A NOTE ON A BIJECTION OF ALDRED, ATKINSON, AND MCCAUGHAN

JEAN-CHRISTOPHE AVAL, MARK DUKES, AND YVAN LE BORGNE

Abstract. We present a recursive description of a bijection due to Aldred, Atkinson, and McCaughan. The calculation of this bijection by hand is shown to correspond to traversing the permutation diagram in a particular zig-zag manner. A recursive description of the inverse is also given.

1. Introduction

Let S_n be the set of all permutations of the set $\{1,\dots,n\}$. Given two permutations $\pi = (\pi_1,\dots,\pi_n) \in S_n$ and $\sigma = (\sigma_1,\dots,\sigma_k) \in S_k$, we say that π contains the consecutive pattern σ if there exists a number $1 \leq a \leq n$ such that the sequence $(\pi_a, \pi_{a+1},\dots,\pi_{a+k-1})$ is order isomorphic to σ. If no such subsequence exists then we say that π avoids the consecutive pattern σ.

For example, the permutation $(4,2,6,8,1,3,5,7)$ contains the consecutive pattern $(3,1,2)$ (in the form of $(8,1,3)$) but does not contain either of the consecutive patterns $(1,3,2)$ or $(3,2,1)$. Throughout this paper we will mostly write a permutation $\pi = (\pi_1,\dots,\pi_n) \in S_n$ as the word $\pi_1 \cdots \pi_n$ so that $(1,3,2)$ will be written as 132.

Let Cyc_n be the set of permutations in S_n that have 1 as a fixed point and avoid each of the three consecutive patterns $\{123, 231, 312\}$, and let $\mathsf{Cyc} = \cup_{n\geq 1}\mathsf{Cyc}_n$. Let Alter_n be the set of alternating permutations in S_n that have 1 as a fixed point and define $\mathsf{Alter} = \cup_{n\geq 1}\mathsf{Alter}_n$. Equivalently Alter_n is the set of permutations in S_n that have 1 as a fixed point and avoid both of the consecutive patterns $\{123, 321\}$.

Aldred, Atkinson, and McCaughan [1] presented and proved a bijection between the sets Cyc_n and Alter_n. Their bijection involved embedding the elements of Cyc and Alter into an infinite tree T which has the following property: the sequence of child-degree sequence of nodes at level n of T contains no two equal sequences. Here the child-degree sequence of a node v at level n is the weakly increasing of the down degrees of v's children, denoted $\mathsf{childseq}(v)$.

For example, in Figure 1, $\mathsf{childseq}(1423) = (3,4)$ and $\mathsf{childseq}(1324) = (2,3,4)$. Therefore the sequence of child-degree sequences for level 3 is $((3,4),(2,3,4))$. See Figures 1 and 2 for the embeddings into the top five levels of the tree and Figures 4 and 5 for one further level.

While theirs is a perfectly valid and correct bijection, it remains somewhat difficult to compute the image of a permutation under this correspondence without constructing the tree. This paper solves this construction issue by giving a recursive bijection from Cyc_n to Alter_n and also for its inverse. The calculation of this bijection by hand is shown to correspond to traversing the permutation diagram in a particular zig-zag manner.

2. The bijection f

If $x \leq y$ then let $[x,y] = \{x, x+1,\dots,y\}$. Given a permutation $\pi \in S_n$ and a value $x \in [1, n+1]$, let $\pi \oplus x$ be the permutation $\pi' \in S_{n+1}$ where $\pi'_{n+1} = x$ and for all $1 \leq i \leq n$;

$$\pi'_i = \begin{cases} \pi_i + 1 & \text{if } \pi_i \geq x \\ \pi_i & \text{if } \pi_i < x. \end{cases}$$

For example, $2431 \oplus 1 = 35421$, $2431 \oplus 2 = 35412$, $2431 \oplus 3 = 25413$, and so on. Given $\pi = \pi_1 \dots \pi_n \in S_n$, let us write $\pi = \langle x_1,\dots,x_n\rangle$ where x_i is one plus the number of entries to the left

J.-C. Aval is supported by ANR - PSYCO project (ANR-11-JS02-001).

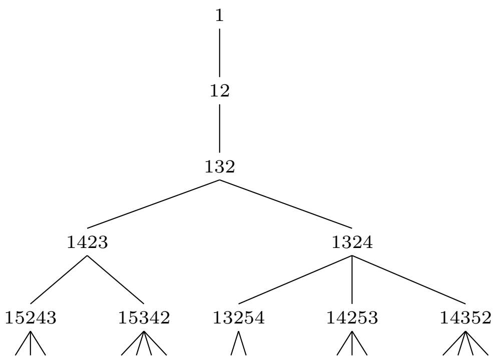

FIGURE 1. The tree showing Alter_i for $i \in \{1, \ldots, 5\}$.

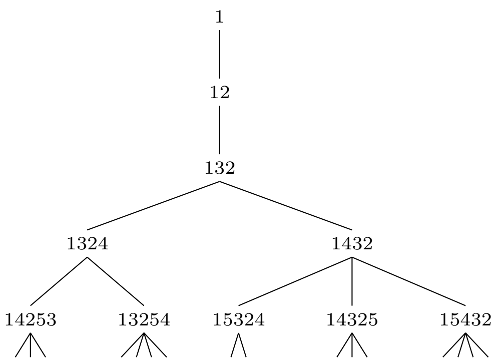

FIGURE 2. The tree showing Cyc_i for $i \in \{1, \ldots, 5\}$.

of π_i in π which are less than π_i. For example, $12345 = \langle 1,2,3,4,5\rangle$ and $54321 = \langle 1,1,1,1,1\rangle$. Using this notation we have

$$S_n = \{\pi = \langle x_1, \ldots, x_n\rangle \ : \ (x_1, \ldots, x_n) \in [1,1] \times [1,2] \times \cdots \times [1,n]\}.$$

The notation $\pi = \langle x_1, \ldots, x_n\rangle$ is equivalent to $\pi = (\cdots((x_1 \oplus x_2) \oplus x_3) \cdots) \oplus x_n$.

The sets Alter_n and Cyc_n have the following characterizations in terms of the angle bracket notation.

Characterization 1. *A permutation $\pi = \langle x_1, \ldots, x_n\rangle \in \mathsf{Alter}_n$ iff $x_1 = 1$, $x_2 = 2$, and for all $2 \leq i < n$:*

$$x_{i+1} \in \begin{cases} [2, x_i] & \textit{for all odd } i+1 \\ [1+x_i, i+1] & \textit{for all even } i+1. \end{cases}$$

Characterization 2. *A permutation $\pi = \langle x_1, \ldots, x_n\rangle \in \mathsf{Cyc}_n$ iff $x_1 = 1$, $x_2 = 2$, and for all $2 \leq i < n$:*

$$x_{i+1} \in \begin{cases} [1+x_{i-1}, x_i] & \textit{if } x_{i-1} < x_i \\ [2, x_i] \cup [2+x_{i-1}, i+1] & \textit{if } x_{i-1} \geq x_i. \end{cases}$$

We will make use of the following simple property of permutations in Cyc_n in our proofs.

Lemma 3. *The permutation $\pi = \langle x_1, \ldots, x_n\rangle \in \mathsf{Cyc}_n$ is such that*

$$\pi \textit{ ends in the consecutive pattern } \begin{cases} 321 & \textit{iff } x_{n-2} \geq x_{n-1} \geq x_n \\ 132 & \textit{iff } x_{n-2} < x_n \leq x_{n-1} \\ 213 & \textit{iff } x_{n-1} < 1 + x_{n-2} < x_n. \end{cases}$$

Proof. It is straightforward to see that the permutation $\pi = \langle x_1, \ldots, x_n\rangle$ ends in the consecutive pattern 12 iff $x_n > x_{n-1}$, and π ends in the consecutive pattern 21 iff $x_n \leq x_{n-1}$. Iterating this observation for the penultimate pair of elements in x, (x_{n-2}, x_{n-1}) and comparing where x_n lies with respect to x_{n-2} yields the result. □

Now we are in a position to state the bijection.

Definition 4. Given $\pi = \langle x_1, \ldots, x_n \rangle \in \mathsf{Cyc}_n$, let $f(\pi) = \langle z_1, \ldots, z_n \rangle$ where $z_1 = 1$ and for all $i \in [2, n]$,

$$z_i = \begin{cases} 1 + x_i - x_{i-1} & \text{if } i \text{ is even and } x_i > x_{i-1} \\ i + x_i - x_{i-1} & \text{if } i \text{ is even and } x_i \leq x_{i-1} \\ i + 1 + x_{i-1} - x_i & \text{if } i \text{ is odd and } x_i > x_{i-1} \\ 2 + x_{i-1} - x_i & \text{if } i \text{ is odd and } x_i \leq x_{i-1}. \end{cases}$$

Theorem 5. $f : \mathsf{Cyc}_n \to \mathsf{Alter}_n$ *is a bijection.*

Proof. The statement of the theorem is true for $n = 2$ and $n = 3$. Let us suppose the claim is true for $n \in [2, m]$ where $m \geq 3$. We will show it to be true for $n = m + 1$. This will be done in three steps. First we will show that $f(\mathsf{Cyc}_n) \subseteq \mathsf{Alter}_n$. Then we will show that f is injective. Finally we will show that f is surjective.

Step 1: Suppose $\pi = \langle x_1, \ldots, x_{m+1} \rangle \in \mathsf{Cyc}_{m+1}$. To show $f(\pi) \in \mathsf{Alter}_{m+1}$ we will have to condition on the parity of m and also on whether $(\pi_{m-1}, \pi_m, \pi_{m+1})$ is order isomorphic to the patterns 132, 321 or 213.

Let $\pi' = \langle x_1, \ldots, x_m \rangle$. It is clear from Characterization 2 that $\pi' \in \mathsf{Cyc}_m$. By the induction hypothesis we also have that $f : \mathsf{Cyc}_m \to \mathsf{Alter}_m$ is a bijection.

$m + 1$ **even and** π **ends in pattern 321:** In this case we have $x_{m-1} \geq x_m \geq x_{m+1}$ by Lemma 3. Using Definition 4 with $i = m + 1$ and $i = m$ we find that

$$z_{m+1} = m + 1 + x_{m+1} - x_m \text{ and } z_m = 2 + x_{m-1} - x_m.$$

Characterization 1 tells us that $f(\pi) \in \mathsf{Alter}_{m+1}$ iff $z_{m+1} \in [1 + z_m, m + 1]$. Using the expressions for z_m and z_{m+1} just derived, this is equivalent to $x_{m-1} \leq m - 2 + x_{m+1}$ and $x_{m+1} \leq x_m$. The first inequality is valid because $x_{m-1} \leq (m - 2) + 1 \leq (m - 2) + x_{m+1}$. The second inequality holds since $x_{m-1} \geq x_m \geq x_{m+1}$.

$m + 1$ **even and** π **ends in pattern 132:** Using Lemma 3 we have $x_{m-1} < x_{m+1} \leq x_m$. Definition 4 with $i = m + 1$ and $i = m$ gives $z_{m+1} = m + 1 + x_{m+1} - x_m$ and $z_m = m + 1 + x_{m-1} - x_m$. Characterization 1 tell us that $f(\pi) \in \mathsf{Alter}_{m+1}$ iff $z_{m+1} \in [1 + z_m, m + 1]$. By using the new expressions for z_{m+1} and z_m, and simplifying, this previous condition holds true iff $x_{m-1} < x_{m+1} \leq x_m$, which is stated at the beginning of this case.

$m + 1$ **even and** π **ends in pattern 213:** Using Lemma 3 we have $x_{m-1} < 1 + x_{m+1} < x_m$. Definition 4 with $i = m + 1$ and $i = m$ gives $z_{m+1} = 1 + x_{m+1} - x_m$ and $z_m = 2 + x_{m-1} - x_m$. Again $f(\pi) \in \mathsf{Alter}_{m+1}$ iff $z_{m+1} \in [1 + z_m, m + 1]$. By using the new expressions for z_{m+1} and z_m, and simplifying, this previous condition holds true iff (a) $x_{m-1} < x_{m+1} - 1$ and (b) $x_{m+1} \leq m + x_m$. Since $x_{m-1} < 1 + x_{m+1} < x_m$, inequality (a) holds. Also, since $x_{m+1} \leq m + 1$ and $m + 1 \leq m + x_m$, inequality (b) also holds.

$m + 1$ **odd and** π **ends in pattern 321:** Using Lemma 3 we have $x_{m-1} \geq x_m \geq x_{m+1}$. Definition 4 with $i = m + 1$ and $i = m$ gives $z_{m+1} = 2 + x_m - x_{m+1}$ and $z_m = m + x_m - x_{m-1}$. Characterization 1 tells us that $f(\pi) \in \mathsf{Alter}_{m+1}$ iff $z_{m+1} \in [2, z_m]$ (since $m + 1$ is now odd). By using the new expressions for z_{m+1} and z_m, and simplifying, this previous condition holds true iff (a) $x_m \geq x_{m+1}$ and (b) $2 + x_{m-1} \leq x_{m+1} + m$. Inequality (a) holds since $x_{m-1} \geq x_m \geq x_{m+1}$. Since $x_{m-1} \leq (m - 2) + 1 \leq (m - 2) + x_{m+1}$, inequality (b) also holds.

$m + 1$ **odd and** π **ends in pattern 132:** Using Lemma 3 we have $x_{m-1} < x_{m+1} \leq x_m$. Definition 4 with $i = m + 1$ and $i = m$ gives $z_{m+1} = 2 + x_m - x_{m+1}$ and $z_m = 1 + x_m - x_{m-1}$. Again, $f(\pi) \in \mathsf{Alter}_{m+1}$ iff $z_{m+1} \in [2, z_m]$. By using the new expressions for z_{m+1} and z_m, and simplifying, this previous condition holds true iff $x_m \geq x_{m+1}$ and $x_{m-1} < x_{m+1}$ which are stated at the beginning of this case.

$m + 1$ **odd and** π **ends in pattern 213:** Using Lemma 3 we have $x_m < 1 + x_{m-1} \leq x_{m+1}$. Definition 4 with $i = m + 1$ and $i = m$ gives $z_{m+1} = m + 2 + x_m - x_{m+1}$ and $z_m = m + x_m - x_{m-1}$. Again $f(\pi) \in \mathsf{Alter}_{m+1}$ iff $z_{m+1} \in [2, z_m]$. By using the new expressions for z_{m+1} and z_m, and simplifying, this previous condition holds true iff $x_{m+1} \leq m + x_m$

and $1 + x_{m-1} < x_{m+1}$. The first inequality holds since $x_{m+1} \le m + 1 \le m + x_m$, and the second inequality holds from the inequality at the start of this case.

In all cases $f(\pi) \in \mathsf{Alter}_{m+1}$.

Step 2: We will now show that $f : \mathsf{Cyc}_{m+1} \to \mathsf{Alter}_{m+1}$ is injective.

Let $\alpha = \langle \alpha_1, \dots, \alpha_{m+1} \rangle$ and $\beta = \langle \beta_1, \dots, \beta_{m+1} \rangle$ be in Cyc_{m+1}. Let $\alpha' = \langle \alpha_1, \dots, \alpha_m \rangle$ and $\beta' = \langle \beta_1, \dots, \beta_m \rangle$. Both α' and β' are in Cyc_m by Characterization 2.

Suppose that $f(\alpha) = f(\beta) = \langle z_1, \dots, z_{m+1} \rangle$. Since $f(\alpha) = f(\alpha') \oplus z_{m+1}$ and $f(\beta) = f(\beta') \oplus z_{m+1}$ we have $f(\alpha') \oplus z_{m+1} = f(\beta') \oplus z_{m+1}$. This implies that $f(\alpha') = f(\beta')$. Both of these are in Cyc_m and therefore $f(\alpha') = f(\beta')$, according to the induction hypothesis, is true iff $\alpha' = \beta'$

At this point we have $\alpha = \alpha' \oplus \alpha_{m+1}$, $\beta = \alpha' \oplus \beta_{m+1}$, and $f(\alpha') = \langle z_1, \dots, z_m \rangle$. We now condition on the parity of m. In several cases we will come upon the condition $m+\alpha_{m+1} = \beta_{m+1}$. This cannot hold because $\alpha_{m+1}, \beta_{m+1} \in [2, m + 1]$ which gives $m + 2 \le m + \alpha_{m+1} = \beta_{m+1}$, a contradiction. The same is true of $m + \beta_{m+1} = \alpha_{m+1}$.

- $m + 1$ **even and** α **ends in pattern 321:** Using Lemma 3 we have $\alpha_{m-1} \ge \alpha_m \ge \alpha_{m+1}$. Definition 4 with $i = m + 1$ gives $z_{m+1} = m + 1 + \alpha_{m+1} - \alpha_m$ $(*_1)$. Since $\beta = \langle \alpha_1, \dots, \alpha_m, \beta_{m+1} \rangle \in \mathsf{Cyc}_{m+1}$ and $\alpha_{m-1} \ge \alpha_m$, by Characterization 2 we must have $\beta_{m+1} \in [2, \alpha_m] \cup [2 + \alpha_{m-1}, m + 1]$. If $\beta_{m+1} \le \alpha_m$ then Definition 4 gives $z_{m+1} = m+1+\beta_{m+1}-\alpha_m$. This is compatible with $(*_1)$ iff $\beta_{m+1} = \alpha_{m+1}$. Otherwise $\beta_{m+1} > \alpha_m$ which gives $z_{m+1} = 1 + \beta_{m+1} - \alpha_m$. This is compatible with $(*_1)$ iff $m + \alpha_{m+1} = \beta_{m+1}$, which cannot happen.
- $m + 1$ **even and** α **ends in pattern 132:** Using Lemma 3 we have $\alpha_{m-1} < \alpha_{m+1} \le \alpha_m$. Definition 4 with $i = m + 1$ gives $z_{m+1} = m + 1 + \alpha_{m+1} - \alpha_m$ $(*_2)$. Since $\beta = \langle \alpha_1, \dots, \alpha_m, \beta_{m+1} \rangle \in \mathsf{Cyc}_{m+1}$ and $\alpha_{m-1} < \alpha_m$, by Characterization 2 we must have $\beta_{m+1} \in [1 + \alpha_{m-1}, \alpha_m]$. Since $\beta_{m+1} < \alpha_m$ we have $z_{m+1} = m + 1 + \beta_{m+1} - \alpha_m$ by Definition 4. Comparing this to $(*_2)$ gives $\beta_{m+1} = \alpha_{m+1}$.
- $m + 1$ **even and** α **ends in pattern 213:** Using Lemma 3 we have $\alpha_m < 1 + \alpha_{m-1} < \alpha_{m+1}$. Definition 4 with $i = m + 1$ gives $z_{m+1} = 1 + \alpha_{m+1} - \alpha_m$ $(*_3)$. Since $\beta = \langle \alpha_1, \dots, \alpha_m, \beta_{m+1} \rangle \in \mathsf{Cyc}_{m+1}$ and $\alpha_{m-1} \ge \alpha_m$, by Characterization 2 we must have $\beta_{m+1} \in [2, \alpha_m] \cup [2 + \alpha_{m-1}, m + 1]$. If $\beta_{m+1} \le \alpha_m$ then Definition 4 gives $z_{m+1} = m + 1 + \beta_{m+1} - \alpha_m$. This implies $\beta_{m+1} + m = \alpha_{m+1}$ which is impossible. If $\beta_{m+1} > \alpha_m$ then Definition 4 gives $z_{m+1} = 1 + \beta_{m+1} - \alpha_m$ which implies $\beta_{m+1} = \alpha_{m+1}$ by $(*_3)$.
- $m + 1$ **odd and** α **ends in pattern 321:** Using Lemma 3 we have $\alpha_{m-1} \ge \alpha_m \ge \alpha_{m+1}$. Definition 4 with $i = m+1$ gives $z_{m+1} = 2+\alpha_m-\alpha_{m+1}$ $(*_4)$. Since $\beta = \langle \alpha_1, \dots, \alpha_m, \beta_{m+1} \rangle \in \mathsf{Cyc}_{m+1}$ and $\alpha_{m-1} \ge \alpha_m$, by Characterization 2 we must have $\beta_{m+1} \in [2, \alpha_m] \cup [2 + \alpha_{m-1}, m + 1]$. If $\beta_{m+1} \le \alpha_m$ then Definition 4 gives $z_{m+1} = 2 + \alpha_m - \beta_{m+1}$. Comparing this to $(*_4)$ gives $\beta_{m+1} = \alpha_{m+1}$. Alternatively, if $\beta_{m+1} > \alpha_m$ then $z_{m+1} = m + 2 + \alpha_m - \beta_{m+1}$. Combining this with $(*_4)$ gives $\beta_{m+1} = m + \alpha_{m+1}$, which is not possible.
- $m + 1$ **odd and** α **ends in pattern 132:** Using Lemma 3 we have $\alpha_{m-1} < \alpha_{m+1} \le \alpha_m$. Definition 4 with $i = m + 1$ gives $z_{m+1} = 2 + \alpha_m - \alpha_{m+1}$ $(*_5)$. Since $\beta = \langle \alpha_1, \dots, \alpha_m, \beta_{m+1} \rangle \in \mathsf{Cyc}_{m+1}$ and $\alpha_{m-1} < \alpha_m$, by Characterization 2 we must have $\beta_{m+1} \in [1 + \alpha_{m-1}, \alpha_m]$. This forces $\beta_{m+1} \le \alpha_m$ so that $z_{m+1} = 2 + \alpha_m - \beta_{m+1}$. Combining this with $(*_5)$ gives $\beta_{m+1} = \alpha_{m+1}$.
- $m + 1$ **odd and** α **ends in pattern 213:** Using Lemma 3 we have $\alpha_m < 1 + \alpha_{m-1} < \alpha_{m+1}$. Definition 4 with $i = m + 1$ gives $z_{m+1} = 2 + \alpha_m - \alpha_{m+1}$ $(*_6)$. Since $\beta = \langle \alpha_1, \dots, \alpha_m, \beta_{m+1} \rangle \in \mathsf{Cyc}_{m+1}$ and $\alpha_{m-1} \ge \alpha_m$, by Characterization 2 we must have $\beta_{m+1} \in [2, \alpha_m] \cup [2 + \alpha_{m-1}, m + 1]$. If $\beta_{m+1} \le \alpha_m$ then Definition 4 gives $z_{m+1} = 2+\alpha_m-\beta_{m+1}$. Comparing this to $(*_6)$ gives $\beta_{m+1} = \alpha_{m+1}$. Alternatively, if $\beta_{m+1} > \alpha_m$ then $z_{m+1} = m+2+\alpha_m-\beta_{m+1}$. Compare this to $(*_6)$ to get $m+\alpha_{m+1} = \beta_{m+1}$, which is not possible.

In each of these cases it was shown that $\alpha_{m+1} = \beta_{m+1}$. Therefore $f : \mathsf{Cyc}_{m+1} \to \mathsf{Alter}_{m+1}$ is injective.

Step 3: We will now show that given $z = \langle z_1, \ldots, z_{m+1} \rangle \in \mathsf{Alter}_{m+1}$ there exists $x = \langle x_1, \ldots, x_{m+1} \rangle$ such that $f(x) = z$. Let $z' = \langle z_1, \ldots, z_m \rangle \in \mathsf{Alter}_m$ so that $z = z' \oplus z_{m+1}$. Let $x' = \langle x_1, \ldots, x_m \rangle$ be the unique $x' \in \mathsf{Cyc}_m$ such that $f(x') = z'$. By the induction hypothesis we need only show that for any valid value of z_{m+1} such that $z' \oplus z_{m+1} \in \mathsf{Cyc}_{m+1}$ there exists a valid x_{m+1} such that $f(x' \oplus x_{m+1}) = z$. We do this by conditioning on the parity of $m+1$ and the value of x_m relative to x_{m-1}.

$m+1$ **even:** We must show that there exists a value of x_{m+1} such that z_{m+1} can take all values in $[1+z_m, m+1]$. Since $m+1$ is even we must have $z_m \in [2, z_{m-1}]$.

- Suppose that $x_m > x_{m-1}$. Using Definition 4 with $i = m$ we have $z_m = m + 1 + x_{m-1} - x_m$. Characterization 2 tells us that x_{m+1} takes values in the set $[1+x_{m-1}, x_m]$. As $x_{m+1} \leq x_m$ Definition 4 tells us that $z_{m+1} = m+1+x_{m+1}-x_m$. Since $x_{m+1} \in [1+x_{m-1}, x_m]$, we see that $z_{m+1} \in [m+2+x_{m-1}-x_m, m+1] = [1+z_m, m+1]$, as required.
- Suppose that $x_m \leq x_{m-1}$. Using Definition 4 with $i = m$ we have $z_m = 2 + x_{m-1} - x_m$. Characterization 2 tells us that x_{m+1} takes values in the set $[2, x_m] \cup [2+x_{m-1}, m+1]$. If $x_{m+1} \leq x_m$ then Definition 4 tells us that $z_{m+1} = m + 1 + x_{m+1} - x_m$. Since $x_{m+1} \in [2, x_m]$ for this part, z_{m+1} takes values in the set $[m+3-x_m, m+1]$. Alternatively, if $x_{m+1} > x_m$ then Definition 4 tells us that $z_{m+1} = 1 + x_{m+1} - x_m$. Since $x_{m+1} \in [2+x_{m-1}, m+1]$ for this part, z_{m+1} takes values in the set $[3+x_{m-1}-x_m, m+2-x_m]$. Combining both of these cases shows us that z_{m+1} can take values in the set $[3+x_{m-1}-x_m, m+1] = [1+z_m, m+1]$, as required.

$m+1$ **odd:** We must show that there exists a value of x_{m+1} such that z_{m+1} can take all values in $[2, z_m]$. Since $m+1$ is odd we must have $z_m \in [1+z_{m-1}, m]$.

- Suppose that $x_m > x_{m-1}$. Using Definition 4 with $i = m$ we have $z_m = 1 + x_m - x_{m-1}$. We want to show that we can always find a valid x_{m+1} to produce every z_{m+1} in $[2, z_m] = [2, 1+x_m-x_{m-1}]$. From Characterization 2 x_{m+1} takes values in $[1+x_{m-1}, x_m]$. Note that we have $x_{m+1} \leq x_m$ for each of these, so z_{m+1} takes values in $[2+x_m-x_m, 2+x_m-(1+x_{m-1})] = [2, z_m]$, as required.
- Suppose that $x_m \leq x_{m-1}$. Using Definition 4 with $i = m$ we have $z_m = m + x_m - x_{m-1}$. We want to show that we can always find a valid x_{m+1} to produce every z_{m+1} in $[2, z_m] = [2, m+x_m-x_{m-1}]$. From Characterization 2 x_{m+1} takes values in $[2, x_m] \cup [2+x_{m-1}, m+1]$. Consider the relative values of x_m and x_{m+1}. If $x_{m+1} \leq x_m$ then $z_{m+1} = 2+x_m-x_{m+1}$. Since x_{m+1} takes values in $[2, x_m]$, z_{m+1} takes values in the set $[2, x_m]$. Alternatively, if $x_{m+1} > x_m$ then $z_{m+1} = m+2+x_m-x_{m+1}$. Since x_{m+1} takes values in $[2+x_{m-1}, m+1]$ we find that z_{m+1} takes values in $[m+2+x_m-(m+1), m+2+x_m-2-x_{m-1}] = [1+x_m, z_m]$. Combining both these subcases we find that z_{m+1} takes values in the set $[2, x_m] \cup [1+x_m, z_m] = [2, z_m]$, as required.

Therefore $f : \mathsf{Cyc}_{m+1} \to \mathsf{Alter}_{m+1}$ is surjective. By the principle of induction, we thus have that $f : \mathsf{Cyc}_n \to \mathsf{Alter}_n$ is a bijection for all n. □

3. A graphical method for calculating f

The calculation of $f(\pi)$ in Definition 4 involves the repeated use of four cases. The sequence of numbers $(z_1, \ldots, z_n)$ can be quickly calculated by hand by traversing the permutation diagram in a way that we will describe below.

Suppose that we are given $\pi = (1, 8, 4, 10, 5, 2, 7, 3, 9, 6) \in \mathsf{Cyc}_{10}$ and wish to calculate $f(\pi)$. Draw the permutation diagram of π as illustrated in Figure 3. Insert down arrows on those vertical lines which have a column of odd index immediately to their left, and insert up arrows on those vertical lines which have a column of even index immediately to their left.

Let us now define a sequence of numbers $(a_3, \ldots, a_n)$ where the integer a_i is defined as follows: start at the point (i, π_i) and move in the direction indicated by the arrows on the line immediately to the left of the point. Count the number of points seen to the left up to and including the point $(i-1, \pi_{i-1})$, and at this point we stop. We move cyclically so that if the point $(i-1, \pi_{i-1})$ has not been encountered upon reaching position $(i, 1)$ (respectively (i, n)) then move to (i, n) (respectively $(i, 1)$) and continue until it is encountered. From this sequence of numbers we have the corresponding permutation $f(\pi) = \langle 1, 2, 1 + a_3, 1 + a_4, \ldots, 1 + a_n \rangle$ in Alter_n.

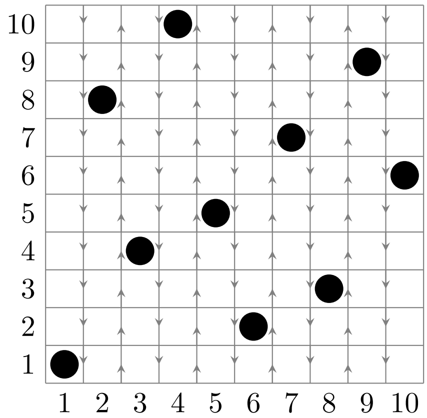

FIGURE 3.

Suppose $\pi = (1, 8, 4, 10, 5, 2, 7, 3, 9, 6)$. To calculate a_9 we start at $(i = 9, \pi_i = 9)$, and move upwards (because i is odd). First we see the point $(4, 10)$ to the left, and we are now stuck at the top of the column but have not yet seen to the left the ending point $(8, 3) = (i-1, \pi_{i-1})$. We therefore move to the bottom of the column and continue moving upwards again. We see the point $(1, 1)$ to the left, next the point $(6, 2)$, and the next point we see to the left is $(8, 3) = (i-1, \pi_{i-1})$, which is our final point. We have encountered 4 points on our journey so $a_9 = 4$.

Doing the same for $i = 3, 4, \ldots, 8, 10$ we get $(a_3, \ldots, a_{10}) = (1, 2, 2, 4, 4, 5, 4, 7)$. Finally,

$$\begin{aligned}
f(\pi) &= \langle 1, 2, 1 + a_3, 1 + a_4, \ldots, 1 + a_{10} \rangle \\
&= \langle 1, 2, 2, 3, 3, 5, 5, 6, 5, 8 \rangle \\
&= (((((((((1) \oplus 2) \oplus 2) \oplus 3) \oplus 3) \oplus 5) \oplus 5) \oplus 6) \oplus 5) \oplus 8) \\
&= ((((((((1, 2) \oplus 2) \oplus 3) \oplus 3) \oplus 5) \oplus 5) \oplus 6) \oplus 5) \oplus 8) \\
&= (((((((1, 3, 2) \oplus 3) \oplus 3) \oplus 5) \oplus 5) \oplus 6) \oplus 5) \oplus 8) \\
&= ((((((1, 4, 2, 3) \oplus 3) \oplus 5) \oplus 5) \oplus 6) \oplus 5) \oplus 8) \\
&= (((((1, 5, 2, 4, 3) \oplus 5) \oplus 5) \oplus 6) \oplus 5) \oplus 8) \\
&= ((((1, 6, 2, 4, 3, 5) \oplus 5) \oplus 6) \oplus 5) \oplus 8) \\
&= (((1, 7, 2, 4, 3, 6, 5) \oplus 6) \oplus 5) \oplus 8) \\
&= ((1, 8, 2, 4, 3, 7, 5, 6) \oplus 5) \oplus 8) \\
&= ((1, 9, 2, 4, 3, 8, 6, 7, 5) \oplus 8) \\
&= (1, 10, 2, 4, 3, 9, 6, 7, 5, 8).
\end{aligned}$$

4. Proof of equivalence with Aldred et al. and the inverse of f

It is not immediately obvious that f is in fact the same bijection that is described in Aldred et al. [1]. To show that this is the case, let us make some observations followed by a theorem.

Given $\pi = \langle x_1, \dots, x_n \rangle \in \mathsf{Alter}$ and $\pi' = \langle x_1, \dots, x_n, x_{n+1} \rangle \in \mathsf{Alter}_{n+1}$, we say that π' is the child of π in Alter since π may be recovered from π' by removing its rightmost element and renumbering the other elements with the first n natural numbers. We use the same terminology for members of Cyc.

Aldred et al. proved two results [1, Lemmas 2 and 3] to show that their mapping was indeed a bijection. They showed that in their trees for both Alter and Cyc, the children of a node at level n with down degree d have down degrees $n-d+1, n-d+2, \dots, n$. This, along with the fact that the tops of the trees were the same, shows that there is a one-to-one correspondence between both embeddings.

The next theorem shows that f respects this correspondence.

Theorem 6. *Let $\pi = \langle x_1, \dots, x_n \rangle \in \mathsf{Cyc}_n$ and $\sigma = \langle z_1, \dots, z_n \rangle \in \mathsf{Alter}_n$ be such that $\sigma = f(\pi)$ and both π and σ have d children in the sets Cyc_{n+1} and Alter_n, respectively. Suppose that x_{n+1} is chosen such that $\pi' = \langle x_1, \dots, x_n, x_{n+1} \rangle \in \mathsf{Cyc}_{n+1}$ has j children (for some $j \in [n-d+1, n]$). Then $f(\pi \oplus x_{n+1})$ also has j children.*

Proof. Let π, σ and π' be as stated in the theorem. There are 4 cases to consider: (a) n even and $x_n > x_{n-1}$, (b) n odd and $x_n > x_{n-1}$, (c) n even and $x_n \le x_{n-1}$, and (d) n odd and $x_n \le x_{n-1}$.

For case (a) we have that $z_n = 1 + x_n - x_{n-1}$. Since π has d children $|[1 + x_{n-1}, x_n]| = d$ and since σ has d children $|[2, z_n]| = d$. This gives us the identity $x_n - x_{n-1} = d = z_n - 1$ so that $x_n = d + x_{n=1}$. The value x_{n+1} is chosen so that π' has j children. Because $x_{n-1} < x_n$ we must have $x_{n+1} \in [1+x_{n-1}, x_n]$ which shows that $x_{n+1} \le x_n$. This means $|[2, x_{n+1}] \cup [2+x_n, n+2]| = j$ which gives the identity $j = x_{n+1} - x_n + n$, or equivalently $x_{n+1} = j + x_n - n$.

Now $f(\pi') = f(\pi \oplus x_{n+1}) = f(\pi) \oplus z_{n+1} = \sigma \oplus z_{n+1}$ where $z_{n+1} = 2 + x_n - x_{n+1}$. The number of children of $\langle z_1, \dots, z_{n+1} \rangle$ is

$$\begin{aligned} |[1 + z_{n+1}, n+2]| &= n + 2 - z_{n+1} \\ &= n + 2 - (2 + x_n - x_{n+1}) \\ &= n - d - x_{n-1} + j + d + x_{n-1} - x_n \\ &= j. \end{aligned}$$

The other three cases reach the same conclusion. □

The inverse of f is the function g and is derived from the Definition of f. In order to derive an expression for x_i in terms of z_i and x_{i-1}, we must compare the relative values of x_{i-1} and x_{i-2}.

Definition 7 (Inverse of f)**.** Given $\pi = \langle z_1, \dots, z_n \rangle \in \mathsf{Alter}_n$, let $g(\pi) = \langle x_1, \dots, x_n \rangle$ where $x_1 = 1$, $x_2 = 2$, and for all $i \in [3, n]$:

$$x_i = \begin{cases} x_{i-1} + z_i - 1 & \text{if } i \text{ even, } x_{i-1} \le x_{i-2} \text{ and } z_i \in [3 + x_{i-2} - x_{i-1}, 1 + i - x_{i-1}] \\ x_{i-1} + z_i - i & \text{if } i \text{ even, } x_{i-1} \le x_{i-2} \text{ and } z_i \in [i + 2 - x_{i-1}, i] \\ x_{i-1} + z_i - i & \text{if } i \text{ even, } x_{i-1} > x_{i-2} \text{ and } z_i \in [i + 1 + x_{i-2} - x_{i-1}, i] \\ x_{i-1} - z_i + 2 & \text{if } i \text{ odd, } x_{i-1} > x_{i-2} \text{ and } z_i \in [2, 1 + x_{i-1} - x_{i-2}] \\ x_{i-1} - z_i + 2 & \text{if } i \text{ odd, } x_{i-1} \le x_{i-2} \text{ and } z_i \in [2, x_{i-1}] \\ x_{i-1} - z_i + i + 1 & \text{if } i \text{ odd, } x_{i-1} \le x_{i-2} \text{ and } z_i \in [1 + x_{i-1}, i - 1 + x_{i-1} - x_{i-2}]. \end{cases}$$

References

[1] R.E.L. Aldred, M.D. Atkinson, D.J. McCaughan. Avoiding consecutive patterns in permutations. *Advances in Applied Mathematics* **45** (2010) 449–461.

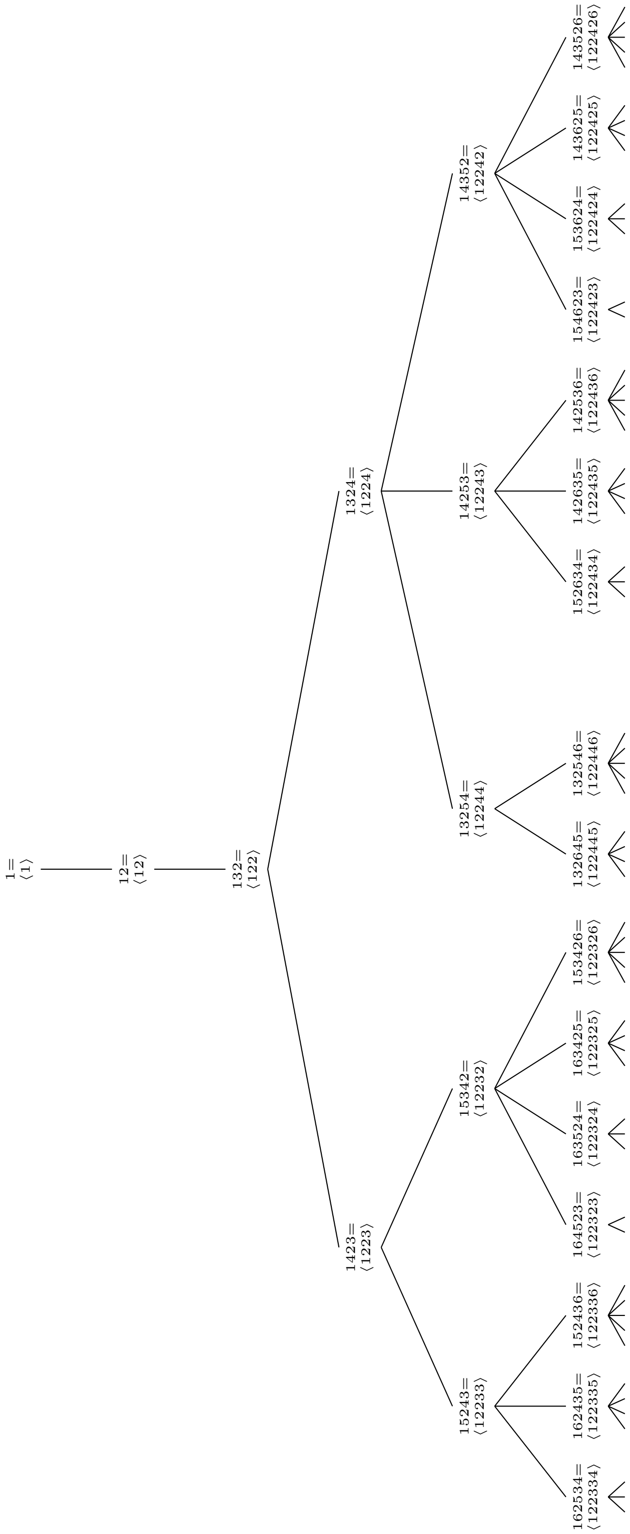

FIGURE 4. The tree showing Alter_i for $i \in \{1, \ldots, 6\}$. We have omitted commas from the angle bracket notation to save on space.

LaBRI, CNRS, Université de Bordeaux, 351 Cours de la Libération, Bordeaux, France.

Department of Computer and Information Sciences, University of Strathclyde, Glasgow G1 1XH, United Kingdom.

CNRS, PIMS SFU, Simon Fraser University, Burnaby BC, Canada.

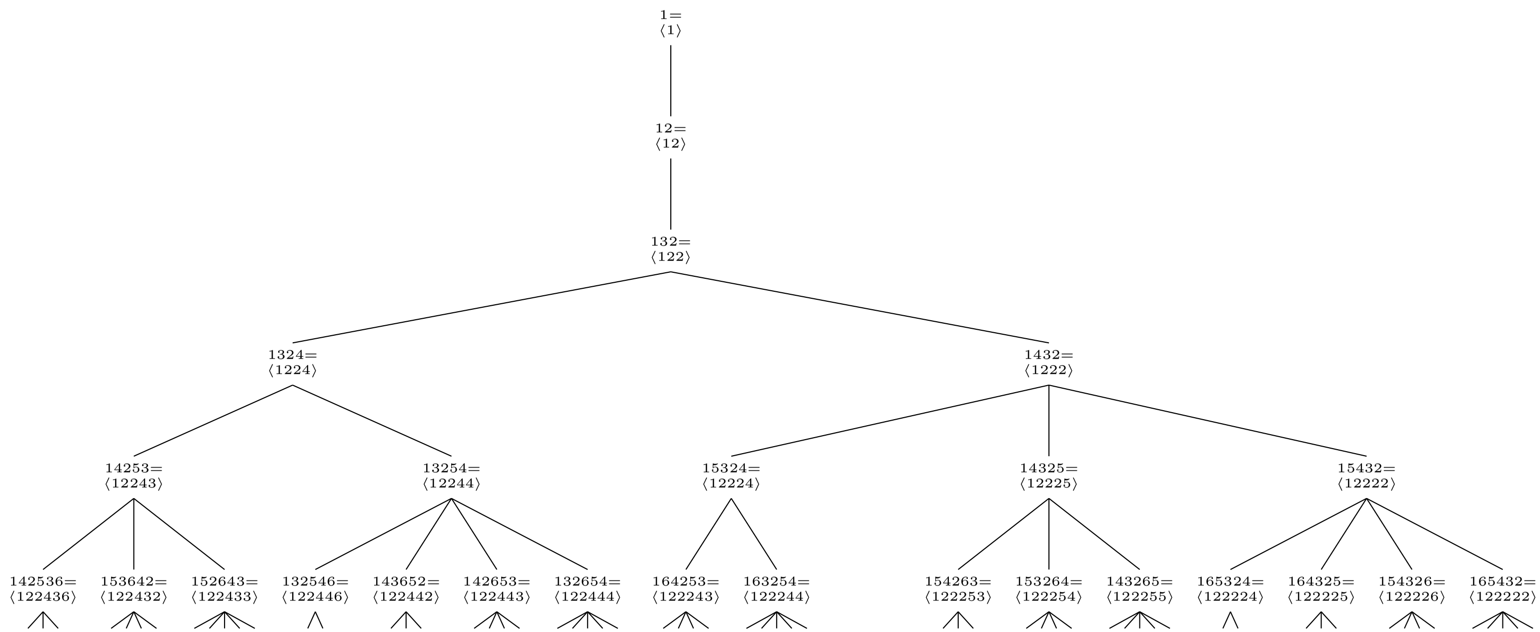

FIGURE 5. The tree showing Cyc_i for $i \in \{1, \ldots, 6\}$. We have omitted commas from the angle bracket notation to save on space.