A note on the lacking polynomial of the complete bipartite graph

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Abstract. The lacking polynomial is a graph polynomial introduced by Chan, Marckert, and Selig in 2013 that is closely related to the Tutte polynomial of a graph. It arose by way of a generalization of the Abelian sandpile model and is essentially the generating function of the level statistic on the set of recurrent configurations, called stochastically recurrent states, for that model. In this note we consider the lacking polynomial of the complete bipartite graph. We classify the stochastically recurrent states of the stochastic sandpile model on the complete bipartite graphs $K_{2,n}$ and $K_{m,2}$ where the sink is always an element of the set counted by the first index. We use these characterizations to give explicit formulae for the lacking polynomials of these graphs. Log-concavity of the sequence of coefficients of these two lacking polynomials is proven, and we conjecture log-concavity holds for this general class of graphs.

The stochastic sandpile model is a variant of the Abelian sandpile model [3] that was introduced by Chan, Marckert and Selig [4] in 2013. In the stochastic sandpile model the classical sandpile toppling rule is replaced with an alternative rule whereby, on toppling an unstable vertex, a grain may (but does not have to) be sent to each neighbouring vertex. It can be viewed as a Markov chain on the set of non-negative configurations where at each time step a grain is added to a random node followed by a complete stochastic stabilization.

One consequence of this is that the toppling of a vertex does not necessarily result in it becoming stable. As part of their paper the authors introduced a notion of stochastically recurrent states as the analog of recurrent states for the classical model. They provided a characterization for such states in terms of *orientations compatible with configurations*. Further research into this model can be found in the papers [8, 9].

Let G be a finite, unoriented, connected and loop-free graph with a distinguished vertex, s, called the sink. A *stable configuration* on G is an assignment of non-negative integers to each non-sink vertex such that the number of grains at a given vertex is less than its degree. First, we will recall a definition from Chan et al. [4] which explains what it means for an orientation to be compatible with a configuration.

Definition 1 (Chan et al. [4]). Let c be a configuration on G. An orientation $\mathcal{O}$ of G is an assignment of a direction to each of the edges of G. We say that configuration c *is compatible with* $\mathcal{O}$ (and likewise $\mathcal{O}$ *is compatible with* c) if for all non-sink vertices v in G,

$$\mathrm{in}_{\mathcal{O}}(v) \;\geq\; d(v) - c_i,$$

where $\mathrm{in}_{\mathcal{O}}(v)$ is the number of incoming edges to v w.r.t. $\mathcal{O}$ and $d(v)$ is the degree of vertex v.

We denote by $\mathrm{comp}(\mathcal{O})$ the set of stable configurations on G that are compatible with $\mathcal{O}$. Note that, in comparison to the paper [4], the inequality in Definition 1 is missing one on the right hand side. This is because of a subtle change to the model. In [4]

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they considered a vertex unstable if the number of grains at a vertex exceeds its degree, whereas we consider a vertex unstable if the number of grains at vertex is not less than the degree which is in line with the definition of the ASM in [5]. The two notions are equivalent.

Theorem 2 (Chan et al. [4])**.** A stable configuration c on G is *stochastically recurrent* if and only if there exists an orientation $\mathcal{O}$ on G such that $c \in \mathrm{comp}(\mathcal{O})$. It implies that $\mathsf{Sto}(G)$, the set of all stochastically recurrent states on G, may be written

$$\mathsf{Sto}(G) = \bigcup_{\substack{\text{orientations } \mathcal{O} \\ \text{of } G}} \mathrm{comp}(\mathcal{O})$$

where the union is taken over all orientations on G.

Chan et al. [4] also introduced the lacking polynomial of a graph to be the generating function counting the stochastically recurrent configurations according to the number of grains by which a given configuration differs from the maximally stable configuration.

Definition 3 (Chan et al. [4])**.** The lacking polynomial $L_G(x)$ is

$$L_G(x) := \sum_{c \in \mathsf{Sto}(G)} x^{\ell(c)},$$

where

$$\ell(c) := \sum_{v \in V(G)\setminus\{s\}} l(v)$$

and $l(v) := d(v) - c(v) - 1$ is the *lacking number* of vertex v.

Readers familiar with the sandpile model literature will notice that the lacking polynomial is essentially the level polynomial (see e.g. Cori and Le Borgne [2]) of the graph over the set of stochastically recurrent states with the sequence of coefficients reversed.

In this note we consider the stochastic sandpile model on the complete bipartite graph $K_{m,n}$ with vertex set $\{v_0, v_1, \ldots, v_{m+n-1}\}$. We will treat v_0 as the sink s and this graph has edges connecting vertices in the sets $\{v_0, v_1, v_2, \ldots, v_{m-1}\}$ and $\{v_m, v_{m+1}, \ldots, v_{m+n-1}\}$.

We characterise stochastically recurrent states on the graphs $K_{2,n}$ and $K_{m,2}$ and use these characterisations to give expressions for the lacking polynomials $L_{2,n}(x)$ and $L_{m,2}(x)$ on those graphs. We also prove that the sequence of coefficients of both $L_{2,n}(x)$ and $L_{m,2}(x)$ are log-concave. This note is motivated by Alofi and Dukes [1] that considers rectangular tableaux representations of recurrent states of the Abelian sandpile model on the complete bipartite graph, and transformations upon them.

Theorem 2 allows us to write an expression for stochastically recurrent states on the complete bipartite graph$K_{m,n}$:

Proposition 4. The set of stochastically recurrent states of the stochastic sandpile model on $K_{m,n}$ is

$$\mathsf{Sto}(K_{m,n}) = \bigcup_{\substack{\text{orientations } \mathcal{O} \\ \text{of } K_{m,n}}} \{(c_1, \ldots, c_{m+n-1}) \ : \ \mathrm{out}_{\mathcal{O}}(v_i) \leq c_i < d(v_i), \ \forall 1 \leq i \leq m+n-1\}.$$

Proof. Let $c = (c_1, c_2, \ldots, c_{m+n-1})$ be a stable configuration on $K_{m,n}$. Suppose it is compatible with an orientation $\mathcal{O}$ where $c_i < d(v_i)$ for all $1 \leq i \leq m+n-1$. According to the Definition 1 we must have:

$$\mathrm{in}_{\mathcal{O}}(v_i) \geq d(v_i) - c_i.$$

This means $c_i \geq d(v_i) - \text{in}_{\mathcal{O}}(v_i) = \text{out}_{\mathcal{O}}(v_i)$. When we combine the application of Definition 1 with Theorem 2 for $G = K_{m,n}$ we find

$$\mathsf{Sto}(K_{m,n}) = \bigcup_{\substack{\text{orientations } \mathcal{O} \\ \text{of } K_{m,n}}} \{(c_1, \ldots, c_{m+n-1}) \ : \ \text{out}_{\mathcal{O}}(v_i) \leq c_i < d(v_i), \ \forall 1 \leq i \leq m+n-1\}.$$

□

Example 5. Consider the graph $K_{2,2}$. To determine the stochastically recurrent states compatible with each orientation $\mathcal{O}$ of graph $K_{2,2}$ first we find $\text{out}_{\mathcal{O}}(v_i)$ for all $1 \leq i \leq 3$, and then when we apply Prop. 4. See Table 1 for a listing of the configurations that are compatible with each orientation. It follows that

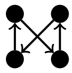
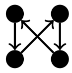
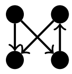
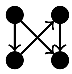
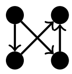
v_0 v_1  v_2 v_3	$\text{out}_{\mathcal{O}}(v_1) = 1, \text{out}_{\mathcal{O}}(v_2) = 1, \text{out}_{\mathcal{O}}(v_3) = 1.$ $\implies 1 \leq c_1 < 2$, $1 \leq c_2 < 2$, and $1 \leq c_3 < 2$. The contribution to $\mathsf{Sto}(K_{2,2})$ is $\{(1,1,1)\}$.
v_0 v_1  v_2 v_3	$\text{out}_{\mathcal{O}}(v_1) = 1, \text{out}_{\mathcal{O}}(v_2) = 1, \text{out}_{\mathcal{O}}(v_3) = 1.$ $\implies 1 \leq c_1 < 2$, $1 \leq c_2 < 2$, and $1 \leq c_3 < 2$. The contribution to $\mathsf{Sto}(K_{2,2})$ is $\{(1,1,1)\}$.
v_0 v_1  v_2 v_3	$\text{out}_{\mathcal{O}}(v_1) = 1, \text{out}_{\mathcal{O}}(v_2) = 0, \text{out}_{\mathcal{O}}(v_3) = 1.$ $\implies 1 \leq c_1 < 2$, $0 \leq c_2 < 2$, and $1 \leq c_3 < 2$. The contribution to $\mathsf{Sto}(K_{2,2})$ is $\{(1,0,1),(1,1,1)\}$.
v_0 v_1  v_2 v_3	$\text{out}_{\mathcal{O}}(v_1) = 1, \text{out}_{\mathcal{O}}(v_2) = 1, \text{out}_{\mathcal{O}}(v_3) = 0.$ $\implies 1 \leq c_1 < 2$, $1 \leq c_2 < 2$, and $0 \leq c_3 < 2$. The contribution to $\mathsf{Sto}(K_{2,2})$ is $\{(1,1,0),(1,1,1)\}$.
v_0 v_1  v_2 v_3	$\text{out}_{\mathcal{O}}(v_1) = 0, \text{out}_{\mathcal{O}}(v_2) = 1, \text{out}_{\mathcal{O}}(v_3) = 1.$ $\implies 0 \leq c_1 < 2$, $1 \leq c_2 < 2$, and $1 \leq c_3 < 2$. The contribution to $\mathsf{Sto}(K_{2,2})$ is $\{(0,1,1),(1,1,1)\}$.

TABLE 1. Checking all orientations of $K_{2,2}$.

$$\mathsf{Sto}(K_{2,2}) = \{(0,1,1),(1,0,1),(1,1,0),(1,1,1)\}.$$

We can provide crude lower and upper bounds on the number of stochastically recurrent configurations by making use of the fact [4, Prop. 2.3] that $\mathsf{Sto}(K_{m,n})$ properly contains the set of classically recurrent states, and such states are in 1-1 correspondence with the number of spanning trees of the underlying graph. Fieldler and Sedlacek [6] showed the number of spanning trees of the complete bipartite graph $K_{m,n}$ is $n^{m-1}m^{n-1}$. Thus the number of stochastically recurrent configurations on $K_{m,n}$ is at least $n^{m-1}m^{n-1}$. A trivial upper bound is achieved by noting that the stochastically recurrent states are stable configurations, of which there are $n^{m-1}m^n$ many. Thus

$$n^{m-1}m^{n-1} \ \leq \ |\mathsf{Sto}(K_{m,n})| \ \leq \ n^{m-1}m^n. \tag{1}$$

Note that the upper bound differs from the lower bound only by a factor of m.

Question 6. Can it be determined whether or not the number of stochastically recurrent states dominates the set of stable states? I.e. can it be decided

$$|\mathsf{Sto}(K_{m,n})| \lesseqgtr \frac{n^{m-1}m^n}{2}?$$

The set of stable configurations on $K_{2,n}$ is

$$\{(c_1, c_2, \ldots, c_{n+1}) \ : \ 0 \leq c_1 < n \text{ and } c_i \in \{0, 1\}, \ \forall\ 2 \leq i \leq n+1\}.$$

In order to calculate the lacking polynomial $L_{2,n}(x)$ we first need an explicit characterization of the set $\mathsf{Sto}(K_{2,n})$.

Proposition 7.

$$\mathsf{Sto}(K_{2,n}) = \{(c_1, c_2, \ldots, c_{n+1}) : c_2, \ldots, c_{n+1} \in \{0, 1\} \text{ and } c_1 \geq |j \in [2, n+1] : c_j = 0|\}.$$

Proof. Using Proposition 4 we know

$$\mathsf{Sto}(K_{2,n}) = \{c = (c_1, \ldots, c_{n+1}) \ : \ \exists \text{ an orientation } \mathcal{O} \text{ of } K_{2,n} \text{ compatible with } c\}.$$

Let us suppose $c = (c_1, \ldots, c_{n+1})$ is a member of $\mathsf{Sto}(K_{2,n})$. This means

$$n - \text{in}_{\mathcal{O}}(v_1) = \text{out}_{\mathcal{O}}(v_1) \leq c_1 < n$$

and

$$2 - \text{in}_{\mathcal{O}}(v_j) = \text{out}_{\mathcal{O}}(v_j) \leq c_j < 2 \text{ for all } 2 \leq j \leq n+1.$$

Let us now see under what conditions one can construct an orientation $\mathcal{O}$ on $K_{2,n}$ that is compatible with a given c. Let X be the set of indices j in $[2, n+1]$ for which $c_j = 0$, and where we use the notation $[a, b] := \{a, a+1, \ldots, b\}$. For any j in X the number of outgoing edges at vertex v_j is zero ($\text{out}_{\mathcal{O}}(v_j) = 0$), because we know that $\text{out}_{\mathcal{O}}(v_j) \leq c_j < 2$, so if $c_j = 0$ then $\text{out}_{\mathcal{O}}(v_j) = 0$. Therefore, for all j in X there is one outgoing edge from v_1 to v_j, and hence vertex v_j has one incoming edge from v_1. So the number of outgoing edges from v_1 is greater than or equal to the number of elements in X.

For any j in $[2, n+1]\backslash X$, we know that there is at most one outgoing edge from v_j as $\text{out}_{\mathcal{O}}(v_j) \leq c_j < 2$. So if $c_j = 1$ then $\text{out}_{\mathcal{O}}(v_j) \leq 1$. With these considerations in mind, we can construct an orientation $\mathcal{O}$ on $K_{2,n}$ that is compatible with a stable configuration c:

(i) If $c_j = 1$ with $2 \leq j \leq n+1$ then there is at most one outgoing edge from v_j.
(ii) If $c_j = 0$ and $2 \leq j \leq n+1$ then there are no outgoing edges from v_j.
(iii) The number of outgoing edges from vertex v_1 is greater than or equal the number of vertices v_j when $c_j = 0$ for all $2 \leq j \leq n+1$. We know that the $\text{out}_{\mathcal{O}}(v_1)$ is less than or equal to c_1 and greater than or equal to the number of vertices v_j when $c_j = 0$ for all $2 \leq j \leq n+1$. Therefore c_1 is greater than or equal to the number of vertices v_j when $c_j = 0$ for all $2 \leq j \leq n+1$.

There are no other restrictions that forbid us from constructing such an orientation $\mathcal{O}$. Therefore we can write down the following self-contained expression for the set $\mathsf{Sto}(K_{2,n})$ that does not depend on an orientation $\mathcal{O}$:

$$\mathsf{Sto}(K_{2,n}) = \{(c_1, c_2, \ldots, c_{n+1}) : c_2, \ldots, c_{n+1} \in \{0, 1\} \text{ and } c_1 \geq |j \in [2, n+1] : c_j = 0|\}.$$

□

Theorem 8. The lacking polynomial of the graph $K_{2,n}$ is

$$L_{2,n}(x) = \sum_{k=0}^{n-1} \sum_{i=0}^{k} \binom{n}{i} x^k.$$

Proof. Definition 3 gives

$$L_{2,n}(x) = \sum_{c \in \mathsf{Sto}(K_{2,n})} x^{\ell(c)}$$

where

$$\ell(c) := \sum_{v_i \in V(K_{2,n}) \setminus \{v_0\}} l(v_i) \qquad \text{and} \qquad l(v_i) := d(v_i) - c(v_i) - 1.$$

Proposition 7 provides an explicit expression for the set $\mathsf{Sto}(K_{2,n})$ that can be used to calculate the lacking polynomial. Let c be in $\mathsf{Sto}(K_{2,n})$. Let $l_1 = n - c_1 - 1$ be the lacking number at vertex v_1, and let $l_j = 1 - c_j$ be the lacking number at vertices $v_2, \ldots, v_{n+1}$ for all $j \in [2, n+1]$, so $l_j = 0$ when $c_j = 1$ and $l_j = 1$ when $c_j = 0$. Let i be the number of vertices v_j with $j \in [2, n+1]$ and $c_j = 0$ for which $l_j = 1$. The remaining vertices have $l_j = 0$. Then $x^{\ell(c)}$ factors as $x^{\ell(c)} = x^i x^{l_1}$. Since $c_1 \geq |j \in [2, n+1] : c_j = 0|$, we conclude that $c_1 \geq i$, therefore the lacking number at v_1 will be between 0 and $n - 1 - i$. Since there are $\binom{n}{i}$ combinations of i vertices with lacking number 1 among $\{v_2, \ldots, v_{n+1}\}$, we obtain

$$L_{2,n}(x) = \sum_{i=0}^{n} \binom{n}{i} x^i \sum_{l_1=0}^{n-1-i} x^{l_1}.$$

Now setting $k = i + l_1$ we have

$$L_{2,n}(x) \;=\; \sum_{k=0}^{n-1} \sum_{i=0}^{k} \binom{n}{i} x^k. \quad \square$$

Note that $L_{2,n}(1) = n2^{n-1}$.

A configuration $c = (c_1, c_2, \ldots, c_{m+1})$ on $K_{m,2}$ is stable precisely when $c_i \in \{0, 1\}$ for all $i \in \{1, \ldots, m-1\}$ and $0 \leq c_j < m$ for all $j \in \{m, m+1\}$. To calculate the lacking polynomial $L_{m,2}(x)$ requires an explicit characterization of the set $\mathsf{Sto}(K_{m,2})$.

Proposition 9.

$$\mathsf{Sto}(K_{m,2}) = \{(c_1, c_2, \ldots, c_{m+1}) \;:\; c_1, \ldots, c_{m-1} \in \{0, 1\} \text{ and } c_m + c_{m+1} \geq m - 1 + |X|\}.$$

Proof. From Proposition 4 we have

$$\mathsf{Sto}(K_{m,2}) = \{c = (c_1, \ldots, c_{m+1}) \;:\; \exists \text{ orientation } \mathcal{O} \text{ of } K_{m,2} \text{ compatible with } c\}.$$

Suppose $c = (c_1, \ldots, c_{m+1})$ is a member of $\mathsf{Sto}(K_{m,2})$. This means that

$$n - \text{in}_{\mathcal{O}}(v_i) = \text{out}_{\mathcal{O}}(v_i) \leq c_i < 2 \text{ for all } i \in \{1, \ldots, m-1\}$$

and

$$2 - \text{in}_{\mathcal{O}}(v_j) = \text{out}_{\mathcal{O}}(v_j) \leq c_j < m \text{ for all } j \in \{m, m+1\}.$$

Let us see under what conditions one can construct an orientation $\mathcal{O}$ on $K_{m,2}$ that is compatible with a given c. Let X be the set of indices i in $[1, m-1]$ for which $c_i = 0$. For any $i \in X$ the number of outgoing edges at vertex v_i is zero ($\text{out}_{\mathcal{O}}(v_i) = 0$) since $\text{out}_{\mathcal{O}}(v_i) \leq c_i < 2$. So if $c_i = 0$ then $\text{out}_{\mathcal{O}}(v_i) = 0$.

Therefore, for all i in X there are is one outgoing edge from each of v_m and v_{m+1} to v_i. Moreover, vertex v_i has one incoming edge from each of v_m and $v_m + 1$. So the number of outgoing edges from v_m is greater than or equal to the number of elements in X. Also, the number of outgoing edges from v_{m+1} is greater than or equal to the number of elements in X. Therefore $c_m \geq |X|$ and $c_{m+1} \geq |X|$.

For any $i \in [1, m-1] \setminus X$, we know that there is at most one outgoing edge from v_i since $\text{out}_{\mathcal{O}}(v_j) \leq c_j < 2$, so if $c_j = 1$ then $\text{out}_{\mathcal{O}}(v_j) \leq 1$. So the total number of outgoing edges from v_m and v_{m+1} to v_i must at least equal $m - 1 - |X|$. Therefore we can then construct such an orientation $\mathcal{O}$ of $K_{m,2}$ that is compatible with a stable configuration c precisely when $c_m + c_{m+1} \geq 2|X| + m - 1 - |X| = m - 1 + |X|$. There are no other restrictions that forbid us from constructing such an orientation $\mathcal{O}$. Therefore we can write down the following self-contained expression for the set $\mathsf{Sto}(K_{m,2})$ that does not depend on an orientation $\mathcal{O}$:

$$\mathsf{Sto}(K_{m,2}) = \{(c_1, c_2, \ldots, c_{m+1}) \ : \ c_1, \ldots, c_{m-1} \in \{0,1\} \text{ and } c_m + c_{m+1} \geq m - 1 + |X|\}.$$

$\square$

Theorem 10. The lacking polynomial for the graph $K_{m,2}$ is

$$L_{m,2}(x) = \sum_{k=0}^{m-1} S(m-1,k)x^k$$

where

$$S(m-1,k) = \sum_{q=0}^{k}\sum_{r=0}^{q}\binom{m-1}{r} \text{ for all } \ 0 \leq k \leq m-1.$$

Proof. The lacking polynomial $L_{m,2}(x)$, given in Definition 3, is

$$L_{m,2}(x) = \sum_{c \in \mathsf{Sto}(K_{m,2})} x^{\ell(c)}$$

where

$$\ell(c) = \sum_{v_i \in V(K_{m,2}) \setminus \{v_0\}} l(v_i), \qquad \text{and} \qquad l(v_i) = d(v_i) - c(v_i) - 1.$$

Proposition 9 provides an explicit expression for the set $\mathsf{Sto}(K_{m,2})$ that can now be used to calculate $L_{m,2}(x)$. Let c be in $\mathsf{Sto}(K_{m,2})$. Let $l_i = 1 - c_i$ be the lacking number of vertex v_i for all $i \in [1, m-1]$, so that $l_i = 0$ when $c_i = 1$ and $l_i = 1$ when $c_i = 0$. For the vertices v_m and v_{m+1} the lacking numbers are $l_m = m - 1 - c_m$ and $l_{m+1} = m - 1 - c_{m+1}$, respectively. We factor $x^{\ell(c)} = x^{|X|}x^{l_m + l_{m+1}}$ and can now write $c_m + c_{m+1} \geq m - 1 + |X|$ in terms of lacking numbers as:

$$m - 1 - l_m + m - 1 - l_{m+1} \geq m - 1 + |X|$$

which is equivalent to

$$m - 1 - |X| \geq l_m + l_{m+1}.$$

Now suppose that $r = l_m + l_{m+1}$, then r ranges in between 0 and $m - 1 - |X|$. For each choice of r we have exactly $r + 1$ choices for (l_m, l_{m+1}). Now let $j = |X|$ be the number of vertices v_i with $i \in [1, m-1]$ and $c_i = 0$ for which then $l_i = 1$. The remaining vertices will have $l_i = 0$. So there are $\binom{m-1}{j}$ combinations of j vertices with lacking number 1 among $\{v_1, \ldots, v_{m-1}\}$, and we obtain

$$L_{m,2}(x) = \sum_{j=0}^{m-1}\binom{m-1}{j}x^j \sum_{r=0}^{m-1-j}(r+1)x^r = \sum_{j=0}^{m-1}\sum_{r=0}^{m-1-j}\binom{m-1}{j}(r+1)x^{r+j}.$$

Set $k = j + r$. Then k runs from 0 to $m - 1$ so that j can run from 0 to k and, with $r = k - j$, we obtain the sum

$$L_{m,2}(x) = \sum_{k=0}^{m-1}\sum_{j=0}^{k}\tbinom{m-1}{j}(k - j + 1)x^k.$$

This is equal to

$$L_{m,2}(x) = \sum_{k=0}^{m-1} S(m-1,k)x^k$$

where

$$S(m-1,k) = \sum_{q=0}^{k}\sum_{r=0}^{q}\binom{m-1}{r} \text{ for all } 0 \le k \le m-1. \quad \square$$

Note that the sequence $(L_{m,2}(1)_{m\ge 1}$ corresponds to sequence A084851 in the OEIS [7].

A sequence $a_0, a_1, \ldots, a_n$ of non-negative real numbers is said to be *logarithmically concave*, or *log-concave*, if for all $0 < k < n$, $a_k^2 - a_{k-1}a_{k+1} \ge 0$.

Lemma 11. Suppose $(x_0, x_1, \ldots, x_n)$ is a log-concave sequence. Then the sequence $(z_0, \ldots, z_n)$ of partial sums defined by

$$z_k = \sum_{i=0}^{k} x_i$$

is also log-concave.

Proof. Wang and Yeh [10] proved that if sequences (x_k) and (y_k) are log-concave, then the sequence (z_k) where z_k is the ordinary convolution

$$z_k = \sum_{i=0}^{k} x_i y_{n-i}$$

is log-concave. The sequence $(y_0, \ldots, y_k) = (1, \ldots, 1)$ is trivially log-concave. The statement of the lemma follows since it is the special case with all y_j's replaced with 1s. $\square$

Theorem 12.

(i) The sequence of coefficients of the lacking polynomial $L_{2,n}(x)$ is log-concave.

(ii) The sequence of coefficients of the lacking polynomial $L_{m,2}(x)$ is log-concave.

Proof. (i) From Theorem 8 we have

$$L_{2,n}(x) = \sum_{k=0}^{n-1} T(n,k)x^k \quad \text{where} \quad T(n,k) := \sum_{i=0}^{k}\binom{n}{i}.$$

It is well-known that the sequence of binomial coefficients $\left(\binom{n}{k}\right)_{k=0,1,2,\ldots,n}$ is log-concave. By Lemma 11, the sequence $(T(n,k))_{k=0,\ldots,n}$ is log-concave. Therefore the sequence of coefficients of the lacking polynomial $L_{2,n}(x)$ is log-concave.

(ii) From Theorem 10 we have

$$L_{m,2}(x) = \sum_{k=0}^{m-1} S(m-1,k)x^k$$

where

$$S(m-1,k) := \sum_{q=0}^{k}\sum_{r=0}^{q}\binom{m-1}{r} \text{ for all } 0 < k \le m-1.$$

Let $R(m-1,q) := \sum_{r=0}^{q}\binom{m-1}{r}$. By Lemma 11 the sequence $(R(m-1,q))_{q=0,\ldots,m-1}$ is log-concave. Then, again by an application of Lemma 11, the sequence $(S(m-1,k))_{k=0,\ldots,m-1}$ $= \left(\sum_{q=0}^{k} R(m-1,q)\right)_{k=0,\ldots,m-1}$ is also log-concave. Therefore the sequence of coefficients of the lacking polynomial $L_{m,2}(x)$ is log-concave. $\square$

$$L_{2,2}(x) = 1 + 3x$$
$$L_{2,3}(x) = 1 + 4x + 7x^2$$
$$L_{2,4}(x) = 1 + 5x + 11x^2 + 15x^3$$
$$L_{2,5}(x) = 1 + 6x + 16x^2 + 26x^3 + 31x^4$$

$$L_{3,2}(x) = 1 + 4x + 8x^2$$
$$L_{3,3}(x) = 1 + 5x + 15x^2 + 30x^3 + 39x^4$$
$$L_{3,4}(x) = 1 + 6x + 21x^2 + 52x^3 + 100x^4 + 148x^5 + 158x^6$$
$$L_{3,5}(x) = 1 + 7x + 28x^2 + 79x^3 + 175x^4 + 320x^5 + 490x^6 + 610x^7 + 585x^8$$

$$L_{4,2}(x) = 1 + 5x + 12x^2 + 20x^3$$
$$L_{4,3}(x) = 1 + 6x + 21x^2 + 53x^3 + 105x^4 + 162x^5 + 189x^6$$
$$L_{4,4}(x) = 1 + 7x + 28x^2 + 84x^3 + 203x^4 + 413x^5 + 716x^6 + 1068x^7 + 1344x^8 + 1336x^9$$

$$L_{5,2}(x) = 1 + 6x + 17x^2 + 32x^3 + 48x^4$$
$$L_{5,3}(x) = 1 + 7x + 28x^2 + 80x^3 + 182x^4 + 347x^5 + 561x^6 + 756x^7 + 837x^8$$

Table 2. Lacking polynomials $L_{m,n}(x)$ for some small m and n.

The two results in Theorem 12 suggest that log-concavity might be a property of these lacking polynomials for the general complete bipartite graph. We have also verified this for all (m, n) with $m, n \geq 2$ and $m+n \leq 8$ (see Table 2). Since log-concavity of a sequence implies unimodality of the sequence, we may also conjecture this latter property in the event that log-cavity does not hold.

Conjecture 13. Let $m, n \geq 2$.

(i) The sequence of coefficients of $L_{m,n}(x)$ is log-concave.
(ii) The sequence of coefficients of $L_{m,n}(x)$ is unimodal.

It is worth mentioning that that more general property of these polynomials being real-rooted, which implies log-concavity, does not hold. This is easily verified by looking at the roots of $L_{2,3}(x) = 0$ or $L_{3,2}(x) = 0$.

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