The Spectral Method (MAPH 40260)

Part 3: Spherical Harmonics

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Outline

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Laplacian

Eigenfunctions of the Laplacian

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The trigonometric functions

$$f_{k\ell} = \begin{pmatrix} \sin \\ \cos \end{pmatrix} kx \begin{pmatrix} \sin \\ \cos \end{pmatrix} \ell y$$

are clearly eigenfunctions of ∇^2:

$$\nabla^2 f_{k\ell} = -(k^2 + \ell^2) f_{k\ell}$$

with eigenvalues $\lambda_{k\ell} = -(k^2 + \ell^2)$.

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$$0 \leq x \leq L_x \qquad \text{and} \qquad 0 \leq y \leq L_y$$

with **homogeneous boundary conditions:**

$$f(0, y) = f(L_x, y) = f(x, 0) = f(x, L_y) = 0$$

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This **quantizes** the wave numbers: eigensolutions are

$$f_{mn}(x, y) = \sin \frac{m\pi}{L_x} x \sin \frac{n\pi}{L_y} y$$

with eigenvalues

$$\lambda_{mn} = -\left(\left[\frac{m\pi}{L_x} \right]^2 + \left[\frac{n\pi}{L_y} \right]^2 \right) .$$

Now we consider Spherical coordinates. Then

$$\nabla^2 f = \left(\frac{1}{\cos^2\phi}\frac{\partial^2 f}{\partial\lambda^2} + \frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\cos\phi\frac{\partial f}{\partial\phi} \right) = -\kappa f$$

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We define $\mu = \sin\phi$ and write this as

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Then the eigenproblem is

$$\frac{d^2\Lambda}{d\lambda^2}\cdot\Phi + \Lambda\cdot(1-\mu^2)\frac{d}{d\mu}(1-\mu^2)\frac{d\Phi}{d\mu} = -(1-\mu^2)\kappa\Lambda\Phi\,.$$

Dividing by $\Lambda\Phi$, the problem separates into two parts

$$\frac{1}{\Lambda}\frac{d^2\Lambda}{d\lambda^2} = -\frac{1-\mu^2}{\Phi}\left[\frac{d}{d\mu}(1-\mu^2)\frac{d\Phi}{d\mu} + \kappa\Phi\right] .$$

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Since the l.h.s. depends only on λ and the r.h.s. only on μ, they must both be constants:

$$\begin{aligned}\frac{1}{\Lambda}\frac{d^2\Lambda}{d\lambda^2} &= -m^2 \\ -\frac{1-\mu^2}{\Phi}\left[\frac{d}{d\mu}(1-\mu^2)\frac{d\Phi}{d\mu} + \kappa\Phi\right] &= -m^2 .\end{aligned}$$

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The zonal structure is given by

$$\frac{d^2\Lambda}{d\lambda^2} + m^2\Lambda = 0$$

which is immediately solved: $\Lambda = \exp(im\lambda)$.

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The complete eigensolutions are the spherical harmonics

$$Y_n^m(\lambda,\mu) = P_n^m(\mu)\exp(im\lambda)\,.$$

The derivation above is standard and may be found in many books on *Mathematical Methods of Physics*.

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The important result for us is the following:

$$\nabla^2 Y_n^m(\lambda, \mu) = -n(n+1) Y_n^m(\lambda, \mu) \,.$$

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$$\nabla^2 Y_n^m(\lambda, \mu) = -\left[\frac{n(n+1)}{a^2}\right] Y_n^m(\lambda, \mu) .$$

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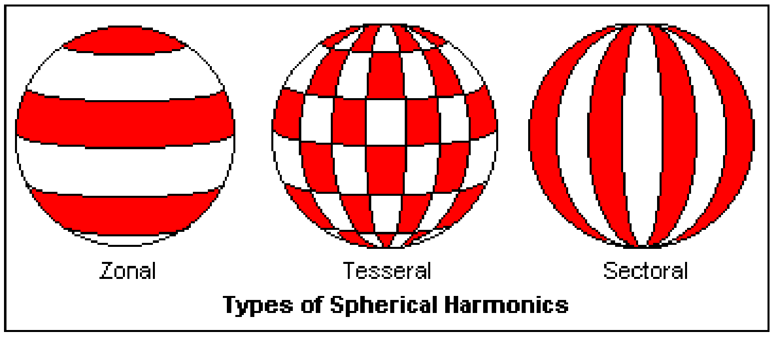
The spherical harmonics $Y_n^m(\lambda, \mu)$ are the eigenfunctions of the Laplacian on the sphere, with eigenvalues $-n(n+1)/a^2$.

Alternating regions of positive and negative values.



Zonal: $m = 0, n > 0$.
Tesseral: $0 < m < n$.
Sectoral: $m = n$.

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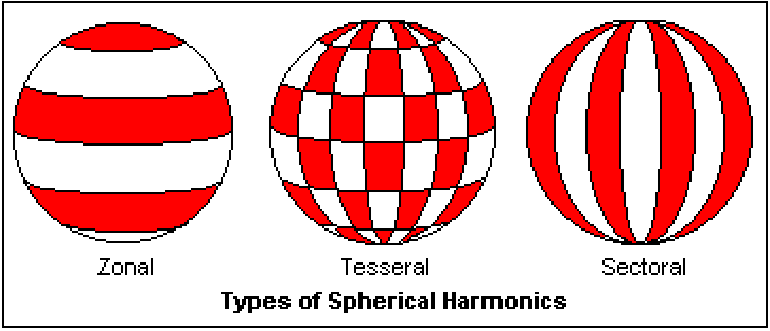

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$|m|$ **zeros in** $0 \leq \lambda < 2\pi$
$n - |m|$ **zeros in** $-1 < \mu < +1$.

Orthogonality & Completeness

The spherical harmonics are an orthogonal set:

$$\frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1}[Y_q^p(\lambda,\mu)]^* \cdot Y_s^r(\lambda,\mu)\,d\mu\,d\lambda = \delta_{pr}\delta_{qs}\,.$$

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Any (reasonable) function $f(\lambda,\mu,t)$ on the sphere can be expanded in spherical harmonics:

$$f(\lambda,\mu,t) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} f_n^m(t)\, Y_n^m(\lambda,\mu)\,.$$

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The coefficients $f_n^m(t)$ are given by

$$f_n^m(t)=\frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1}[Y_n^m(\lambda,\mu)]^*\cdot f(\lambda,\mu)\,d\mu\,d\lambda$$

Truncation

In practice, we have to replace the infinite summation

$$f(\lambda, \mu, t) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} f_n^m(t) Y_n^m(\lambda, \mu) .$$

by a finite summation.

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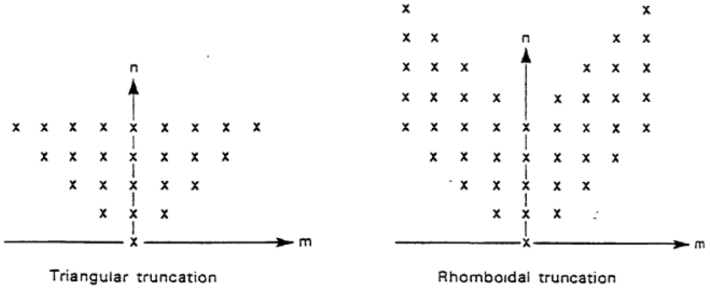
$$f(\lambda, \mu, t) = \sum_{n=0}^{N} \sum_{m=-n}^{n} f_n^m(t) Y_n^m(\lambda, \mu) \,.$$

It can be shown that this gives uniform resolution throughout the sphere.

The ECMWF model uses triangular truncation.

Permissible vales of m and n for triangular and rhomboidal truncation.



Note that m can be positive or negative, and $|m| \leq n$.

End of Part 3