Adiabatic evolution of the constants of motion in resonance

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Based on the collaboration

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Old strategy for computing $\langle dQ/d\lambda \rangle$

Mino 0302075

Kerr metric is invariant under this transformation

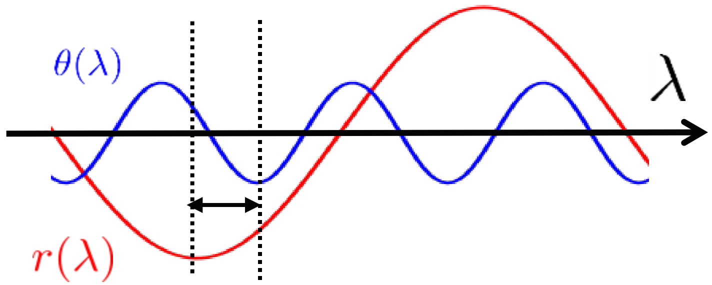
$$(t, r, \theta, \phi) \Rightarrow (-t, r, \theta, -\phi)$$

If the resonance is absent, the geodesic is also invariant with $\lambda \to -\lambda$

$$P_i^{(0)} \Rightarrow P_i^{(0)}$$

Constants of motion

This invariance validates the use of **radiative field**



$$\Delta\lambda := \lambda_{\theta:\mathrm{min}} - \lambda_{r:\mathrm{min}}$$

In the resonance case,
the geodesic is not invariant.

S.I. et al.,1302.4035

$$(P_i^{(0)}, \Delta\lambda) \Rightarrow (P_i^{(0)}, -\Delta\lambda)$$

Go back to the original R-part description.

$$\left\langle \frac{dP_i}{d\lambda} \right\rangle^{(R)} = \underbrace{\frac{1}{2}\left\{ \left\langle \frac{dP_i}{d\lambda} \right\rangle^{(\mathrm{ret})} - \left\langle \frac{dP_i}{d\lambda} \right\rangle^{(\mathrm{adv})} \right\}}_{\text{Radiative part}} + \underbrace{\frac{1}{2}\left\{ \left\langle \frac{dP_i}{d\lambda} \right\rangle^{(\mathrm{ret})} + \left\langle \frac{dP_i}{d\lambda} \right\rangle^{(\mathrm{adv})} \right\} - \left\langle \frac{dP_i}{d\lambda} \right\rangle^{(S)}}_{\text{Symmetric part}}$$

The Hamiltonian of the geodesic : $H := \frac{1}{2} g^{\alpha\beta} p_\alpha p_\beta$

A particle moves along the geodesic on a smooth perturbed space time. Detweiler and Whiting 0202086

$$\frac{H}{\mu^2} = -\frac{1}{2} - \frac{1}{2\mu^2}[h^{\mu\nu\ (R)}(x) p_\mu^{(0)} p_\nu^{(0)}] + O\left(\frac{\mu^2}{M^2}\right)$$

$g_{(0)}^{\alpha\beta} p_\alpha p_\beta = -\mu^2$: **“background” geodesics**

$$H_{\rm int}(x) := -\frac{1}{2} h^{\mu\nu (R)}(x) p_\mu^{(0)} p_\nu^{(0)}$$ **: Interaction Hamiltonian**

Canonical transformation to the "constants of motion" coordinate $(x^\mu, p_\mu) \to (X^\alpha, P_\alpha)$

Carter Phys. Rev. **174** 1559

$$P_\alpha := \left\{ -\mu^2/2,\ \xi^\mu_{(t)} p_\mu,\ \xi^\mu_{(\phi)} p_\mu,\ K^{\mu\nu} p_\mu p_\nu \right\}$$

$\left(\approx \{-\mu^2/2, E, L_z, Q\}\right)$ **at the background geodesics**

Rewrite the interaction Hamiltonian in (x^μ, P_α)

$$H_{\mathrm{int}}(x^\mu, P_\alpha) := H_{\mathrm{int}}(x^\mu, p_\mu(x^\mu, P_\alpha))$$

Hamilton equation in the transformed coordinate

$$p_\sigma^{(0)} = \frac{\partial \mathcal{W}(x^\mu_{(0)}, P^{(0)}_\alpha)}{\partial x^\sigma_{(0)}} \qquad X^\gamma_{(0)} = \frac{\partial \mathcal{W}(x^\mu_{(0)}, P^{(0)}_\alpha)}{\partial P^{(0)}_\gamma}$$

Canonical transformation

$$\begin{aligned}
\mu \frac{dP_\gamma}{d\tau} &= -\left(\frac{\partial x^\sigma}{\partial X^\gamma}\frac{\partial H(x^\mu, P_\alpha)}{\partial x^\sigma}\right)_{P_\alpha} \\
&= -\left(\frac{\partial P^{(0)}_\gamma}{\partial p^{(0)}_\sigma}\right)_{x_\mu}\left(\frac{\partial H_{\rm int}(x^\mu, P^{(0)}_\alpha)}{\partial x^\sigma}\right)_{P^{(0)}_\alpha} + O\left(\frac{\mu^2}{M^2}\right)
\end{aligned}$$

Rewrite them in the more familiar form.:

$$\mu \frac{dE}{d\tau} = \frac{\partial H_{\rm int}}{\partial t} \qquad \mu \frac{dL_z}{d\tau} = -\frac{\partial H_{\rm int}}{\partial \phi} \qquad \mu \frac{dQ}{d\tau} = -2K^{\rho\sigma} p^{(0)}_\rho \frac{\partial H_{\rm int}}{\partial x^\sigma}$$

Mode decomposition of the metric perturbation:

$$h_{\mu\nu}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} d\omega \sum_{m=-\infty}^{+\infty} \tilde{h}^{m}_{\mu\nu}(\omega; r, \theta) e^{-i(\omega t + m\phi)}$$

Symmetry of the Kerr space time admits at least the separation of variables (t, ϕ)

Gauge transformation

$$\delta_\xi \tilde{H}^m_{\rm int} = -\mu^2 \frac{D}{d\tau}\left([\tilde{\xi}^\alpha u_\alpha]^m\right)$$

We find that $\tilde{H}^m_{\rm int}$ is **"gauge invariant"** if

✔ the gauge vector is **physically reasonable**.

Barack and Sago 1101.3331

$$\tilde{\xi}^m_\alpha[z(\lambda)] = \tilde{\xi}^m_\alpha[z(\lambda+\Lambda)]$$

✔ and take **the average over one period**

Comment 1.

✔ The **averaged** change rate of the constants of motion are also “gauge invariant”.

✔ $\tilde{\xi}^{m}_{\alpha}$ must be regular at the particle location.

Long time average = **average over one period**

$$\lim_{t\to\infty}\frac{1}{2T}\int_{-T}^{T}d\lambda\cdots\Leftrightarrow\frac{1}{\Lambda}\int_{0}^{\Lambda}d\lambda\ldots$$

For example, the Carter constant becomes

$$\mu\frac{dQ}{d\tau}^{(R)}=-2K^{\rho\sigma}p_{\rho}^{(0)}\frac{\partial H_{\rm int}}{\partial x^{\sigma}}^{(R)}$$

$$\mu\left\langle\frac{dQ}{d\lambda}\right\rangle^{(R)}=\frac{1}{\Lambda}\int_{0}^{\Lambda}d\lambda\left\{\left(V_{tr}(r)\partial_t+V_{\phi r}(r)\partial_\phi+\frac{dr(\lambda)}{d\lambda}\partial_r\right)\left[\Sigma(x)H_{\rm int}^{(R)}(x)\right]\right\}_{x=z(\lambda)}$$

$$\frac{dt}{d\lambda}=V_{t\theta}(\theta)+V_{tr}(r)\qquad\frac{d\phi}{d\lambda}=V_{\phi\theta}(\theta)+V_{\phi r}(r)$$ **Kerr geodesic equation**

Decompose the R-part integrands into the **radiative** and **symmetric** via the “Green functions”

$$H_{\mathrm{int}}(x) = \int_{-\infty}^{\infty} d\lambda' \Sigma[z(\lambda')]\, I(x,x')_{x'=z(\lambda')}$$

$$I(x,x') := G_{\alpha\beta\gamma'\delta'}(x,x') p^{\alpha}_{(0)}(x) p^{\beta}_{(0)}(x) p^{\gamma'}_{(0)}(x') p^{\delta'}_{(0)}(x')$$

Radiative part

$$I^{(\mathrm{rad})}(x,x') := \frac{1}{2}\left[I^{(\mathrm{ret})}(x,x') - I^{(\mathrm{adv})}(x,x')\right]$$

Symmetric (and S-part)

$$I^{(\mathrm{sym})}(x,x') - I^{(S)}(x,x') := \frac{1}{2}\left[I^{(\mathrm{ret})}(x,x') + I^{(\mathrm{adv})}(x,x')\right] - I^{(S)}(x,x')$$

In the rest of the talk, I will limit my attention to discuss the **symmetric (minus-S) part** only.

Radiative part is given our preparing paper or Falanagan,Hughes and Ruangsri 1208.3906

The translation invariance in the Killing direction

$$F(t,\phi;t'\phi') = F(t-t',\phi-\phi') \quad \Longrightarrow \quad \begin{aligned} \partial_t I^{(\mathrm{sym})}(x,x') &= -\partial_{t'} I^{(\mathrm{sym})}(x,x') \\ \partial_\phi I^{(\mathrm{sym})}(x,x') &= -\partial_{\phi'} I^{(\mathrm{sym})}(x,x') \end{aligned}$$

The symmetric (minus S-)part is simplified as,

$$\left\langle \frac{dQ}{d\lambda} \right\rangle^{(\mathrm{sym}-S)} = -\frac{1}{2}\frac{d}{d\Delta\lambda}\Phi(\Delta\lambda)^{(\mathrm{sym}-S)}$$

with **the potential function** (= averaged $H_{\mathrm{int}}^{(\mathrm{sym}-S)}[z(\lambda)]$)

$$\Phi(\Delta\lambda)^{(\mathrm{sym}-S)} := \frac{1}{\Lambda}\int_0^\lambda \Sigma[z(\lambda)] \int_{-\infty}^{\infty} d\lambda' \Sigma[z(\lambda')] I^{(\mathrm{sym}-S)}(z(\lambda),z(\lambda'))$$

Comment 2.

✔The symmetric (and S-part) "Green function" diverges at the coincidence limit.

$$I^{(\mathrm{sym})}(z(\lambda), z(\lambda')) \Rightarrow I^{(\mathrm{sym})}(z_+(\lambda), z_-(\lambda'))$$

Point splitting regularization into Killing direction

$$z^\mu_\pm(\lambda) := z^\mu(\lambda) \pm \frac{\epsilon}{2}\xi^\mu(\zeta) \qquad \xi^\mu(\zeta) := \cos\zeta\, \xi^\mu_{(t)} + (\Omega_\phi \cos\zeta - \Omega \sin\zeta)\, \xi^\mu_{(\phi)}$$

The potential function admits **the (m,N)-mode decomposition** with Teukolsky formalism.

$$\Phi(\Delta\lambda)^{(\mathrm{sym}-S)} = \lim_{\epsilon\to 0} \sum_{N=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} e^{-i\Omega(\epsilon_1 N+\epsilon_2 m)} \left[\Phi^{(\mathrm{sym})}_{mN}(\Delta\lambda) - \Phi^{(S)}_{mN}(\Delta\lambda)\right]$$

$$(\epsilon_1, \epsilon_2) := (\epsilon\cos\zeta, \epsilon\sin\zeta)$$

The frequency spectrum for the bounded geodesics is discretized.

Drasco and Hughes 0308479

$$\omega_{mN} := m\Omega_\phi + N\Omega$$

Common resonance frequency

Work in the **(half-ingoing) radiation gauge**

Chrzanowski Phys. Rev. D11 2042 (1975)

Wald Phys.Rev.Lett. 41 203 (1978)

$$\ell^\mu h^{(\mathrm{IRG})}_{\mu\nu} = g^{\mu\nu} h^{(\mathrm{IRG})}_{\mu\nu} = 0$$

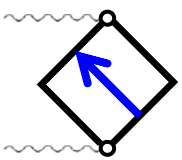
Project the potential function onto the Teukolsky variables (with s=2)

$$H_{\mathrm{int}}(x) := -\frac{1}{2} h^{\mu\nu(R)}(x) p^{(0)}_\mu p^{(0)}_\nu$$

$$\begin{aligned}
\Phi(\Delta\lambda)^{(\mathrm{sym}-S)} &= \frac{\mu^2}{2}\int d^4x\sqrt{-g_0}\, h^{(\mathrm{sym}-S)}_{\alpha\beta}(x) T^{\alpha\beta}[x; z(\lambda)] \\
&= \frac{\mu^2}{2\Lambda}\int_0^\Lambda d\lambda \Sigma[z_+(\lambda)]\, {}_2\mathcal{T}[z_+(\lambda)]\mathcal{D}_0^{-4}\left\{ {}_2\Psi^{(\mathrm{sym})}[z_+(\lambda)] - {}_2\Psi^{(S)}[z_+(\lambda)]\right\}
\end{aligned}$$

$$\mathcal{D}_0^{-1} := (\ell^\mu \partial_\mu)^{-1}$$

Integration in the (in)going null direction



(m,N) mode decomposition

heuristic argument

$\Phi^{(\mathrm{sym})}_{mN}(\Delta\lambda)$ is finite even at the particle location.

$$\sum_l \Phi(\Delta\lambda)^{(\mathrm{sym}-S)}_{lmN} \approx \lim_{x\to z_+(\lambda)} \sum_l \int_0^\Lambda d\lambda \Sigma(x)\, {}_2\mathcal{T}_{lmN}(x) (\mathcal{D}_0^{-4})_{lmN} \left\{ {}_2\Psi^{(\mathrm{sym})}_{lmN}[x, z_-(\lambda)] \right\}$$
$$\approx \sum_l O\left(\frac{1}{l^2}\right)$$

$$\approx (\partial_r)^{-1}$$

We can treat the symmetric and S-part separately at the (m,N) mode level.

(m,N) mode decomposition

heuristic argument

Derive the S-part in the Lorentz gauge at first.

$$\Phi(\Delta\lambda;\epsilon,\zeta)^{(S-\mathrm{Lor})} \approx \int d^4x\sqrt{-g_0}\, h_{\alpha\beta}^{(S-\mathrm{Lor})}(x)T^{\alpha\beta}[x;z_+(\lambda)]$$
$$= -\frac{f(\zeta)}{\epsilon} + O(\epsilon)$$

Decompose it via **inverse Fourier transformation**.

$$\Phi_{mN}^{(S-\mathrm{Lor})}(\Delta\lambda) = \frac{\Omega^2}{4\pi^2}\int_{-\frac{\pi}{\Omega}}^{\frac{\pi}{\Omega}} d\epsilon_1 \int_{-\frac{\pi}{\Omega}}^{\frac{\pi}{\Omega}} d\epsilon_2 e^{i\Omega(\epsilon_1 N+\epsilon_2 m)}\Phi^{(S-\mathrm{Lor})}(\Delta\lambda)$$

$\Phi_{mN}^{(S-\mathrm{Lor})}(\Delta\lambda)$ is also finite at the particle location.

Intermediate gauge approach

heuristic argument

Formal gauge transformation at (m,N)-modes

$$\Phi(\Delta\lambda)^{(\mathrm{sym}-S)} = \lim_{\epsilon\to 0}\sum_{N=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} e^{-i\Omega(\epsilon_1 N+\epsilon_2 m)} \times\left\{\Phi_{mN}[h^{(\mathrm{sym}-\mathrm{IRG})}] - \Phi_{mN}\left[h^{(S-\mathrm{Lor})} - 2\nabla\xi^{(\mathrm{Lor}\to\mathrm{IRG})}[h^{(S-\mathrm{Lor})}]\right]\right\}$$

Contribution from gauge transformation vanishes

$$\Rightarrow\quad \Phi(\Delta\lambda)^{(\mathrm{sym}-S)} = \sum_{N=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}\left[\Phi_{mN}^{(\mathrm{sym})}(\Delta\lambda) - \Phi_{mN}^{(S)}(\Delta\lambda)\right]$$

S-part can be subtracted mode-by–mode thanks to the **“gauge invariance”** at (m,N)-modes level.

Comment 3.

✔ $\xi_\mu^{(\mathrm{Lor}\to\mathrm{IRG})}$ itself has string-like singularity begins at the particle, and diverges logarithmically.

(m,N) mode decomposition is crucial.

✔ $l = 0, 1$ modes on the metric perturbation give rise only the phase errors that scales as $O\left((\mu/M)^0\right)$

Irrelevant here.

终

(Fin.)

The leading orbital evolution in the adiabatic regime:

Tanaka 0508144

Hinderer and Falanagan 0805.3337

1) $P^i_{(0)} := \{E_{(0)}, L_{z(0)}, Q_{(0)}\}$ $\Rightarrow$ $P^i(\lambda) := \{E(\lambda), L_z(\lambda), Q(\lambda)\}$

Long time average.

Common resonance frequency in Mino time

$$\frac{\Upsilon_r}{j_r} = \frac{\Upsilon_\theta}{j_\theta} := \Upsilon$$

2) $$\frac{dP^i}{d\lambda} = \underbrace{\left\langle \frac{dP^i}{d\lambda} \right\rangle_{N=0}} + \underbrace{\sum_{N\neq 0} I^i_N \left[P^j_{(0)}\right] e^{iN\Upsilon\lambda}}$$

Accumulate for long time

Not accumulate

The averaged value dominates the whole evolution.

Tricks for the simplification:

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}d\lambda\cdots=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}d\lambda_\theta\int_{-\infty}^{+\infty}d\lambda_r\,\delta(\lambda_r-\lambda_\theta+\Delta\lambda)\ldots$$

Eliminate the r-derivative terms with the identity that holds at the particle's location.

$$\frac{d}{d\lambda_r}F\left(\bar z(\lambda_r,\lambda_\theta)\right)=\left[\left(\boxed{\frac{dr(\lambda_r)}{d\lambda_r}\partial_r}+\frac{d\Delta t_r(\lambda_r)}{d\lambda_r}\partial_t+\frac{d\Delta\phi_r(\lambda_r)}{d\lambda_r}\partial_\phi\right)F(x)\right]_{x=\bar z(\lambda_r,\lambda_\theta)}$$

$$\frac{d\Delta t_r(\lambda_r)}{d\lambda_r}=V_{tr}[r(\lambda_r)]-\langle V_{tr}\rangle$$

r-oscillatory part of the motion in the t-direction

After integrating by parts, the **radiative** parts becomes

In Takahiro's talk

Sago et al. 0506092

Falanagan,Hughes and Ruangsri 1208.3906

$$\left\langle \frac{dQ}{d\lambda} \right\rangle^{(\mathrm{rad})} = \frac{-\mu^2}{\Lambda} \int_0^\Lambda d\lambda \Sigma[z(\lambda)] \int_{-\infty}^{\infty} d\lambda' \Sigma[z(\lambda')] \left[\left(\langle V_{tr} \rangle \partial_t + \langle V_{\phi r} \rangle \partial_\phi \right) I^{(\mathrm{rad})}(x, z(\lambda')) \right]_{x=z(\lambda)}$$
$$+ \frac{\mu^2}{\Lambda} \int_0^\Lambda d\lambda \int_{-\infty}^{\infty} d\lambda' \Sigma(x') \left[\frac{d}{d(\Delta\lambda)} \left\{ \Sigma[z(\lambda)] I^{(\mathrm{rad})}(z(\lambda), x') \right\} \right]_{x'=z(\lambda')}$$

$\Downarrow$ **Teukolsky formalism**

$$\left\langle \frac{dQ}{dt} \right\rangle = 2 \left\langle \frac{(r^2+a^2)P(r)}{\Delta} \right\rangle \left\langle \frac{dE}{dt} \right\rangle - 2 \left\langle \frac{aP(r)}{\Delta} \right\rangle \left\langle \frac{dL}{dt} \right\rangle + 2 \underline{\sum_{l,m,n_r,n_\theta} \frac{n_r \Omega_r}{\omega} \left| A_{l,m,n_r,n_\theta} \right|^2}$$

$$2 \sum_{l,m,N} \frac{\Omega_r}{\omega} A_{l,m,N} \overline{B_{l,m,N}}$$

Amplitude of the partial waves

$$A_{l,m,N} = \sum_{n_r \Omega_r + n_\theta \Omega_\theta = N\Omega} A_{l,m,n_r,n_\theta} \qquad B_{l,m,N} = \sum_{n_r \Omega_r + n_\theta \Omega_\theta = N\Omega} n_r A_{l,m,n_r,n_\theta} \qquad \Omega \equiv \frac{\Omega_r}{j_r} = \frac{\Omega_\theta}{j_\theta}$$

Sum for the same frequency is to be taken first.

Harmonic structure

[Schmidt(2002),Drasco+ (2004), Fujita+ (2009)]

- Both radial and polar motions are **periodic**

$$r(\lambda) = r(\lambda + 2\pi\Upsilon_r^{-1}) \qquad \theta(\lambda) = \theta(\lambda + 2\pi\Upsilon_\theta^{-1})$$

- Time and axial motion are linear + **doubly-periodic**

$$t(\lambda) = t_0 + \Upsilon_t\lambda + \boxed{\Delta t_r(\lambda)} + \boxed{\Delta t_\theta(\lambda)}$$
$$\phi(\lambda) = \phi_0 + \Upsilon_\phi\lambda + \boxed{\Delta\phi_r(\lambda)} + \boxed{\Delta\phi_\theta(\lambda)}$$

r-oscillation θ-oscillation

- Only **three parameters** are needed in general.

✔ Reparameterization. $\lambda'_\theta = \lambda_{\theta 0} + \frac{j_\theta \Lambda_\theta}{2}$ $\lambda'_r = \lambda_{r0} + \frac{j_r \Lambda_r}{2}$

$$\{E, L_z, Q, \underbrace{t_0, \phi_0}_{\text{Translation sym.}}, \underbrace{\lambda_{r0}, \lambda_{\theta 0}}_{\text{Periodicity.}}\} \qquad \{E, L_z, Q\}$$

$|\lambda'_\theta - \lambda'_r| \le \epsilon$

Evolution of the resonant orbit

[Flanagan and Hinderer (2010), SI+ (2012)]

$\Delta\lambda$ also evolves during resonance with const. of motion.

$$\Delta\Upsilon := j_\theta \Upsilon_r - j_r \Upsilon_\theta \qquad \Upsilon' = \Upsilon/(j_r j_\theta)$$

Master equation

$$\boxed{\frac{d^2\Delta\lambda}{d\lambda^2} = \frac{\partial(\Upsilon'^{-1}\Delta\Upsilon)}{\partial I^j}\left\langle \frac{dI^i(\Delta\lambda)}{d\lambda}\right\rangle := H(\Delta\lambda, I^i[\Delta\lambda])}$$

1) Transient resonance $H(\Delta\lambda) \neq 0$

$\Rightarrow$ $\Delta\lambda$ varies slowly and leaves the resonance.

2) Sustained resonance $H(\Delta\lambda_c) = 0$

$$\frac{d^2\Delta\lambda}{d\lambda^2} = \underline{-\varpi^2(I^i)(\Delta\lambda - \Delta\lambda_c(I^i))}$$

Harmonic oscillator

$\Rightarrow$ Resonance can last for whole adiabatic regime.