The Spectral Method (MAPH 40260)

Part 4: Barotropic Vorticity Equation

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Outline

Background

Rossby-Haurwitz Waves

Interaction Coefficients

Transform Method

The ECMWF Model

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Triads of RH waves that satisfy conditions for resonance are of critical importance for the distribution of energy in the atmosphere.

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Charney, Fjørtoft & von Neumann (1950) integrated the BVE to produce the earliest numerical weather predictions on the ENIAC.

They integrated the equation on a rectangular domain, in planar geometry.

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Highly truncated versions of the spectral BVE have been analysed to gain understanding of atmospheric phenomena.

Edward Lorenz (1960) introduced what he called the maximum simplification of the system, reducing it to three nonlinear ODEs.

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Interactions are particularly effective when the component parameters are related by **resonance conditions.**

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Such oscillatory response to steady forcing is consistent with forced resonant triads (Lynch, 2009).

Outline

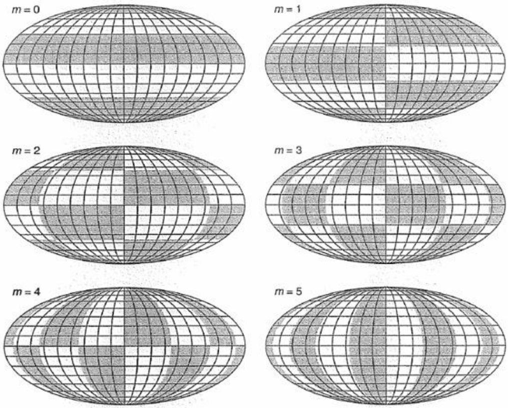

Figure 4.10 Alternating patterns of positives and negatives for spherical functions with $\ell = 5$ and $m = 0, 1, 2, 3, 4, 5$. (Redrawn from Baer 1972.)

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The dynamics of the fluid are governed by conservation of absolute vorticity

$$\frac{d}{dt}(\zeta + f) = 0\,,$$

where $f = 2\Omega \sin\phi$ is the planetary vorticity, and $\zeta = \mathbf{k} \cdot \nabla \times \mathbf{V}$ is the vorticity of the flow.

The time derivative is

$$\frac{d\zeta}{dt} = \frac{\partial\zeta}{\partial t} + \frac{u}{a\cos\phi}\frac{\partial\zeta}{\partial\lambda} + \frac{v}{a}\frac{\partial\zeta}{\partial\phi} .$$

We assume nondivergent flow and introduce a stream-function ψ such that $\mathbf{V} = \mathbf{k} \times \nabla\psi$ and $\zeta = \nabla^2\psi$.

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$$\frac{d\zeta}{dt} = \frac{\partial\zeta}{\partial t} - \frac{1}{a}\frac{\partial\psi}{\partial\phi}\frac{1}{a\cos\phi}\frac{\partial\zeta}{\partial\lambda} + \frac{1}{a\cos\phi}\frac{\partial\psi}{\partial\lambda}\frac{1}{a}\frac{\partial\zeta}{\partial\phi} .$$

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Defining $\mu = \sin\phi$, this may be expressed as

$$\begin{aligned}\frac{d\zeta}{dt} &= \frac{\partial\zeta}{\partial t} + \frac{1}{a^2}\left[-\frac{\partial\psi}{\partial\mu}\frac{\partial\zeta}{\partial\lambda} + \frac{\partial\psi}{\partial\lambda}\frac{\partial\zeta}{\partial\mu}\right]\\ &= \frac{\partial\zeta}{\partial t} + \frac{1}{a^2}\frac{\partial(\psi,\zeta)}{\partial(\lambda,\mu)}\\ &= \frac{\partial\zeta}{\partial t} + \frac{1}{a^2}J(\psi,\zeta)\,.\end{aligned}$$

Since $f = 2\Omega \sin\phi$, the "β-term" may be expressed

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\begin{aligned}
\frac{df}{dt} &= \frac{v}{a}\frac{\partial f}{\partial \phi} \\
&= \frac{1}{a\cos\phi}\frac{\partial \psi}{\partial \lambda}\frac{1}{a}\frac{\partial f}{\partial \phi} \\
&= \frac{1}{a\cos\phi}\frac{\partial \psi}{\partial \lambda}\frac{1}{a}2\Omega\cos\phi = \frac{2\Omega}{a^2}\frac{\partial \psi}{\partial \lambda}
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The barotropic vorticity equation may now be written

$$\frac{\partial \zeta}{\partial t} + \frac{2\Omega}{a^2}\frac{\partial \psi}{\partial \lambda} + \frac{1}{a^2}\frac{\partial(\psi,\zeta)}{\partial(\lambda,\mu)} = 0$$

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Temporarily omitting this, we see that the BVE has solutions of the form

$$\psi = \psi_0 Y_n^m(\lambda, \mu) \exp(-i\sigma t)$$

where ψ_0 is the constant amplitude and the frequency σ is given by the dispersion formula

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Here, m is the zonal wavenumber, n is the total wavenumber (both are integers) and $Y_n^m(\lambda,\mu)$ are the spherical harmonics, eigenfunctions of ∇^2:

$$\nabla^2 Y_n^m = -\frac{n(n+1)}{a^2} Y_n^m \,.$$

We assume the functions Y_n^m to be normalized so that

$$\frac{1}{4\pi}\iint (Y_{n_1}^{m_1})^* Y_{n_2}^{m_2} d\lambda d\mu = \delta_{m_2}^{m_1}\delta_{n_2}^{n_1} .$$

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It is remarkable that, for a single RH wave, the nonlinear Jacobian term vanishes identically, so that such a wave is a solution of the nonlinear BVE.

Outline

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The coefficients ψ_n^m and ζ_n^m are functions of time.

Flows governed by the BVE conserve the total energy and total enstrophy, defined by

$$E = \frac{1}{4\pi a^2} \iint \tfrac{1}{2}\mathbf{V}\cdot\mathbf{V}\,d\lambda\,d\mu = -\frac{1}{4\pi a^2} \iint \tfrac{1}{2}\psi\zeta\,d\lambda\,d\mu$$

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The constancy of energy and enstrophy profoundly influences the energetics of solutions of the BVE.

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We can then write the expansions

$$\psi = \sum_{\gamma} \psi_{\gamma}(t) Y_{\gamma}(\lambda, \mu) \exp(-i\sigma_{\gamma} t)$$

and

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For a pure RH wave, or a collection of non-interacting waves, the coefficients ψ_{γ} and ζ_{γ} are constants.

Their variation is due to nonlinear interactions between the components.

If the expansion

$$\zeta = \sum_{\gamma} \zeta_\gamma(t) Y_\gamma(\lambda, \mu) \exp(-i\sigma_\gamma t)$$

is substituted into the BVE and the orthogonality condition is used, we obtain equations for the evolution of the spectral coefficients in time:

$$\frac{d\zeta_\gamma}{dt} = \tfrac{1}{2} i \sum_{\alpha,\beta} I_{\gamma\beta\alpha} \zeta_\beta \zeta_\alpha \exp(-i\sigma t) \,,$$

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Here $\sigma = \sigma_\alpha + \sigma_\beta - \sigma_\gamma$ and the interaction coefficients are given by

$$I_{\gamma\beta\alpha} = (\kappa_\beta - \kappa_\alpha) K_{\gamma\beta\alpha}\,.$$

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In case $m_\alpha + m_\beta = m_\gamma$, they are given by

$$K_{\gamma\beta\alpha} = \frac{1}{2}\int_{-1}^{+1} P_\gamma \left(m_\beta P_\beta \frac{dP_\alpha}{d\mu} - m_\alpha P_\alpha \frac{dP_\beta}{d\mu} \right) d\mu \, .$$

The interaction coefficients vanish in most cases. For non-vanishing interaction, **selection rules** must be satisfied ...

Selection Rules

$$
\begin{array}{rcl}
m_\alpha + m_\beta & = & m_\gamma \\
m_\alpha^2 + m_\beta^2 & \neq & 0 \\
n_\gamma n_\beta n_\alpha & \neq & 0 \\
n_\alpha & \neq & n_\beta \\
n_\alpha + n_\beta + n_\gamma & \text{is} & \text{odd} \\
(n_\beta - |m_\beta|)^2 + (n_\alpha - |m_\alpha|)^2 & \neq & 0 \\
|n_\alpha - n_\beta| < & n_\gamma & < n_\alpha + n_\beta \\
(m_\beta, n_\beta) \neq (-m_\gamma, n_\gamma) & \text{and} & (m_\alpha, n_\alpha) \neq (-m_\gamma, n_\gamma)
\end{array}
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It is obvious that the following symmetries hold:

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The following redundancy rules are easily proved by integration by parts:

$$K_{\alpha\bar{\beta}\gamma} = K_{\gamma\beta\alpha} \qquad \text{and} \qquad K_{\beta\gamma\bar{\alpha}} = K_{\gamma\beta\alpha} \, ,$$

where $\bar{\alpha} = (-m, n)$ when $\alpha = (m, n)$.

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The Transform Method

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A more efficient spectral technique, the **transform method**, was devised by Eliasen, Machenhauer and Rasmussen (1970) and, independently, by Orszag (1970).

In this approach, the fields are transformed, at each time step, back to the physical domain, the nonlinear terms are calculated, and the result is transformed to spectral space.

Pros and Cons of Spectral Method

Pros:

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- Energy and enstrophy exactly conserved.
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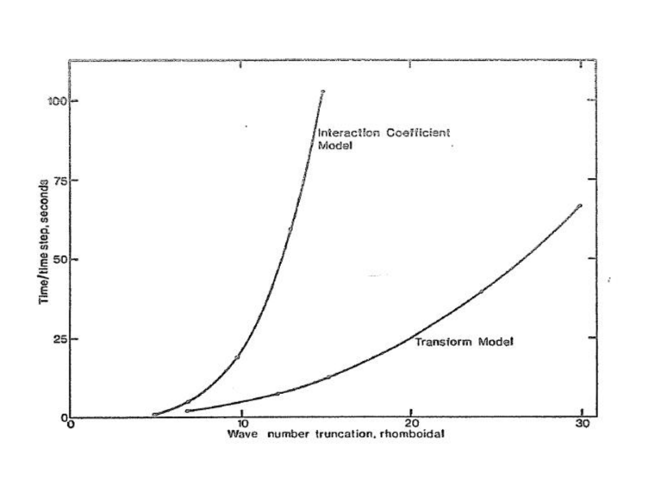
The transform method addresses the last point.


Interaction Coefficient Model
Transform Model
Time/time step, seconds
Wave number truncation, rhomboidal
100
75
50
25
0
0
10
20
30

Derivatives are evaluated exactly in spectral space. The nonlinear terms involve products of derivatives, e.g.,

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The essence of the transform method is this:

- The spatial derivatives are evaluated in spectral space.
- These are then transformed to gridpoint space.
- The multiplications etc. are done in gridpoint space.
- The resulting nonlinear terms are transformed back to spectral space.

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The x-derivative of this is

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We do this for all the terms, do the multiplications, and transform back to spectral space.

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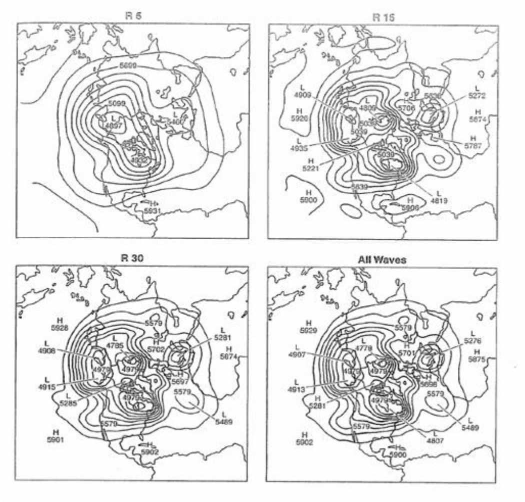
The method is at the heart of most global models of the atmosphere, for example, the ECMWF model known as the IFS code.



R 5
R 15
R 30
All Waves

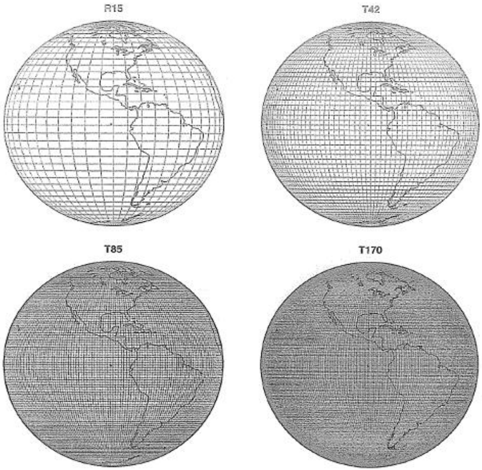


Figure 4.12 Gaussian and triangular grids on the globe for various resolutions: rhomboidal, R15, and triangular, T42, T85 and T170. (David Williamson, personal communication, 2002.)

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The ECMWF Model

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The Centre has been spectacularly successful in fulfilling its mission, and continues to develop forecasts and other products of steadily increasing accuracy and value, maintaining its position as a world leader.

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- Forecasts for the atmosphere out to ten days ahead, based on a T799 (25 km) 91-level (L91) deterministic model are disseminated twice per day.
- Forecasts from the Ensemble Prediction System (EPS) using a T399 (50 km) L62 version of the model and an ensemble of fifty-one members are computed and disseminated twice per day.
- Forecasts out to one month ahead, based on ensembles using a resolution of T255 (78 km) and 62 levels are distributed once per week.
- Seasonal Forecasts out to six months ahead, based on ensembles with a T159 (125 km) L40 model are disseminated once per month.

The Integrated Forecast System

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The IFS uses a *spectral representation* of the meteorological fields. Each field is expanded in series of spherical harmonics; for example,

$$u(\lambda, \phi, t) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} U_n^m(t) Y_n^m(\lambda, \phi)$$

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where the coefficients $U_n^m(t)$ depend only on time, and the spherical harmonics $Y_n^m(\lambda, \phi)$ are as introduced above.

The coefficients U_n^m of the harmonics provide an alternative to specifying the field values $u(\lambda, \phi)$ in the spatial domain.

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When the model equations are transformed to spectral space, they become a set of equations for the spectral coefficients U_n^m.

These are used to advance the coefficients in time, after which the new physical fields may be computed.

Triangular Truncation

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In the IFS model, the expansion is truncated at a fixed total wavenumber N:

$$u(\lambda_i, \phi_j, t) = \sum_{n=0}^{N} \sum_{m=-n}^{n} U_n^m(t) Y_n^m(\lambda_i, \phi_j)$$

This is called *triangular truncation*, and the value of N indicates the resolution of the model.
E.g., if $N = 512$, the resolution is denoted $T512$.

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Since truncation at wavenumber N implies a maximum of N wavelengths around the globe, and since at least two points per wavelength are required, the resolution of the equivalent Gaussian grid is given by the circumference of the Earth divided by twice the truncation N, that is, $\Delta = (2\pi a)/2N$.

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Since $2\pi a = 4 \times 10^7$ m, we get the **simple rule**

$$\Delta = \left(\frac{20{,}000}{N}\right)\ \mathrm{km}\,.$$

Table: Upgrade to the ECMWF Integrated Forecast System in Spring, 2006 (IFS cycle 29r3).

	Deterministic Model		Ensemble Prediction System (EPS)		Monthly Forecast (MOFC)	
	Previous	**Upgrade**	**Previous**	**Upgrade**	**Previous**	**Upgrade**
Spectral Truncation	**T511**	**T799**	**T255**	**T399**	**T159**	**T255**
Gaussian Grid	**N256**	**N400**	**N128**	**N200**	**N80**	**N128**
Model Levels	**L60**	**L91**	**L40**	**L62**	**L40**	**L62**

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The EPS system runs with a horizontal resolution half that of the deterministic model.

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The Centre carries out its operational programme using an IBM High Performance Computing Facility (HPCF). The peak performance is 16.5 TeraFlops for each cluster,

so the complete system has a peak performance of 33 TeraFlops or 33 trillion calculations per second.

End of Part 4