The Spectral Method (MAPH 40260)

Part 3: Spherical Harmonics

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Outline

Laplacian

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Eigenfunctions of the Laplacian

We now investigate the eigenfunctions of the Laplacian operator, i.e., functions f that satisfy

$$\nabla^2 f = \lambda f$$

First, consider Cartesian coordinates. Then

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$$

The trigonometric functions

$$f_{k\ell} = \begin{pmatrix}\sin\\ \cos\end{pmatrix} kx \begin{pmatrix}\sin\\ \cos\end{pmatrix} \ell y$$

are clearly eigenfunctions of ∇^2:

$$\nabla^2 f_{k\ell} = -(k^2 + \ell^2) f_{k\ell}$$

with eigenvalues $\lambda_{k\ell} = -(k^2 + \ell^2)$.

Note that $f_{k\ell}$ is an eigenfunction of ∇^2 for arbitrary values of k and ℓ.

Now we consider a **bounded domain**

$$0 \le x \le L_x \qquad \text{and} \qquad 0 \le y \le L_y$$

with **homogeneous boundary conditions:**

$$f(0,y) = f(L_x,y) = f(x,0) = f(x,L_y) = 0$$

This **quantizes** the wave numbers: eigensolutions are

$$f_{mn}(x,y) = \sin \frac{m\pi}{L_x} x \sin \frac{n\pi}{L_y} y$$

with eigenvalues

$$\lambda_{mn} = -\left(\left[\frac{m\pi}{L_x}\right]^2 + \left[\frac{n\pi}{L_y}\right]^2 \right) .$$

Now we consider **Spherical coordinates**. Then

$$\nabla^2 f = \left(\frac{1}{\cos^2\phi}\frac{\partial^2 f}{\partial\lambda^2} + \frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\cos\phi\frac{\partial f}{\partial\phi} \right) = -\kappa f$$

(for simplicity we have taken $a = 1$).

We define $\mu = \sin\phi$ and write this as

$$\left(\frac{1}{1-\mu^2}\frac{\partial^2 f}{\partial\lambda^2} + \frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial f}{\partial\mu} \right) = -\kappa f$$

We seek a solution by separating the variables:

$$f(\lambda,\mu) = \Lambda(\lambda)\Phi(\mu)$$

Then the eigenproblem is

$$\frac{d^2\Lambda}{d\lambda^2}\cdot\Phi + \Lambda\cdot(1-\mu^2)\frac{d}{d\mu}(1-\mu^2)\frac{d\Phi}{d\mu} = -(1-\mu^2)\kappa\Lambda\Phi .$$

Dividing by $\Lambda\Phi$, the problem separates into two parts

$$\frac{1}{\Lambda}\frac{d^2\Lambda}{d\lambda^2} = -\frac{1-\mu^2}{\Phi}\left[\frac{d}{d\mu}(1-\mu^2)\frac{d\Phi}{d\mu} + \kappa\Phi\right] .$$

Since the l.h.s. depends only on λ and the r.h.s. only on μ, they must both be constants:

$$\begin{aligned} \frac{1}{\Lambda}\frac{d^2\Lambda}{d\lambda^2} &= -m^2 \\ -\frac{1-\mu^2}{\Phi}\left[\frac{d}{d\mu}(1-\mu^2)\frac{d\Phi}{d\mu} + \kappa\Phi\right] &= -m^2 . \end{aligned}$$

The zonal structure is given by

$$\frac{d^2\Lambda}{d\lambda^2} + m^2\Lambda = 0$$

which is immediately solved: $\Lambda = \exp(im\lambda)$.

The meridional structure is given by

$$\left[\frac{d}{d\mu}(1-\mu^2)\frac{d}{d\mu} + \left(\kappa - \frac{m^2}{1-\mu^2}\right)\right]\Phi = 0 .$$

This is called the associated Legendre equation. It has solutions **regular at the poles** ($\mu = \pm 1$) for $\kappa = n(n+1)$ where n is an integer. They are the Legendre functions

$$\Phi = P_n^m(\mu)$$

The complete eigensolutions are the **spherical harmonics**

$$Y_n^m(\lambda,\mu) = P_n^m(\mu)\exp(im\lambda) .$$

The derivation above is standard and may be found in many books on *Mathematical Methods of Physics*.

The important result for us is the following:

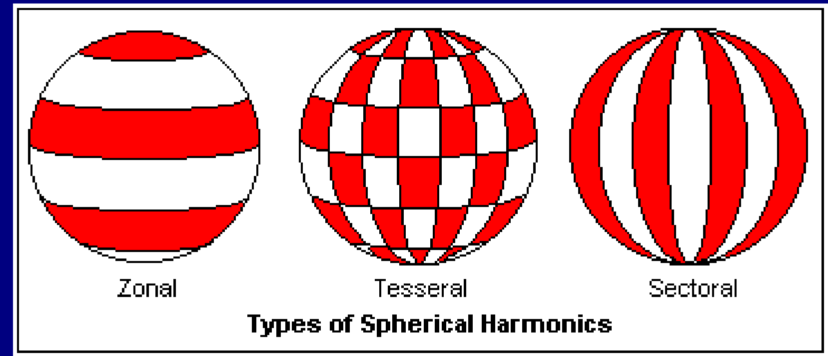
$$\nabla^2 Y_n^m(\lambda,\mu) = -n(n+1)\,Y_n^m(\lambda,\mu)\,.$$

Allowing for a non-unit radius of the sphere, this becomes

$$\nabla^2 Y_n^m(\lambda,\mu) = -\left[\frac{n(n+1)}{a^2}\right] Y_n^m(\lambda,\mu)\,.$$

The spherical harmonics $Y_n^m(\lambda,\mu)$ are the eigenfunctions of the Laplacian on the sphere, with eigenvalues $-n(n+1)/a^2$.

Alternating regions of positive and negative values.



Zonal: $m = 0, n > 0$.
Tesseral: $0 < m < n$.
Sectoral: $m = n$.

$|m|$ zeros in $0 \le \lambda < 2\pi$
$n - |m|$ zeros in $-1 < \mu < +1$.

Orthogonality & Completeness

The spherical harmonics are an orthogonal set:

$$\frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1}[Y_q^p(\lambda,\mu)]^* \cdot Y_s^r(\lambda,\mu)\,d\mu\,d\lambda = \delta_{pr}\delta_{qs}\,.$$

Any (reasonable) function $f(\lambda,\mu,t)$ on the sphere can be expanded in spherical harmonics:

$$f(\lambda,\mu,t) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} f_n^m(t)\,Y_n^m(\lambda,\mu)\,.$$

The coefficients $f_n^m(t)$ are given by

$$f_n^m(t) = \frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{+1}[Y_n^m(\lambda,\mu)]^* \cdot f(\lambda,\mu)\,d\mu\,d\lambda$$

Truncation

In practice, we have to replace the infinite summation

$$f(\lambda,\mu,t) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} f_n^m(t)\,Y_n^m(\lambda,\mu)\,.$$

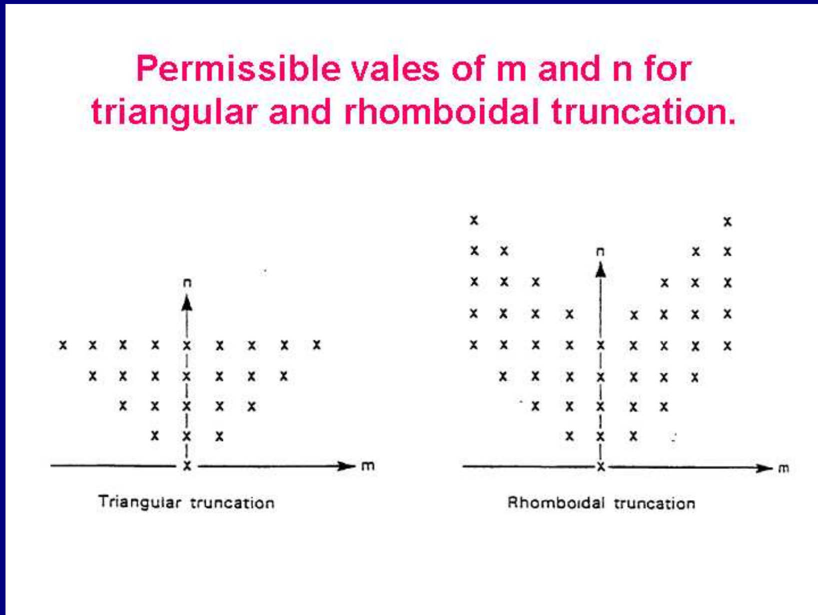
by a finite summation.

There ae a number of ways to truncate the solution. The most common is called triangular truncation:

$$f(\lambda,\mu,t) = \sum_{n=0}^{N}\sum_{m=-n}^{n} f_n^m(t)\,Y_n^m(\lambda,\mu)\,.$$

It can be shown that this gives uniform resolution throughout the sphere.

The ECMWF model uses triangular truncation.



Note that m can be positive or negative, and $|m| \leq n$.

End of Part 3