard Whiting,
Alexandre Le Tiec,
Luc Blanchet.

מכון ויצמן למדע
WEIZMANN INSTITUTE OF SCIENCE

Introduction to Radiation gauge

- We start with the Teukolsky equation which has the form,

$$\mathcal{O}\mathcal{T}(h) = \mathcal{S}\mathcal{E}(h).$$

- From this we want to extract the perturbed metric, h.

Teukolsky equation

Newman-Penrose equations (Bianchi identities):

Derivative operators acting on Weyl scalars

$=$ Derivative operators acting on Ricci tensor

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A combination of them gives us:

Derivative operators acting on ψ_0

= Derivative operators acting on $R_{\mu\nu} \propto T_{\mu\nu}$

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$$\mathcal{O}\mathcal{T}(h) = \mathcal{S}\mathcal{E}(h)$$

Teukolsky equation

$$\mathcal{O}\boxed{\mathcal{T}(h)} = \mathcal{S}\mathcal{E}(h)$$

perturbed Weyl tensor dotted with the background null tetrad, i.e., ψ_0

Teukolsky equation

$$\mathcal{O}\mathcal{T}(h) = \mathcal{S}\boxed{\mathcal{E}(h)}$$

Einstein operator acting on $h = 8\pi\, T_{\mu\nu}$

Teukolsky equation

$$\mathcal{O}\mathcal{T}(h) = \boxed{\mathcal{S}}\mathcal{E}(h)$$

2^{nd} order derivative operator acting on $T_{\mu\nu}$

Teukolsky equation

$$\boxed{\mathcal{O}}\mathcal{T}(h) = \mathcal{S}\mathcal{E}(h)$$

2^{nd} order derivative operator acting on ψ_0

Finding h^{ren} from ψ_0^{ren}

$\boxed{\text{Chrzanowski-Cohen-Kegeles-Wald}}$

Theorem: Suppose $\mathcal{S}\mathcal{E} = \mathcal{O}\mathcal{T}$ holds,

and suppose Ψ satisfies $\mathcal{O}^\dagger \Psi = 0$.

If $\mathcal{E}$ is self-adjoint, then $\mathcal{S}^\dagger \Psi$ satisfies $\mathcal{E}(f) = 0$.

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If $\mathcal{E}$ is self-adjoint, then $\mathcal{S}^\dagger \Psi$ satisfies $\mathcal{E}(f) = 0$.

Proof: Taking adjoint of $\mathcal{S}\mathcal{E} = \mathcal{O}\mathcal{T}$, gives us

$$\mathcal{E}^\dagger \mathcal{S}^\dagger = \mathcal{T}^\dagger \mathcal{O}^\dagger$$

$$\mathcal{E} \mathcal{S}^\dagger = \mathcal{T}^\dagger \mathcal{O}^\dagger$$

If $\mathcal{O}^\dagger \Psi = 0$, then $\mathcal{E}(\mathcal{S}^\dagger \Psi) = 0$, i.e., $h = \mathcal{S}^\dagger \Psi$

Wait a minute...

How do we connect the solution to the Teukolsky equation, ψ_0 or $\mathcal{T}(h)$, to this Ψ?

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Lets substitute $\mathcal{S}^\dagger\Psi$ back into the Teukolsky equation,

$$\begin{aligned}\mathcal{S}\mathcal{E}(\mathcal{S}^\dagger\Psi) &= \mathcal{O}\mathcal{T}(\mathcal{S}^\dagger\Psi)\\ 0 &= \mathcal{O}[\mathcal{T}\mathcal{S}^\dagger\Psi]\end{aligned}$$

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$\mathcal{T}\mathcal{S}^\dagger$ maps solutions of $\mathcal{O}^\dagger\Psi = 0$ to $\mathcal{O}\psi = 0$.

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Intermediate Hertz potential

Summary

Weyl scalar
ψ_0 or $\mathcal{T}(h)$

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$$\boxed{\begin{array}{c}\text{Weyl scalar}\\ \psi_0 \,\text{or}\, \mathcal{T}(h)\end{array}} \xrightarrow[\psi_0 = \mathcal{T}\mathcal{S}^\dagger\Psi]{\text{invert}} \boxed{\begin{array}{c}\text{Hertz Potential}\\ \Psi\end{array}}$$

Summary

$$\begin{tikzcd} \boxed{\begin{array}{c}\text{Weyl scalar}\\ \psi_0 \text{ or } \mathcal{T}(h)\end{array}} \arrow[r, "\text{invert}", "\psi_0 = \mathcal{T}\mathcal{S}^\dagger\Psi"'] & \boxed{\begin{array}{c}\text{Hertz Potential}\\ \Psi\end{array}} \arrow[r, "h = \mathcal{S}^\dagger\Psi"] & \boxed{\begin{array}{c}\text{Perturbed metric}\\ h_{\alpha\beta}\end{array}} \end{tikzcd}$$

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$$+$$

$$\boxed{\begin{array}{c}\text{Non-radiative}\\ \text{contributions to } h_{\alpha\beta}\\ \delta m,\ \delta J\end{array}}$$

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$$u^\alpha u^\beta (g_{\alpha\beta} + h_{\alpha\beta}) = 1$$

$$u^\alpha = [u_0^t + u_1^t + O(\mu^2)]k^\alpha$$

$$\Delta U = u_1^t = u_0^t\, H$$

$$H := \frac{1}{2} h^{\mathrm{R}}_{\alpha\beta} u_0^\alpha u_0^\beta$$

+

$$\boxed{\begin{array}{c}\text{Non-radiative}\\ \text{contributions to } h_{\alpha\beta}\\ \delta m,\ \delta J\end{array}}$$

||

$$\boxed{u_1^t}$$

Blanchet et al

coeff.	value
α_4	$-114.34747(5)$
α_5	$-245.53(1)$
α_6	$-695(2)$
β_6	$+339.3(5)$
α_7	$-5837(16)$

How its done?

MST (Mano-Suzuki-Takasugi) algorithm for the radial harmonics as a sum over known analytic functions with good convergence

Analytical form of the angular harmonics, ${}_sY_{\ell m}(\frac{\pi}{2}, 0)$

We use *Mathematica* which can handle very high precision computations

The accuracy is

$$\boxed{\begin{array}{l} \text{about } 1 \; \textit{part in } 10^{227} \text{ for } r = 10^{20} M \\ \text{about } 1 \; \textit{part in } 10^{242} \text{ for } r = 10^{25} M \\ \text{about } 1 \; \textit{part in } 10^{252} \text{ for } r = 10^{30} M \end{array}}$$

(Expected) pN expansion of u_1^t

$$
\begin{aligned}
u_1^t = & \frac{\alpha_0}{r} + \frac{\alpha_1}{r^2} + \frac{\alpha_2}{r^3} + \frac{\alpha_3}{r^4} + \frac{\alpha_4}{r^5} + \frac{\beta_4 \log(r)}{r^5} \\
& + \frac{\alpha_5}{r^6} + \frac{\beta_5 \log(r)}{r^6} + \frac{\alpha_6}{r^7} + \frac{\beta_6 \log(r)}{r^7} + \cdots
\end{aligned}
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$$+ \frac{\alpha_5}{r^6} + \frac{\beta_5 \log(r)}{r^6} + \frac{\alpha_6}{r^7} + \frac{\beta_6 \log(r)}{r^7} + \cdots$$

Notice the absence of 5.5-pN term

no $\frac{\alpha_{5.5}}{r^{6.5}}$

Analytically known pN coefficients

(from literature)

$$u_1^t = \frac{-1}{r} + \frac{-2}{r^2} + \frac{-5}{r^3} + \frac{(\frac{-121}{3} + \frac{41}{32}\pi^2)}{r^4}$$

$$+ \boxed{\frac{\frac{-592384-196608\gamma+10155\pi^2-393216\log(2)}{7680}}{r^5}}$$

$$+ \frac{\frac{64}{5}\log(r)}{r^5} + \frac{\frac{956}{105}\log(r)}{r^6} + \cdots$$

Bini & Damour '13*

* Thanks to Alexandre Le Tiec

u_1^t at $r = 10^{30}M$

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463025987555823873 [illegible] 41365999702786

5756860740840 [illegible] 49252423

57201651745 [illegible] 553391925

010747719 [illegible] 271980

914443 [illegible] 346

282 X 10^{-30}

Read off the first 4 coefficients to 30 places of accuracy

Monday, July 29, 2013

u_1^t at $r = 10^{30} M$

Lets look at

$$u_1^t - \left(\frac{-1}{r} + \frac{-2}{r^2} + \frac{-5}{r^3} + \frac{\left(\frac{-121}{3} + \frac{41}{32}\pi^2\right)}{r^4} + \frac{\frac{64}{5}\log(r)}{r^5} + \frac{\frac{956}{105}\log(r)}{r^6}\right)$$

= -114.348951367572602952040002444836538764412865284407038869234848092925596369282766597634376193721255230541605421899370456938260020427148255386909790570751 89... X 10^{-150}

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5284407038869234848092925596369282766597634376
1937212552305416054218993704569382600204271482
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= $\boxed{-114.348951367572602952040002444}$83653876441286
528440703886923 … 2827665976343 76
193721255 … 204271482
55386903 …

Again read off α_4 to 30 decimal places

$$+ \boxed{\frac{\frac{-59238 \ldots \gamma \ldots \pi^2 - 393216\log(2)}{7680}}{r^5}}$$

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And we got those terms starting at 5.5 pN in our numerical matching.

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$$5.5\text{pN comes from } \ell = 2 \text{ multipole,}$$

$$6.5\text{pN comes from } \ell = 2, 3 \text{ multipole,}$$

$$7.5\text{pN comes from } \ell = 2, 3, 4 \text{ multipole} \cdots$$

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$$6.5\text{pN comes from } \ell = 2, 3 \text{ multipole,}$$
$$7.5\text{pN comes from } \ell = 2, 3, 4 \text{ multipole} \cdots$$

Specifically,

$$5.5\text{pN comes from } \ell = 2, m = \pm 2 \text{ multipole,}$$
$$6.5\text{pN comes from } \ell = 2, m = \pm 1, \pm 2 \text{ and}$$
$$\ell = 3, m = \pm 1, \pm 3 \text{ multipoles.}$$

Puzzle?

pN theory didn't expect n.5-pN terms.
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This was very puzzling!!

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To confirm its presence, we performed the whole calculation in the RWZ gauge which agreed with the radiation gauge.

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A successful comparison with the UC-Dublin group (compared the source, l=2 multipole, of 5.5pN term):
23-24 digits of agreement at $r = 10^3 M$
43 digits of agreement at $r = 10^6 M$

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To further confirm its presence, we computed analytical value of the 5.5pN coefficient using the self-force recipe in a radiation gauge, and found that the numerically extracted value agrees with it to **113** significant digits!!

Puzzle?

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And we got those terms starting at 5.5 pN.

About a month ago, it was understood that the origin of these n.5pN terms come from the "tails-of-tails" terms in pN theory (Luc Blanchet).

Revised pN-series

$$
\begin{aligned}
u_1^t = & \frac{\alpha_0}{r} + \frac{\alpha_1}{r^2} + \frac{\alpha_2}{r^3} + \frac{\alpha_3}{r^4} + \frac{\alpha_4}{r^5} + \frac{\beta_4 \log(r)}{r^5} \\
& + \frac{\alpha_5}{r^6} + \frac{\beta_5 \log(r)}{r^6} + \frac{\alpha_{5.5}}{r^{6.5}} + \frac{\alpha_6}{r^7} + \frac{\beta_6 \log(r)}{r^7} \\
& + \frac{\alpha_{6.5}}{r^{7.5}} + \frac{\alpha_7}{r^8} + \frac{\beta_7 \log(r)}{r^8} + \frac{\gamma_7 \log^2(r)}{r^8} + \cdots
\end{aligned}
$$

Analytical value of other terms possible from high precision number?

$$u_1^t = \boxed{\frac{\alpha_0}{r}} + \boxed{\frac{\alpha_1}{r^2}} + \boxed{\frac{\alpha_2}{r^3}} + \boxed{\frac{\alpha_3}{r^4}} + \boxed{\frac{\alpha_4}{r^5}} + \boxed{\frac{\beta_4 \log(r)}{r^5}}$$
$$+ \frac{\alpha_5}{r^6} + \boxed{\frac{\beta_5 \log(r)}{r^6}} + \boxed{\frac{\alpha_{5.5}}{r^{6.5}}} + \frac{\alpha_6}{r^7} + \frac{\beta_6 \log(r)}{r^7}$$
$$+ \frac{\alpha_{6.5}}{r^{7.5}} + \frac{\alpha_7}{r^8} + \frac{\beta_7 \log(r)}{r^8} + \frac{\gamma_7 \log^2(r)}{r^8} + \cdots$$

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$$+ \frac{\alpha_5}{r^6} + \boxed{\frac{\beta_5 \log(r)}{r^6}} + \boxed{\frac{\alpha_{5.5}}{r^{6.5}}} + \frac{\alpha_6}{r^7} + \frac{\beta_6 \log(r)}{r^7}$$
$$+ \frac{\alpha_{6.5}}{r^{7.5}} + \frac{\alpha_7}{r^8} + \frac{\beta_7 \log(r)}{r^8} + \frac{\gamma_7 \log^2(r)}{r^8} + \cdots$$

What about other terms?

If possible, it will save us a number of long, tedious, analytical calculations, and help extract further terms with higher accuracy.

Analytical value of other terms possible from high precision number?

$$u_1^t = \boxed{\frac{\alpha_0}{r}} + \boxed{\frac{\alpha_1}{r^2}} + \boxed{\frac{\alpha_2}{r^3}} + \boxed{\frac{\alpha_3}{r^4}} + \boxed{\frac{\alpha_4}{r^5}} + \boxed{\frac{\beta_4 \log(r)}{r^5}}$$
$$+ \frac{\alpha_5}{r^6} + \boxed{\frac{\beta_5 \log(r)}{r^6}} + \boxed{\frac{\alpha_{5.5}}{r^{6.5}}} + \frac{\alpha_6}{r^7} + \frac{\beta_6 \log(r)}{r^7}$$
$$+ \frac{\alpha_{6.5}}{r^{7.5}} + \frac{\alpha_7}{r^8} + \frac{\beta_7 \log(r)}{r^8} + \frac{\gamma_7 \log^2(r)}{r^8} + \cdots$$

What about other terms?

YES!! INDEED!!

Lets look at the value of the 6-pN log term

$$u_1^t = \frac{\alpha_0}{r} + \frac{\alpha_1}{r^2} + \frac{\alpha_2}{r^3} + \frac{\alpha_3}{r^4} + \frac{\alpha_4}{r^5} + \frac{\beta_4 \log(r)}{r^5} + \frac{\alpha_5}{r^6} + \frac{\beta_5 \log(r)}{r^6} + \frac{\alpha_{5.5}}{r^{6.5}} + \frac{\alpha_6}{r^7} + \boxed{\frac{\beta_6 \log(r)}{r^7}} + \frac{\alpha_{6.5}}{r^{7.5}} + \frac{\alpha_7}{r^8} + \frac{\beta_7 \log(r)}{r^8} + \frac{\gamma_7 \log^2(r)}{r^8} + \cdots$$

Numerically extracted value

$$\begin{aligned}\beta_6 = -90.&398589065255731922398589065255731922\\ &398589065255731922398589065255731922\\ &398589065255731922398589048525187995 5\cdots\end{aligned}$$

Numerically extracted value

$$\beta_6 = -90.\boxed{398589065255731922}398589065255731922$$
$$398589065255731922398589065255731922$$
$$398589065255731922398589048525187995 5\cdots$$

Numerically extracted value

$$
\begin{aligned}
\beta_6 = -90.&\boxed{398589065255731922}\boxed{398589065255731922}\\
&\boxed{398589065255731922}\boxed{398589065255731922}\\
&\boxed{398589065255731922}\boxed{3985890}485251879955\cdots
\end{aligned}
$$

More than 5 repetition cycles

$$\beta_6 = \frac{-51256}{567}$$

Another example

$$u_1^t = \frac{\alpha_0}{r} + \frac{\alpha_1}{r^2} + \frac{\alpha_2}{r^3} + \frac{\alpha_3}{r^4} + \frac{\alpha_4}{r^5} + \frac{\beta_4 \log(r)}{r^5}$$
$$+ \frac{\alpha_5}{r^6} + \frac{\beta_5 \log(r)}{r^6} + \frac{\alpha_{5.5}}{r^{6.5}} + \frac{\alpha_6}{r^7} + \frac{\beta_6 \log(r)}{r^7}$$
$$+ \frac{\alpha_{6.5}}{r^{7.5}} + \frac{\alpha_7}{r^8} + \frac{\beta_7 \log(r)}{r^8} + \boxed{\frac{\gamma_7 \log^2(r)}{r^8}} + \cdots$$

Numerically extracted value

$$\gamma_7 = 52.17523809523809523809523809523809$$
$$523809523809523809523809523809$$
$$5238095237538043489164331\cdots$$

Numerically extracted value

$$\gamma_7 = 52.17\boxed{523809}523809523809523809523809$$
$$523809523809523809523809523809$$
$$5238095237538043489164331\cdots$$

Numerically extracted value

$$\gamma_7 = 52.17\boxed{523809}\boxed{523809}\boxed{523809}\boxed{523809}\boxed{523809}$$
$$\boxed{523809}\boxed{523809}\boxed{523809}\boxed{523809}\boxed{523809}$$
$$\boxed{523809}\boxed{523}7538043489164331\cdots$$

More than 11 repetition cycles

$$\gamma_7 = \frac{27392}{525}$$

6.5-pN term

$$
\begin{aligned}
\alpha_{6.5} = 69.&3090904966257595632206088669802055352527628 2\\
&806406929175117896885464782924271210388282 91\\
&65780393465346820430681300807 64\cdots
\end{aligned}
$$

6.5-pN term

$$\alpha_{6.5} = 69.309090496625759563220608866980205535252762828064069291751178968854647829242712103882916578039346534682043068130080764\cdots$$

$$\frac{\alpha_{6.5}}{\pi} = 22.06176870748299319727891156462585034013605442176870748299319727891156462585034013605478708 0\cdots$$

6.5-pN term

$$\alpha_{6.5} = 69.309090496625759563220608866980205535252762828064069291751178968854647829242712103882916578039346534682043068130080764\cdots$$

$$\frac{\alpha_{6.5}}{\pi} = 22.06\boxed{176870748299319727891156462585034013605442}\boxed{1768707482993197278911564625850340136054}787080\cdots$$

Almost 2 repetition cycles here

$$\alpha_{6.5} = \frac{81077\pi}{3675}$$

One is not always lucky to have such repetition cycles...

$$\frac{\alpha_{7.5}}{\pi} = 176.4975875153652931430709208486986264764042541820319598$$
$$0973759881572859646496135482085019945242682178\cdots$$

One is not always lucky to have such repetition cycles...

$$\frac{\alpha_{7.5}}{\pi} = 176.4975875153652931430709208486986264764042541820319598$$
$$0973759881572859646496135482085019945242682178\cdots$$

Multiplying it with appropriate powers of the first few prime numbers clear things up

$$\frac{\alpha_{7.5}}{\pi} \times 5^2 \times 3^5 \times 7 \times 11 =$$
$$\boxed{82561159.00000000000000000000000000000000000000000000000000000000}5286027\cdots$$

One is not always lucky to have such repetition cycles...

$$\frac{\alpha_{7.5}}{\pi} = 176.4975875153652931430709208486986264764042541820319598$$
$$0973759881572859646496135482085019945242682178\cdots$$

Multiplying it with appropriate powers of the first few prime numbers clears things up

$$\frac{\alpha_{7.5}}{\pi} \times 5^2 \times 3^5 \times 7 \times 11 =$$
$$82561159.000000000000000000000000000000000000000000000000000000000000005286027\cdots$$

Consistent with the accuracy with which we extracted this number

Final list of analytically known pN terms

$$\frac{-1}{r} + \frac{-2}{r^2} + \frac{-5}{r^3} + \frac{\frac{-121}{3} + \frac{41\pi^2}{32}}{r^4}$$
$$+ \frac{-592384 - 196608\gamma + 10155\pi^2 - 393216\log(2)}{7680\ r^5}$$
$$+ \frac{64}{5}\frac{\log(r)}{r^5} + \frac{-956}{105}\frac{\log(r)}{r^6} + \frac{-13696\pi}{525 r^{6.5}} + \frac{-51256}{567}\frac{\log(r)}{r^7}$$
$$+ \frac{81077\pi}{3675\ r^{7.5}} + \frac{27392}{525}\frac{\log^2(r)}{r^8} + \frac{82561159\pi}{467775\ r^{8.5}} + \frac{-27016}{2205}\frac{\log^2(r)}{r^9}$$
$$+ \frac{-11723776\pi}{55125}\frac{\log(r)}{r^{9.5}} + \frac{-4027582708}{9823275}\frac{\log^2(r)}{r^{10}} + \frac{99186502\pi}{1157625}\frac{\log(r)}{r^{10.5}}$$
$$+ \frac{23447552}{165375}\frac{\log^3(r)}{r^{11}}$$

Final list of analytically known pN terms

Bini & Damour '13*

$$
\frac{-1}{r} + \frac{-2}{r^2} + \frac{-5}{r^3} + \frac{\frac{-121}{3} + \frac{41\pi^2}{32}}{r^4}
$$
$$
+ \boxed{\frac{-592384 - 196608\gamma + 10155\pi^2 - 393216\log(2)}{7680\ r^5}}
$$
$$
+ \frac{64}{5}\frac{\log(r)}{r^5} + \frac{-956}{105}\frac{\log(r)}{r^6} + \frac{-13696\pi}{525 r^{6.5}} + \frac{-51256}{567}\frac{\log(r)}{r^7}
$$
$$
+ \frac{81077\pi}{3675\ r^{7.5}} + \frac{27392}{525}\frac{\log^2(r)}{r^8} + \frac{82561159\pi}{467775\ r^{8.5}} + \frac{-27016}{2205}\frac{\log^2(r)}{r^9}
$$
$$
+ \frac{-11723776\pi}{55125}\frac{\log(r)}{r^{9.5}} + \frac{-4027582708}{9823275}\frac{\log^2(r)}{r^{10}} + \frac{99186502\pi}{1157625}\frac{\log(r)}{r^{10.5}}
$$
$$
+ \frac{23447552}{165375}\frac{\log^3(r)}{r^{11}}
$$

* Thanks to Alexandre Le Tiec

Final list of analytically known pN terms

$$
\begin{aligned}
&\frac{-1}{r} + \frac{-2}{r^2} + \frac{-5}{r^3} + \frac{\frac{-121}{3} + \frac{41\pi^2}{32}}{r^4} \\
&+ \frac{-592384 - 196608\gamma + 10155\pi^2 - 393216\log(2)}{7680\ r^5} \\
&+ \frac{64}{5}\frac{\log(r)}{r^5} + \frac{-956}{105}\frac{\log(r)}{r^6} + \frac{-13696\pi}{525 r^{6.5}} + \frac{-51256}{567}\frac{\log(r)}{r^7} \\
&+ \frac{81077\pi}{3675\ r^{7.5}} + \frac{27392}{525}\frac{\log^2(r)}{r^8} + \frac{82561159\pi}{467775\ r^{8.5}} + \frac{-27016}{2205}\frac{\log^2(r)}{r^9} \\
&+ \frac{-11723776\pi}{55125}\frac{\log(r)}{r^{9.5}} + \frac{-4027582708}{9823275}\frac{\log^2(r)}{r^{10}} + \frac{99186502\pi}{1157625}\frac{\log(r)}{r^{10.5}} \\
&+ \frac{23447552}{165375}\frac{\log^3(r)}{r^{11}}
\end{aligned}
$$

Flux at future horizon

Recent work by Fujita calculated the pN series of flux at future null infinity to 22pN order!

Tagoshi et al calculated the pN series of flux at future horizon to 6pN order.

Using

Flux lost = Flux at infinity + Flux at horizon

we extract the 7,8,9,10-pN orders of flux at horizon

Flux lost = Flux at infinity + Flux at horizon

pN order	Lost by particle	∞	r_+
1	✓	✓	0
2	✓	✓	0
3	✓	✓	0
4	✓	✓	✓
5	✓	✓	✓
6	✓	✓	✓
7	✓	✓	?
8	✓	✓	?
9	✓	✓	?
10	✓	✓	?

Green tick-marks show what we know from literature and that our result agrees with previous work
Red question marks show what is unknown and how we extract the unknown coefficients.

Flux lost = Flux at infinity + Flux at horizon

pN order	Lost by particle	∞	r_+
1	✓	✓	0
2	✓	✓	0
3	✓	✓	0
4	✓	✓	✓
5	✓	✓	✓
6	✓	✓	✓
7	✓	✓	✓
8	✓	✓	✓
9	✓	✓	✓
10	✓	✓	✓

Green tick-marks show what we know from literature and that our result agrees with previous work
Red question marks show what is unknown and how we extract the unknown coefficients.

Matching in Kerr