I. Numerical method for generic orbits
II. Radial fall evolution

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Plan

I. Numerical method for generic orbits

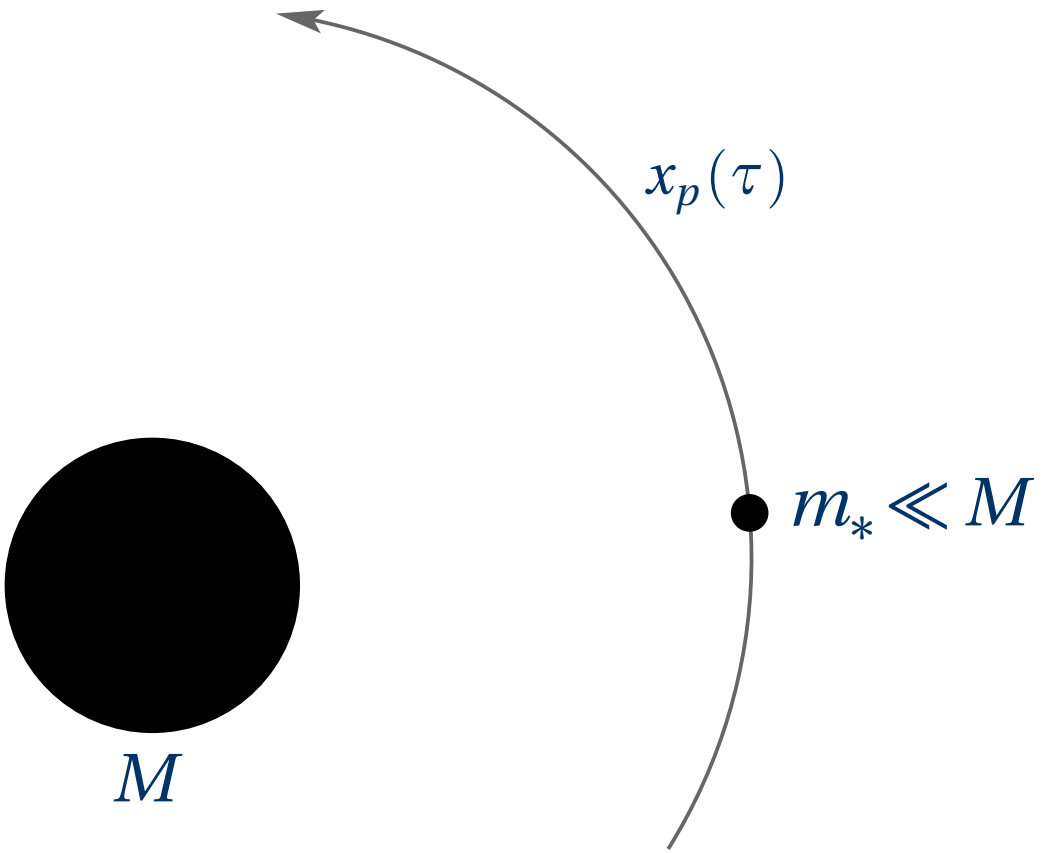
Solve the Regge-Wheeler-Zerilli wave equation in time domain by using jump conditions.

II. Radial fall evolution

Use the code with an osculating method to treat the particular case of radial infall in RW gauge.

Conclusions and perspectives

I.
Numerical method for generic orbits


$x_p(\tau)$
$m_* \ll M$
M

RWZ wave equation

Linear combinations of $h^{(i)\ell m}$ lead to 2 gauge invariant scalar fields (*Moncrief*)

$$\psi^{\ell m}_{\text{even}} = \frac{r}{\lambda+1}\left[K^{\ell m}(r,t) + \frac{r-2M}{\lambda r+3M}\left(H_2^{\ell m}(t,r) - r\partial_r K^{\ell m}(t,r)\right)\right]$$
$$\psi^{\ell m}_{\text{odd}} = \frac{r}{\lambda}\left[r^2\partial_r\left(\frac{h_0^{\ell m}(t,r)}{r^2}\right) - \partial_t h_1^{\ell m}(t,r)\right]$$

The 2 functions satisfy Regge-Wheeler-Zerilli equations

$$\left[-\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial r^{*2}} - V(r)^{\ell}_{e/o}\right]\psi^{\ell m}_{\text{e/o}}(t,r) = P^{\ell m}_{e/o}(t)\frac{\partial}{\partial r}\delta\left(r - r_p(t)\right) + Q^{\ell m}_{e/o}(t)\delta\left(r - r_p(t)\right)$$

$V^{\ell}_{e/o}, P^{\ell m}_{e/o}, Q^{\ell m}_{e/o}$ are known functions
$r^* = r + 2M\ln(r/2M - 1)$ is the tortoise coordinate
$r_p(t)$ particle trajectory
$\lambda = \frac{1}{2}(\ell-1)(\ell+2)$

Knowing $\psi^{\ell m}_{e/o}$, metric reconstruction is still possible : $h^{(i)} = h^{(i)}\left[\psi, \partial\psi, \partial^2\psi\right]$.

Jump conditions

Different ways of jump conditions implementation : *Haas, Barack Sago, Hopper Evans, Sopuerta Laguna, Field et al., Spallicci Aoudia (11), Ritter et al. (11)..*

δ' induces a discontinuity at $r = r_p(t)$

$$\boxed{\psi(t,r) = \psi^+(t,r)\,\Theta_1 + \psi^-(t,r)\,\Theta_2}$$

Jump of the wave function at the location of the particle

$$\boxed{[\![\psi]\!]_{r_p} = \psi^+(t, r_p(t)) - \psi^-(t, r_p(t))}$$

where $\psi^{\pm}(t, r_p(t)) = \lim_{\varepsilon \to 0} \psi(t, r_p \pm \varepsilon)$

$\Theta_1 = \Theta\big(r - r_p\big)$, $\Theta_2 = \Theta\big(r_p - r\big)$ Heaviside distributions

$\delta = \delta\big(r - r_p\big)$, $\delta' = \frac{\partial}{\partial r}\delta\big(r - r_p\big)$ Dirac distribution and its spatial derivative

Jump conditions

$$[\![\psi]\!]_{r_p} = \frac{P(t)}{f(r_p)^2 - \dot{r}_p^2}$$

$$[\![\partial_r \psi]\!]_{r_p} = \frac{1}{f(r_p)^2 - \dot{r}_p^2}\left[Q(t) + \left(f(r_p)\frac{df}{dr}(r_p) - \ddot{r}_p\right)[\![\psi]\!]_{r_p} - 2\dot{r}_p \frac{d}{dt}[\![\psi]\!]_{r_p}\right]$$

$$[\![\partial_t \psi]\!]_{r_p} = \frac{d}{dt}[\![\psi]\!]_{r_p} - \dot{r}_p [\![\partial_r \psi]\!]_{r_p}$$

$$[\![\partial_r^n \partial_t^m \psi]\!]_{r_p} \ldots$$

where $\dot{r}_p = \frac{dr_p}{dt}$, $\ddot{r}_p = \frac{d^2 r_p}{dt^2}$ and $f(r_p) = \left(1 - \frac{2M}{r_p}\right)$

Numerical implementation

For generic orbits $\{t, r_p(t), \theta_p(t), \phi_p(t)\}$

$$P_o^{\ell m}(t) = \frac{8\kappa}{\lambda} r_p \left(\dot{r}_p^2 - f(r_p)^2\right) A^{\ell m \star}$$

$$Q_o^{\ell m}(t) = -\frac{8\kappa}{\lambda} r_p \dot{r}_p \frac{dA^{\ell m \star}}{dt} - \frac{8\kappa}{\lambda}\left[\frac{r_p}{u^t}\frac{d}{dt}(u^t \dot{r}_p) + \left(\dot{r}_p^2 - f(r_p)\right)\right] A^{\ell m \star}$$

$$V_o^{\ell}(r) = 2f(r)\left[(\lambda+1)r^{-2} - 3Mr^{-3}\right]$$

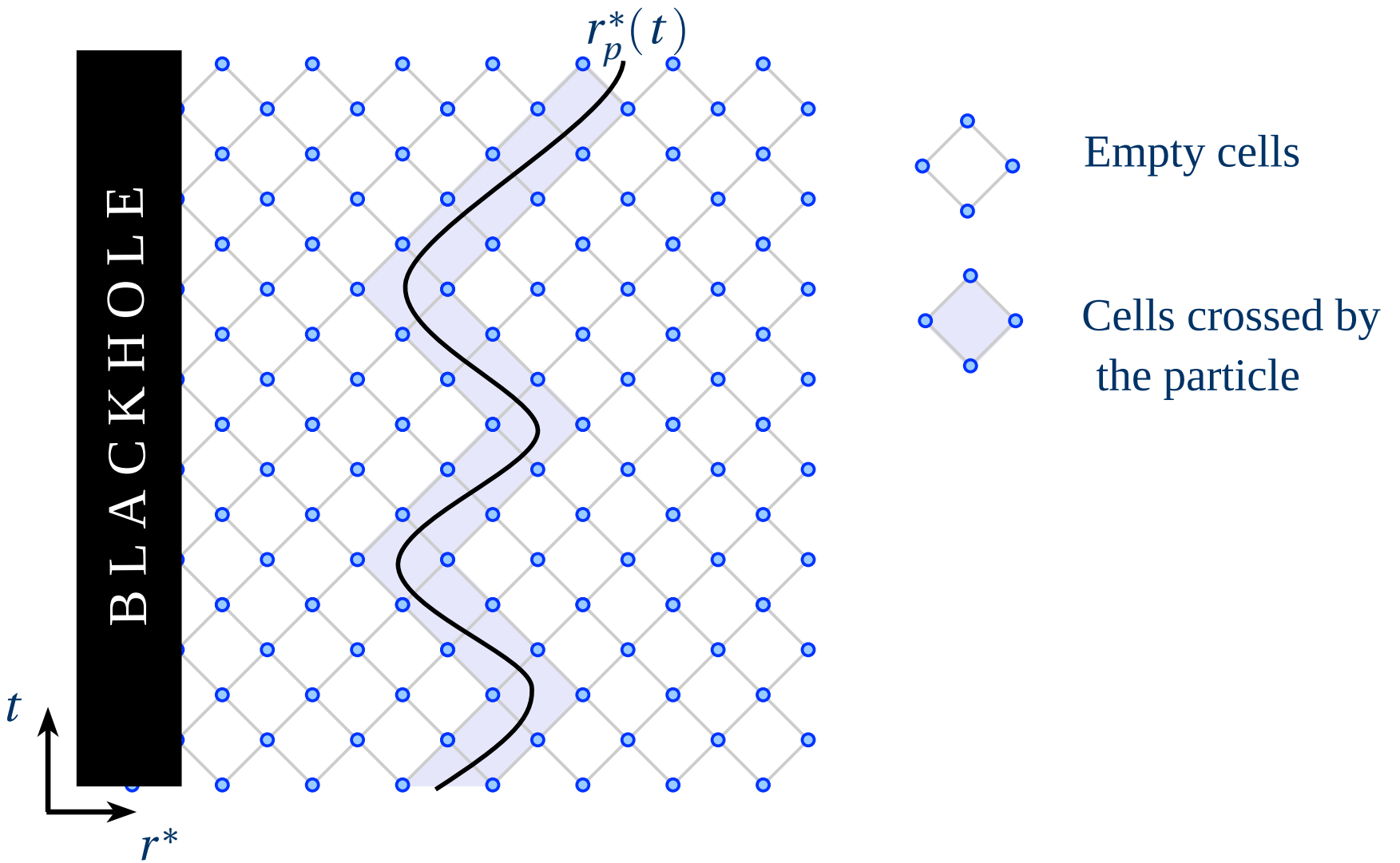
$$P_e^{\ell m}(t) = -8\kappa \frac{r_p f(r_p)\left(\dot{r}_p^2 - f(r_p)^2\right)}{\lambda r_p + 3M} Y^{\ell m \star}$$

$$\begin{aligned} Q_e^{\ell m}(t) = & 16\kappa \frac{r_p \dot{r}_p f(r_p)}{\lambda r_p + 3M}\frac{dY^{\ell m \star}}{dt} - 16\frac{\kappa}{\lambda} r_p f(r_p)\dot{\theta}_p \dot{\phi}_p \partial_\phi \left(\partial_\theta - \cot\theta_p\right) Y^{\ell m \star} + 8\kappa \frac{r_p^2 f(r_p)^2}{\lambda r_p + 3M}\left(\dot{\theta}_p^2 + \sin^2\theta_p \dot{\phi}_p^2\right) Y^{\ell m \star} \\ & -4\kappa\lambda^{-1} r_p f(r_p)\left(\dot{\theta}_p^2 - \sin^2\theta_p \dot{\phi}_p^2\right)\left(\partial_\theta^2 - \cot\theta_p \partial_\theta - \sin^{-2}\theta_p \partial_\phi^2\right) Y^{\ell m \star} \\ & + 8\kappa \frac{\dot{r}_p^2\left[(\lambda+1)(6r_pM + \lambda r_p^2) + 3M^2\right]}{r_p(\lambda r_p + 3M)^2} Y^{\ell m \star} - 8\kappa \frac{f(r_p)^2\left[r_p^2\lambda(\lambda+1) + 6\lambda r_p M + 15M^2\right]}{r_p(\lambda r_p + 3M)^2} Y^{\ell m \star} \end{aligned}$$

$$V_e^{\ell}(r) = 2f(r)\frac{\left[\lambda^2(\lambda+1)r^3 + 3\lambda^2 M r^2 + 9\lambda M^2 r + 9M^3\right]}{r^3(\lambda r + 3M)^2}$$

where $\kappa = (\pi m_* u^t)/(\lambda+1)$, $\lambda = \frac{1}{2}(\ell-1)(\ell+2)$, $f(r) = 1 - 2M/r$ and $A^{\ell m} = \left(\frac{\dot{\theta}_p}{\sin\theta_p}\partial_\phi - \sin\theta_p \dot{\phi}_p \partial_\theta\right) Y^{\ell m}$

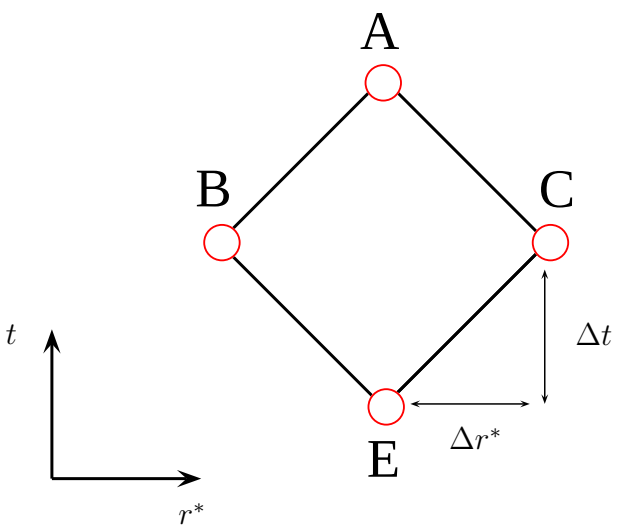
Numerical implementation



Numerical implementation

Empty cells :

2nd order classical finite difference scheme (*Lousto Price, Martel Poisson*)



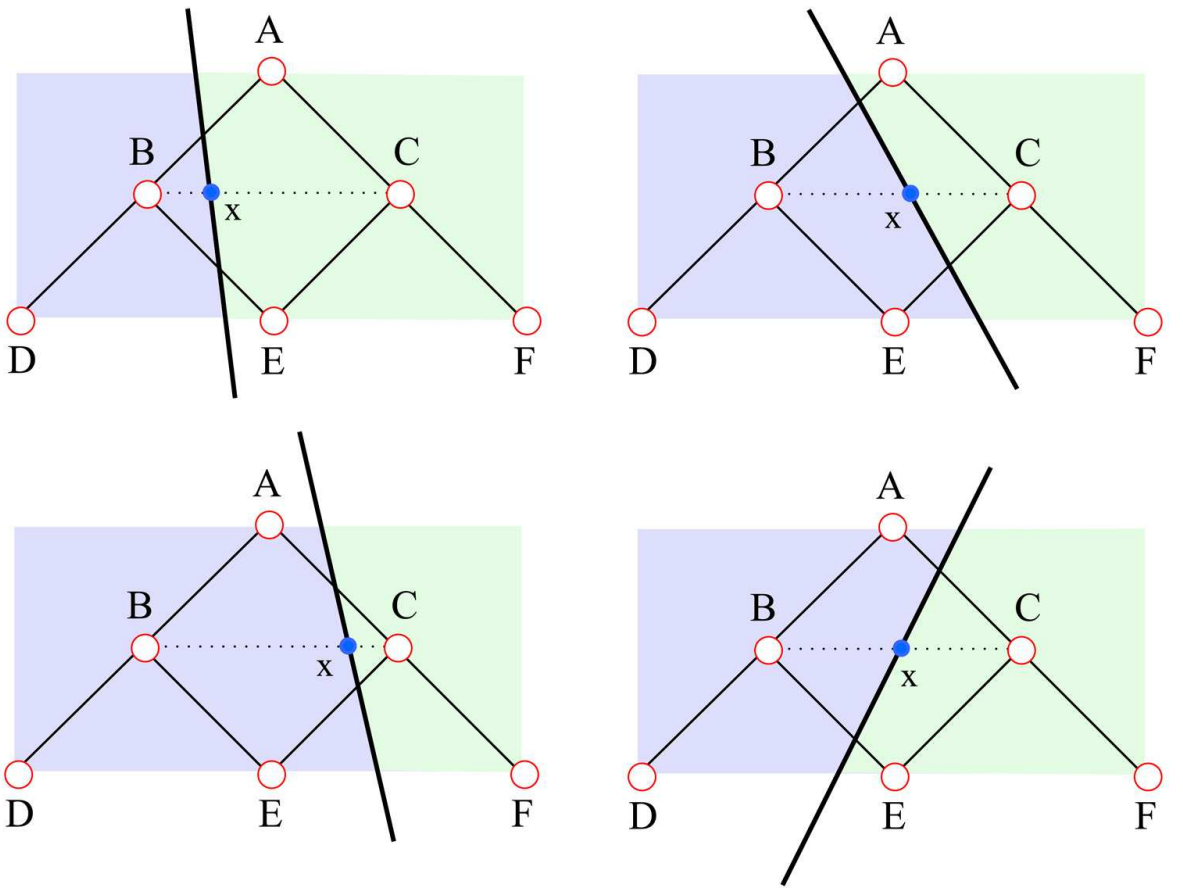
$$\psi_A^{\ell m} = -\psi_E^{\ell m} + \left(\psi_B^{\ell m} + \psi_C^{\ell m}\right)\left(1 - \frac{\Delta r^{*2}}{2} V^\ell(r)\right) + O(\Delta r^{*4})$$

typically $\Delta r^* = \Delta t$

Numerical implementation

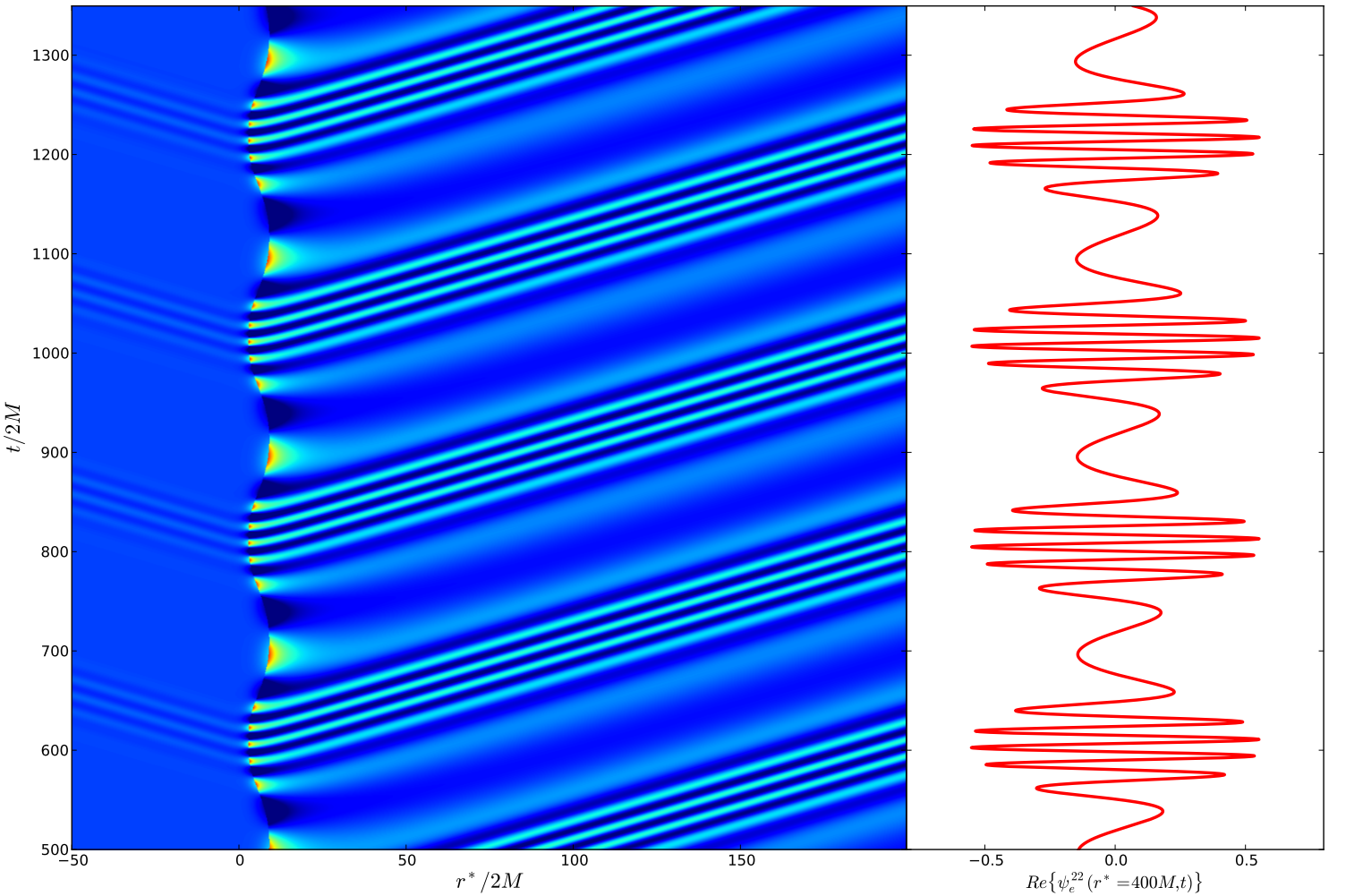
Cells crossed by the world line :

2nd order modified finite difference scheme



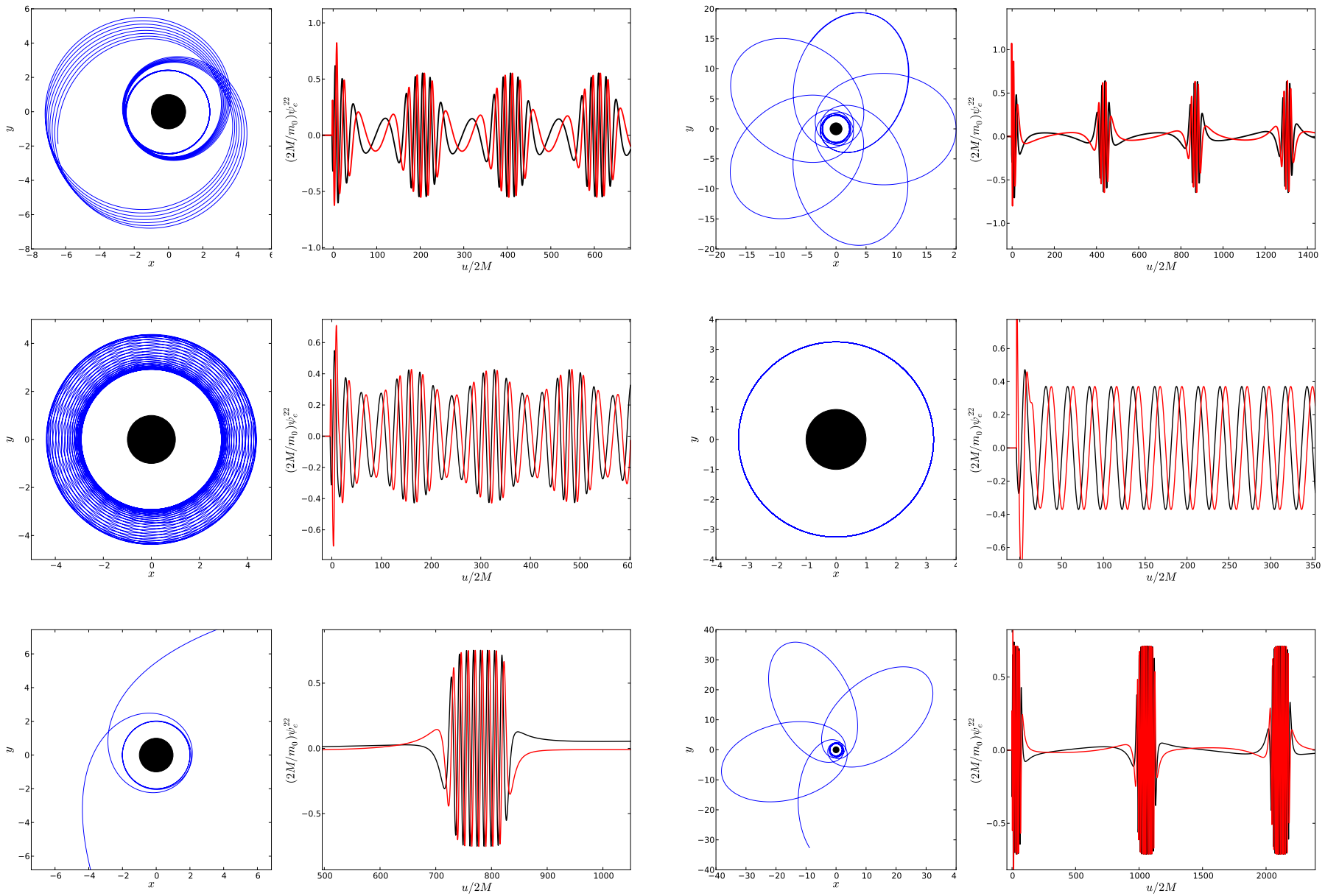
$$\psi_A^{\ell m} = \sum_{n\in\{B,C,D,E,F\}} \alpha_n \psi_n^{\ell m} + \sum_{p+q<3} \beta_{pq} \left[\!\left[\partial_t^p \partial_{r*}^q \psi^{\ell m} \right]\!\right]_{\mathrm{x}} + O(\Delta r^{*3})$$

Numerical implementation



Example : elliptic orbit $(e = 0.5)$ for the quadrupolar mode $(\ell, m) = (2, 2)$

Numerical implementation



Code validation

- 2nd order convergence in time
- Less than 2nd order convergence in space on the trajectory (δ')
- Averaged flux of energy and angular momentum at infinity :

$$\frac{dE}{dt} = \frac{1}{64\pi} \sum_{\ell \geq 2, m} \frac{(\ell+2)!}{(\ell-2)!} \left[\left| \frac{d\psi_e^{\ell m}}{dt} \right|^2 + \left| \frac{d\psi_o^{\ell m}}{dt} \right|^2 \right]$$

$$\frac{dL}{dt} = \frac{im}{64\pi} \sum_{\ell \geq 2, m} \frac{(\ell+2)!}{(\ell-2)!} \left[\psi_e^{\ell m \star} \frac{d\psi_e^{\ell m}}{dt} + \psi_o^{\ell m \star} \frac{d\psi_o^{\ell m}}{dt} \right]$$

Good agreement with previous literature (0.001%<err<1%) *Poisson, Martel, Barack Lousto, Sopuerta Laguna, Cutler et al., Hopper Evans*

II.
Radial fall evolution

Self-force in RW gauge

How to compute the self-force in RW gauge ?

✖ No self-force regularisation procedure in RW gauge for circular or elliptic orbits (not yet) ($h^{\ell m}_{\alpha\beta} \notin \mathscr{C}^0$)

but

✔ mode-sum applicable to a purely radial orbit in RW gauge (*Barack Ori*) i.e

$$\boxed{F^{\alpha(\mathrm{RW})}_{\mathrm{self}} = \sum_{\ell=0}^{\infty}\left[F^{\alpha\ell(\mathrm{RW})}_{\mathrm{ret}} - A^\alpha L - B^\alpha - C^\alpha L^{-1}\right] - D^\alpha}$$

with $L = \ell + 1/2$ and A^α, B^α, C^α and D^α are regularisation parameters

$$F^\alpha[h^{\mathrm{ret}}_{\alpha\beta}] = -\frac{1}{2}\left(g^{\alpha\beta} + u^\alpha u^\beta\right)\left(2\nabla_\mu h^{\mathrm{ret}}_{\beta\nu} - \nabla_\beta h^{\mathrm{ret}}_{\mu\nu}\right) u^\mu u^\nu = \sum_\ell F^{\alpha\ell}_{\mathrm{ret}}$$

[Also ζ-regularisation was used in radial fall (*Lousto, Spallicci Aoudia*)]

Self-force in RW gauge

Radial fall case

- Trajectory : $\dfrac{d^2 r_p}{d t^2} = \dfrac{1}{2} f(r_p) f'(r_p) \left[1 - \dfrac{3 \dot{r}_p^2}{f(r_p)^2} \right], \quad \theta_p = 0$

- Only even perturbations $\psi^\ell \equiv \psi_{\mathrm{e}}^{\ell m=0}$

- $h_{\alpha\beta}^{\mathrm{ret}\ell} = \begin{pmatrix} f H_2^\ell & H_1^\ell \\ H_1^\ell & f^{-1} H_2^\ell \end{pmatrix} Y^{\ell 0} \quad \in \quad \mathscr{C}^0$ on the world line

- $H_1^\ell(t, r) = k_0 \partial_t \psi^\ell + k_1 \partial_{r\,t} \psi^\ell + k_2 \delta' + k_3 \delta$

- $H_2^\ell(t, r) = k_4 \psi^\ell + k_5 \partial_r \psi^\ell + k_6 \partial_r^2 \psi^\ell + k_7 \delta' + k_8 \delta$

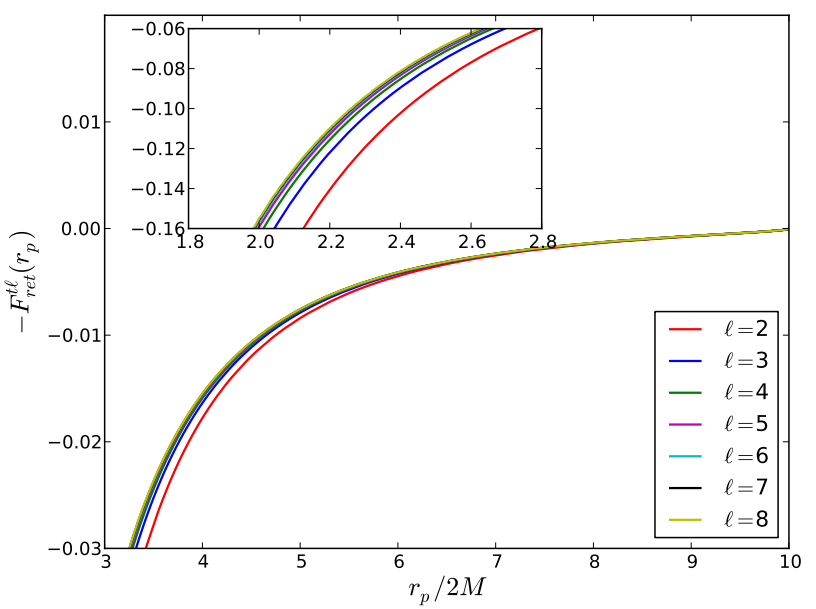
Self-force in RW gauge

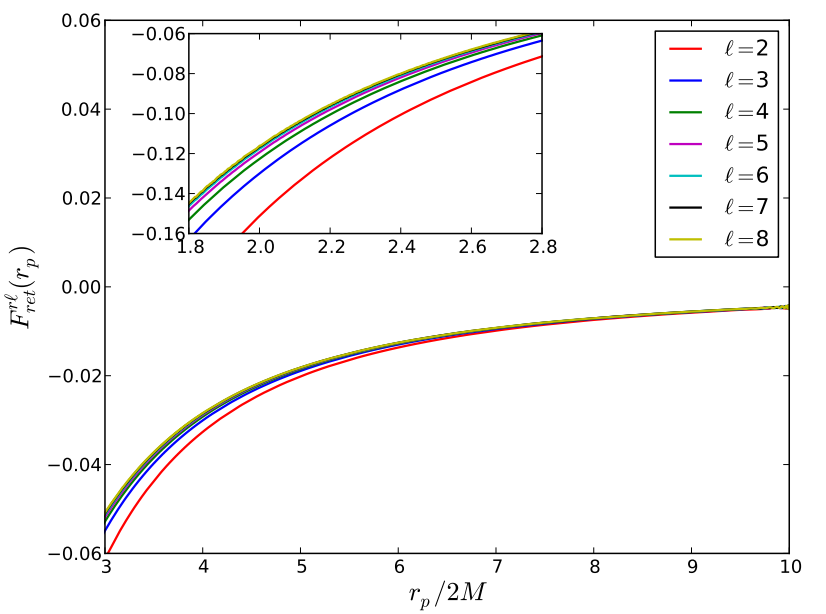
Radial fall case

- $F^{\alpha}[h^{\rm ret}_{\alpha\beta}] = -\frac{1}{2}\left(g^{\alpha\beta} + u^{\alpha}u^{\beta}\right)\left(2\nabla_{\mu}h^{\rm ret}_{\beta\nu} - \nabla_{\beta}h^{\rm ret}_{\mu\nu}\right)u^{\mu}u^{\nu} = \sum_{\ell} F^{\alpha\ell}_{\rm ret}$

$$F^{\alpha\ell}_{\rm ret} = -\frac{1}{2}\left[f^{\alpha}_{0}\left(\frac{\partial H^{\ell}_{2}}{\partial t} - \frac{df}{dr}H^{\ell}_{1}\right) + f^{\alpha}_{1}\left(\frac{\partial H^{\ell}_{1}}{\partial t} - \frac{df}{dr}H^{\ell}_{2}\right) + f^{\alpha}_{2}\frac{\partial H^{\ell}_{2}}{\partial r} + f^{\alpha}_{3}\frac{\partial H^{\ell}_{1}}{\partial r}\right]Y^{\ell 0}$$

where the f^{α}_{i} are functions of r_p and $\dot{r}_p$





Consistent with *Barack Lousto*.

Self-force in RW gauge

Radial fall case

- $F^\alpha[h^{\rm ret}_{\alpha\beta}] = -\frac{1}{2}\left(g^{\alpha\beta} + u^\alpha u^\beta\right)\left(2\nabla_\mu h^{\rm ret}_{\beta\nu} - \nabla_\beta h^{\rm ret}_{\mu\nu}\right) u^\mu u^\nu = \sum_\ell F^{\alpha\ell}_{\rm ret}$

$$F^{\alpha\ell}_{\rm ret} = -\frac{1}{2}\left[f_0^\alpha\left(\frac{\partial H_2^\ell}{\partial t} - \frac{df}{dr}H_1^\ell\right) + f_1^\alpha\left(\frac{\partial H_1^\ell}{\partial t} - \frac{df}{dr}H_2^\ell\right) + f_2^\alpha\frac{\partial H_2^\ell}{\partial r} + f_3^\alpha\frac{\partial H_1^\ell}{\partial r}\right]Y^{\ell 0}$$

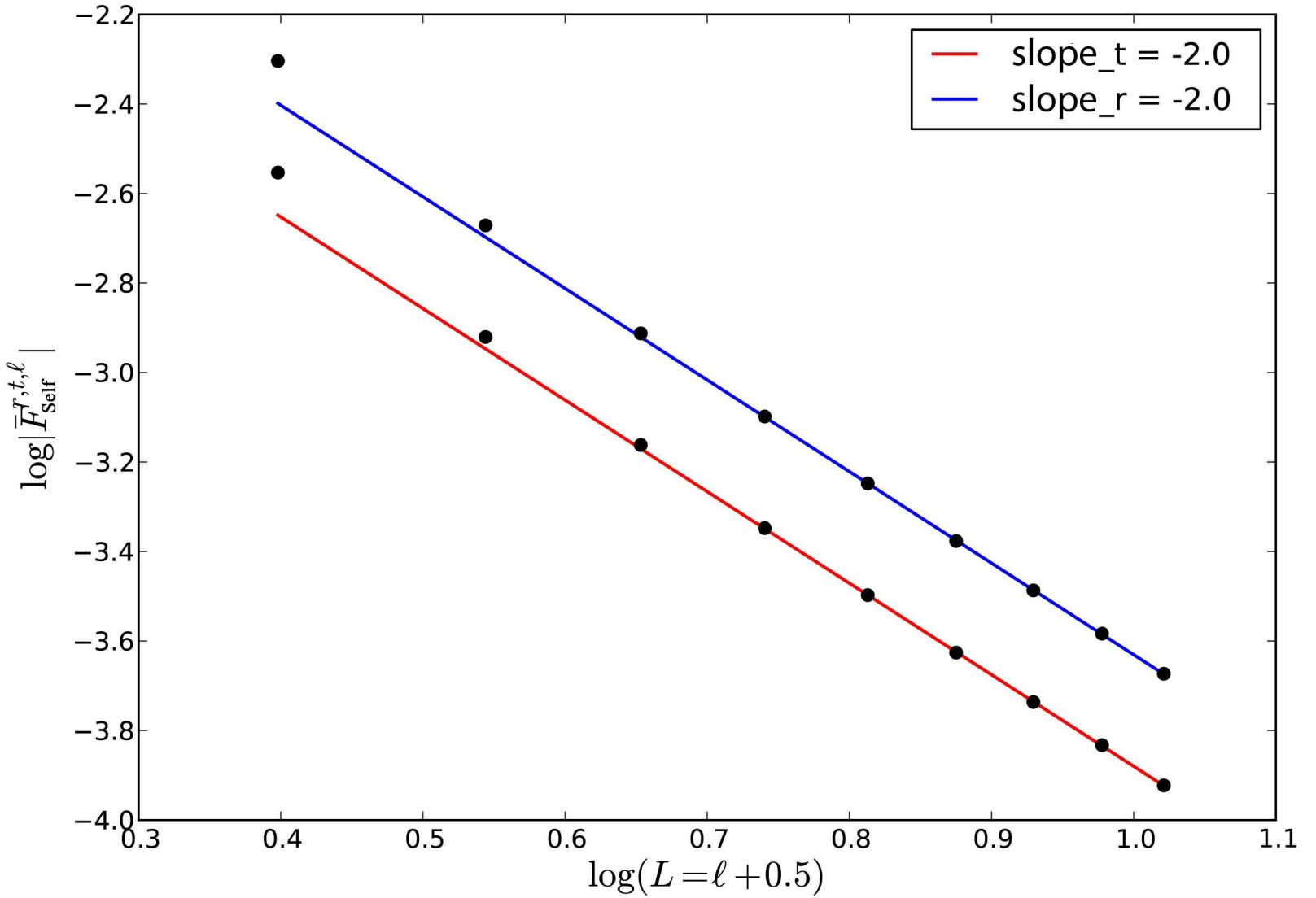
- Regularisation parameters given by $F^{\alpha\ell\to\infty}_{\rm ret}$ at the coincidence limit $r = r_p(t)$

$$A^r = \pm\frac{E}{r_p^2} \quad A^t = \pm\frac{\dot r_p}{f(r_p)r_p^2} \quad B^r = -\frac{E^2}{2r_p^2} \quad B^t = -\frac{E\dot r_p}{2f(r_p)r_p^2}$$

$$C^\alpha = D^\alpha = 0$$

Same expressions as computed in harmonic gauge; consistent with *Barack Ori*.

Self-force in RW gauge



Action of the self-force on the trajectory

$$\frac{d^2 x_p^\alpha}{d\tau^2} + \Gamma^\alpha_{\beta\gamma} \frac{d x_p^\beta}{d\tau} \frac{d x_p^\gamma}{d\tau} = F^\alpha_{\text{Self}}$$

perturbed motion

$$\ddot{r}_p = \frac{1}{2} f(r_p) f'(r_p) \left[1 - \frac{3\dot{r}_p^2}{f(r_p)^2} \right] + \Lambda_{\text{Self}}(r_p, \dot{r}_p)$$

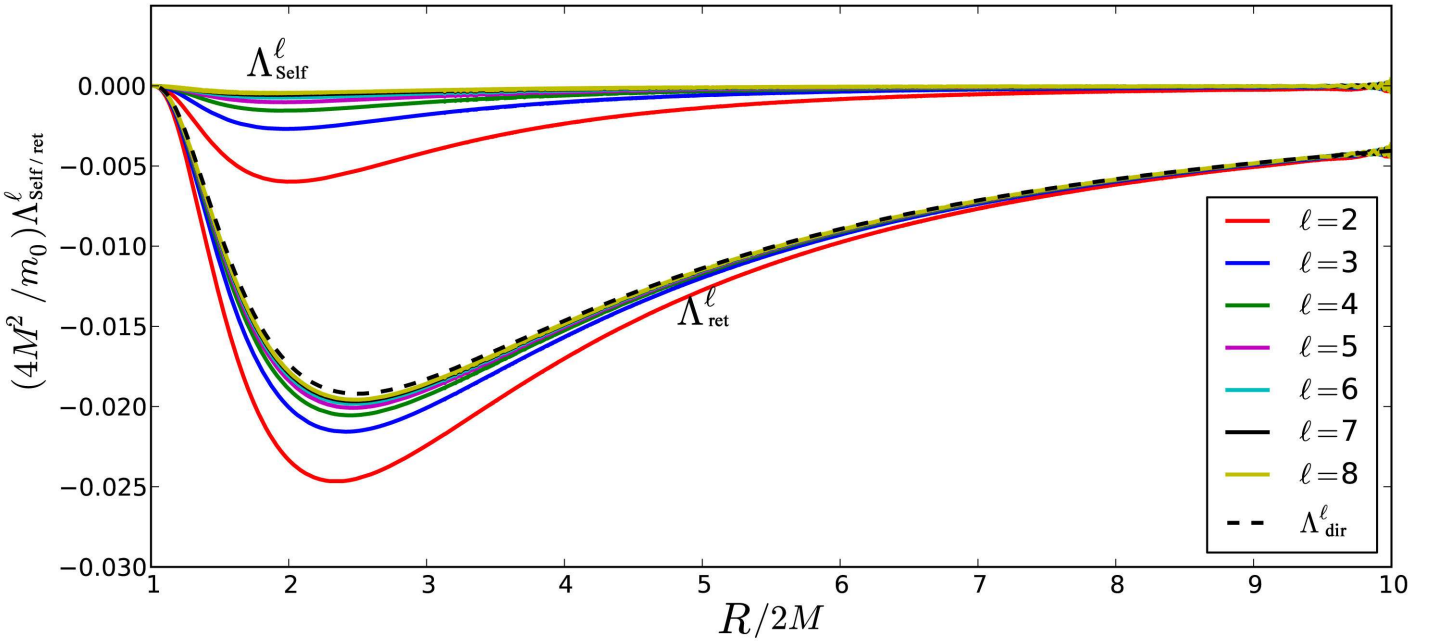
pragmatic approach : $r_p = R + \Delta r$

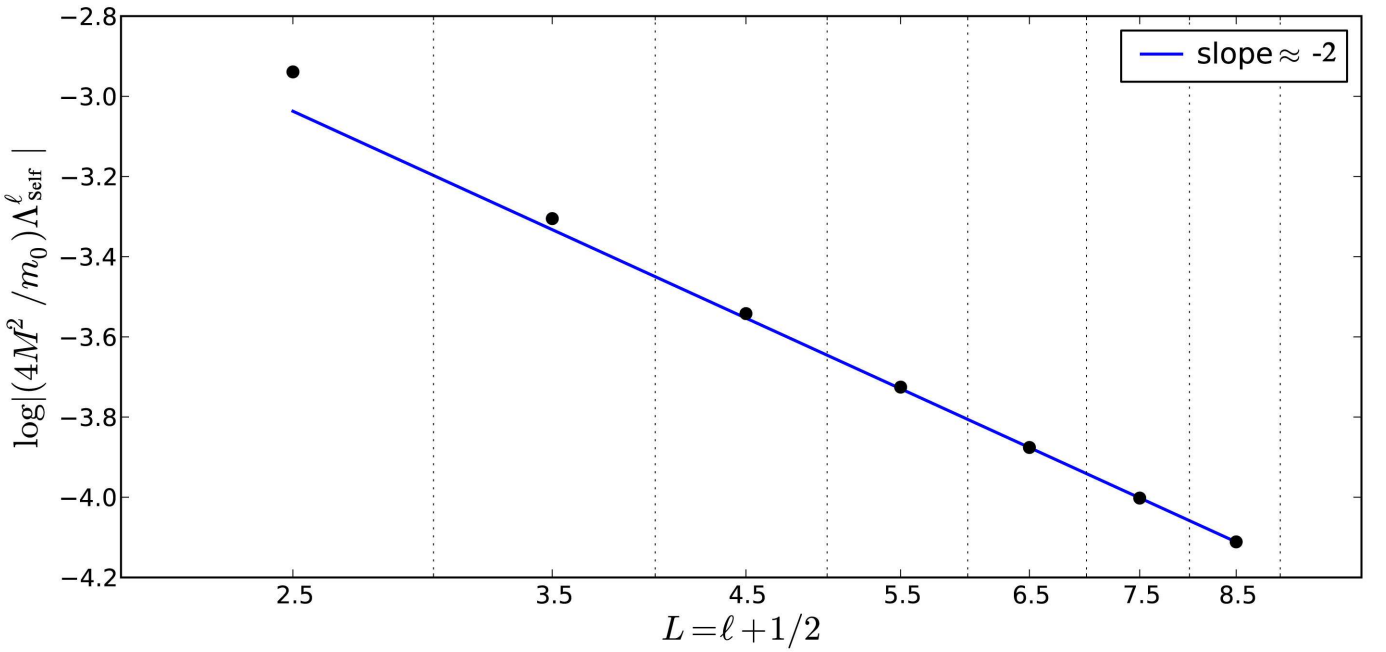
$$\ddot{\Delta} r = \underbrace{\Lambda_0(R, \dot{R}) \Delta r + \Lambda_1(R, \dot{R}) \dot{\Delta} r}_{\text{background geodesic deviation}} + \Lambda_{\text{Self}}(R, \dot{R})$$

where

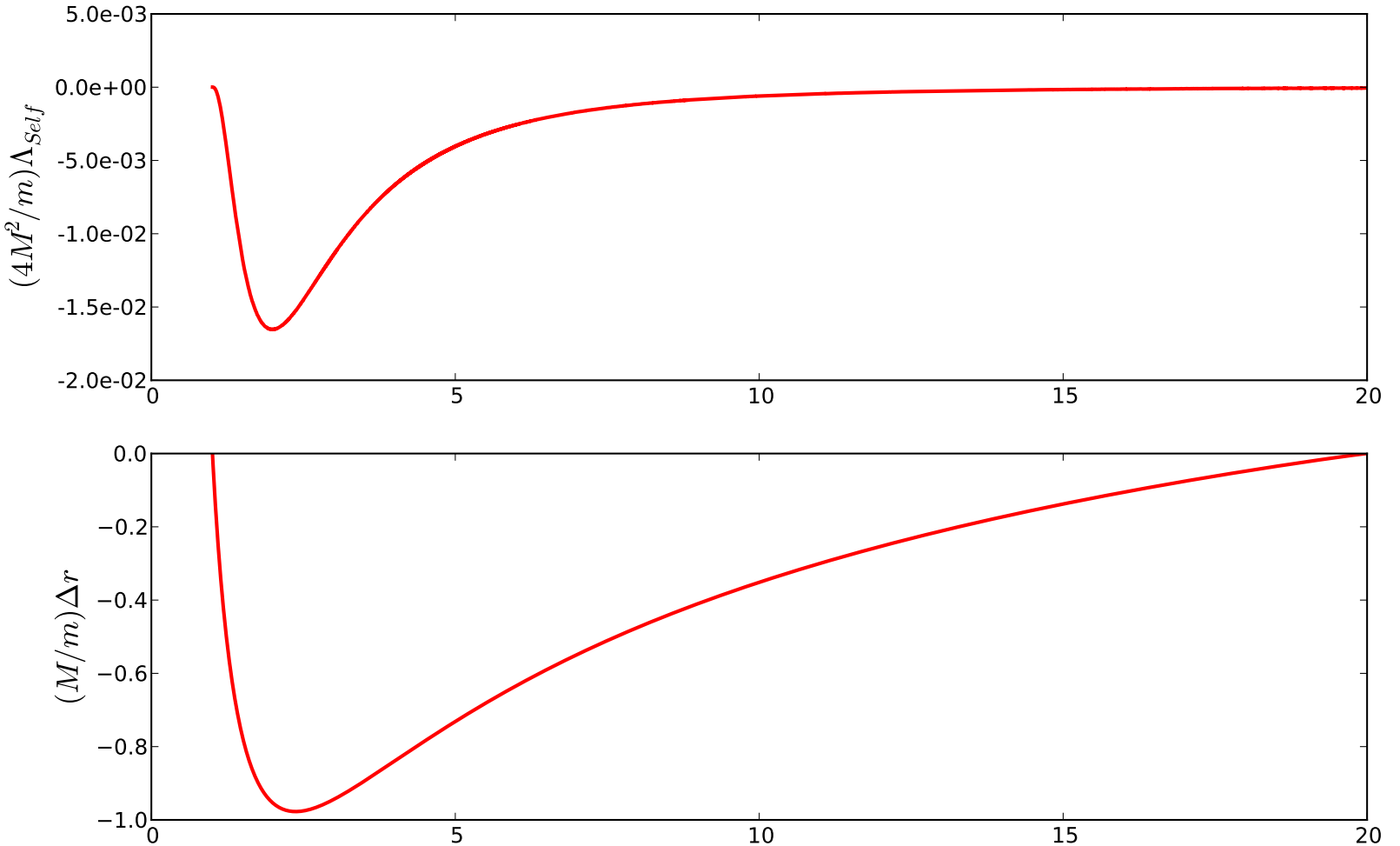
$$\boxed{\Lambda_{\text{Self}}(R, \dot{R}) = \sum_\ell \frac{f(R)^2}{E^2} \left[F^{r\ell}_{\text{Self}} - \dot{R} F^{t\ell}_{\text{Self}} \right]}$$

Action of the self-force on the trajectory





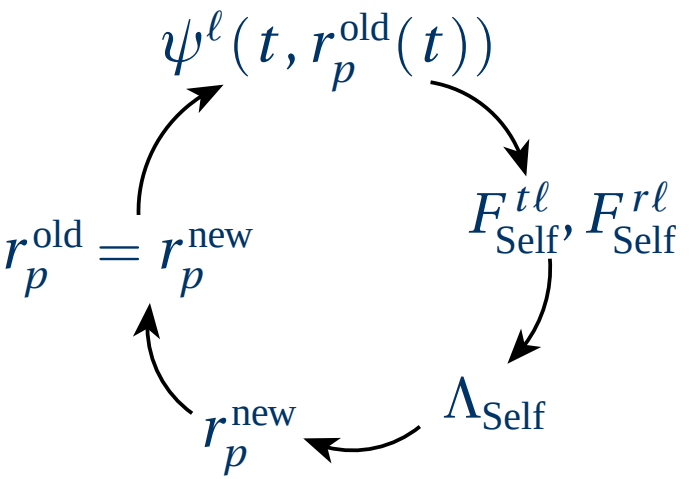
Action of the self-force on the trajectory



$\Delta r < 0 \; \forall \; R \; \rightarrow$ positive work of the SF, $(m_* dE/d\tau = -F_t > 0)$
Consistent with *Barack Lousto 2002* but Λ_{Self} and $\Delta r \neq$ with *Lousto 2000*

Action of the self-force on the trajectory

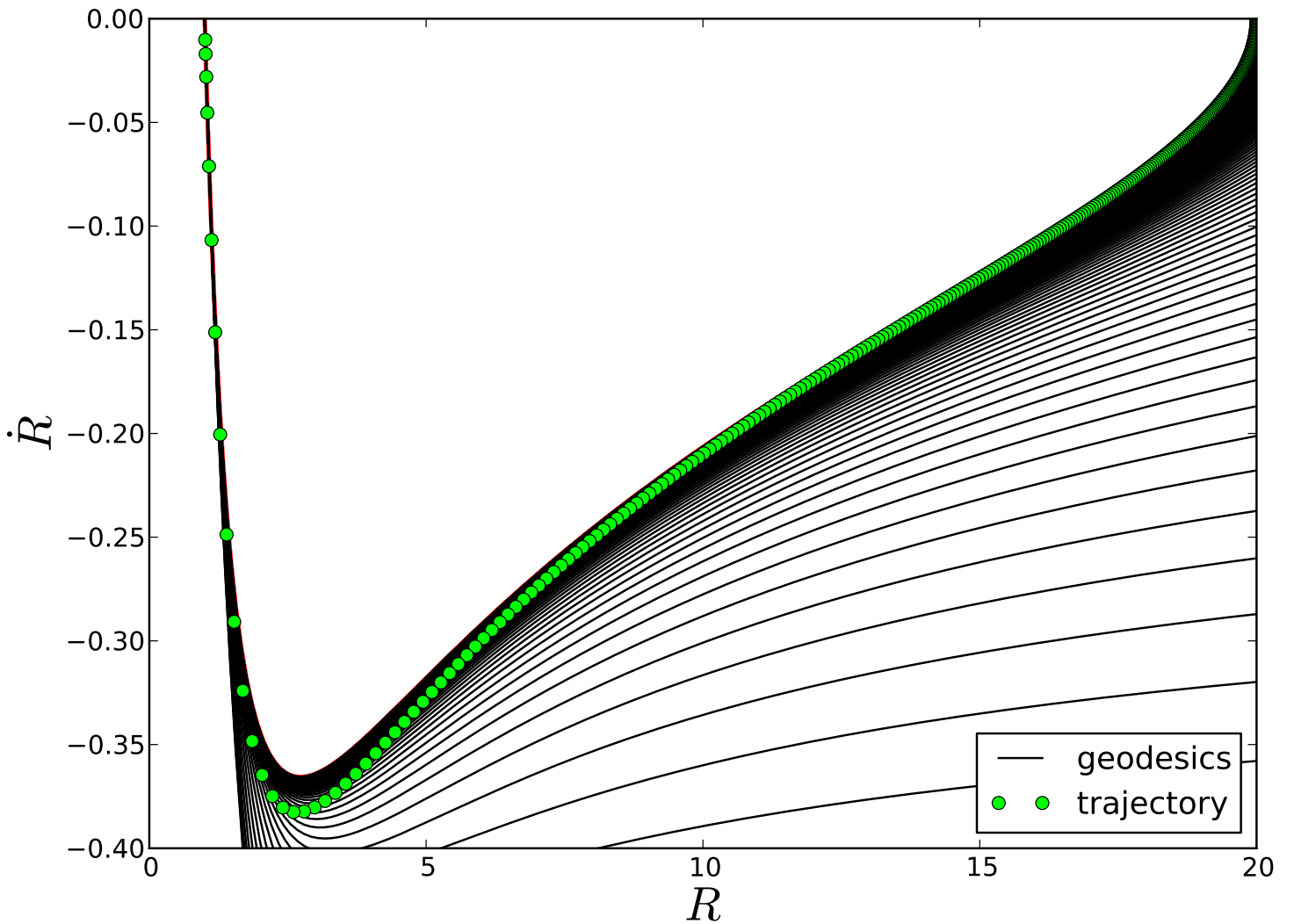
$$\ddot{r}_p = \frac{1}{2} f(r_p) f'(r_p) \left[1 - \frac{3\dot{r}_p^2}{f(r_p)^2} \right] + \Lambda_{\text{Self}}(r_p, \dot{r}_p)$$



But regularisation parameters A^α, B^α and C^α must be computed on a geodesic $\rightarrow$ **osculating orbit approach**.

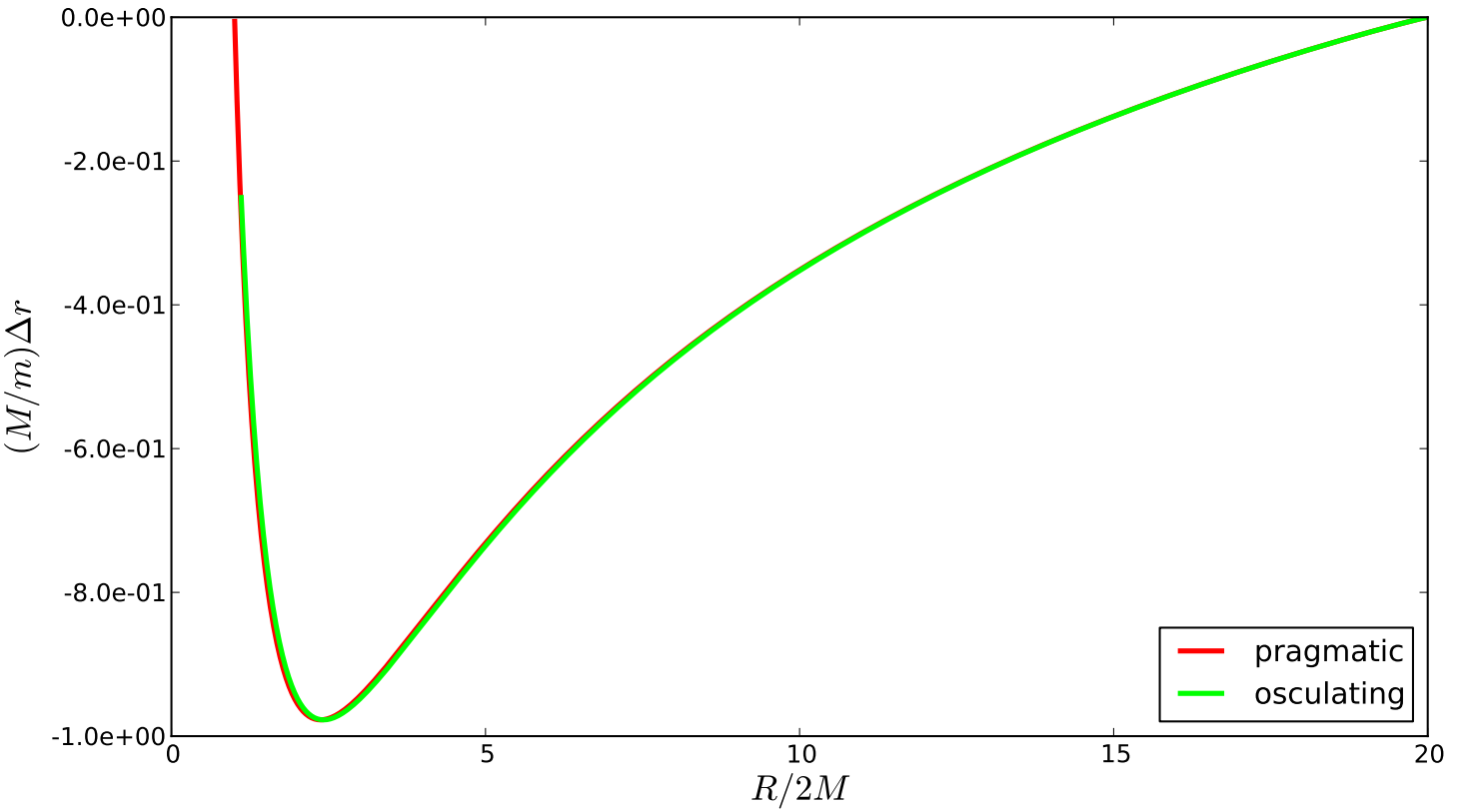
Action of the self-force on the trajectory

phase space



Action of the self-force on the trajectory

Pragmatic vs osculating approach



As expected almost no difference between pragmatic and osculating (maybe for large R_0).
Pragmatic approach good enough to get the perturbed motion.
Good training for more complex orbits in different gauges.

Conclusions

Satisfactory method based on jump conditions applied for radial fall (*Aoudia Spallicci, Ritter et al.*) and generic orbits (submitted) in good agreement with existing literature.

Osculating-iterative scheme applied to radial infall.

Perspectives

For generic orbits, without self-force, we could explore the effect of a third body.

For radial infall, we are still in the phase of evaluation of the results (sensibility to parameters, R_0, m/M, perturbed wave forms, quantify SF errors..)

Numerical investigations, space-time compactification..

Thank you!