The Spectral Method (MAPH 40260)

Part 2: The Advection Equation

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Outline

Advection Equation

Finite Difference Approximation

Spectral Approximation

Solution of Advection Equation

Solution of Burgers' Equation

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The contrast in the results is of great practical importance.

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Substituting this into the equation, we have

$$-ikCU_m + \frac{ic}{\Delta x}\left(\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2i}\right)U_m = 0$$

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Clearly

$$C < c \qquad \textbf{for} \qquad k > 0\,.$$

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For the $4\Delta x$-wave, $k\Delta x = \pi/2$, so

$$C = \left(\frac{\sin \pi/2}{\pi/2}\right) c = \left(\frac{2}{\pi}\right) c \approx \frac{2}{3} c \,,$$

so the wave is **slowed down** by about one third.

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The phase speed is represented exactly.

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Substituting in the expansion, the equation becomes . . .

$$\sum_{n=-N}^{+N}\left[\frac{dU_n}{dt}+\frac{2\pi icn}{\ell}U_n\right]\exp(2\pi inx/\ell)=0\,.$$

$$\sum_{n=-N}^{+N} \left[\frac{dU_n}{dt} + \frac{2\pi i c n}{\ell} U_n \right] \exp(2\pi i n x/\ell) = 0 \, .$$

Now recall the orthogonality relationship

$$\frac{1}{\ell} \int_0^{\ell} \exp(-2\pi i m x/\ell) \cdot \exp(+2\pi i n x/\ell) \, dx = \delta_{mn} \, .$$

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Multiply the equation by $\exp(-2\pi imx/\ell)$ **and integrate:**

$$\sum_{n=-N}^{+N}\left[\frac{dU_n}{dt}+\frac{2\pi icn}{\ell}U_n\right]\ell\delta_{mn}=0\,,\qquad \textbf{or}$$

$$\frac{dU_m}{dt}+\frac{2\pi icm}{\ell}U_m=0\,,\qquad m=-N,-(N-1)\ldots N-1,N\,.$$

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The PDE has been reduced to a set of (independent) ODEs, which can easily be integrated.

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Substituting into the equation, expanding all terms and evaluating spatial derivatives analytically . . .

$$\sum_{n=-N}^{+N} \frac{dU_n}{dt} e^{\frac{2\pi inx}{\ell}} + \sum_{p=-N}^{+N} \sum_{q=-N}^{+N} U_p \left(\frac{2\pi iq}{\ell}\right) U_q \cdot e^{\frac{2\pi ipx}{\ell}} e^{\frac{2\pi iqx}{\ell}}$$

$$= \nu \sum_{n=-N}^{+N} \left(\frac{2\pi in}{\ell}\right)^2 U_n e^{\frac{2\pi inx}{\ell}} .$$

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For simplicity, let us take $\ell = 2\pi$. Then

$$\sum_{n=-N}^{+N} \frac{dU_n}{dt} e^{inx} + \sum_{p=-N}^{+N} \sum_{q=-N}^{+N} iqU_pU_q e^{i(p+q)x} = -\nu \sum_{n=-N}^{+N} n^2 U_n e^{inx} .$$

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We multiply by $\exp(-imx)$ and integrate. The first and last sums reduce to single terms. The double sum reduces to a single sum.

$$\frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{n=-N}^{+N}\frac{dU_n}{dt}e^{inx}\right)e^{-imx}\,dx = \sum_{n=-N}^{+N}\frac{dU_n}{dt}\delta_{mn} = \frac{dU_m}{dt}\,.$$

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$$\frac{1}{2\pi}\int_0^{2\pi}\sum_{p=-N}^{+N}\sum_{q=-N}^{+N}iqU_pU_qe^{i(p+q)x}e^{-imx}\,dx = \sum_{p=-N}^{+N}i(m-p)U_pU_{m-p}$$

Lemma:

$$\sum_{p=-N}^{+N} i(m-p)U_pU_{m-p} = \tfrac{1}{2}im\sum_{p=-N}^{+N} U_pU_{m-p}$$

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Proof:

$$\begin{aligned}
\sum_{p=-N}^{+N} (m-p)U_p U_{m-p} &= m \sum_{p=-N}^{+N} U_p U_{m-p} - \sum_{p=-N}^{+N} pU_p U_{m-p} \\
&= m \sum_{p=-N}^{+N} U_p U_{m-p} - \sum_{q=-N}^{+N} qU_q U_{m-q} \\
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\end{aligned}$$

Taking the last term to the left, the lemma follows.

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If we omit viscosity, we get the inviscid Burgers Equation:

$$\frac{dU_m}{dt} + \tfrac{1}{2} im \sum_{p=-N}^{+N} U_p U_{m-p} = 0 \,.$$

Again, Burgers' Equation in spectral form is:

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We see that **all scales interact.** For any mode U_m, any other mode U_p can change it by interacting with U_{m-p}. Energy can move from any scale to any other scale.

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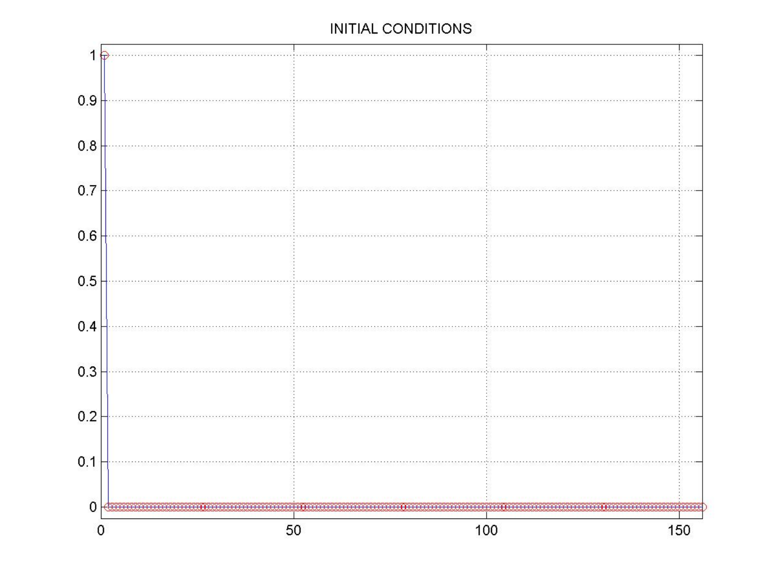
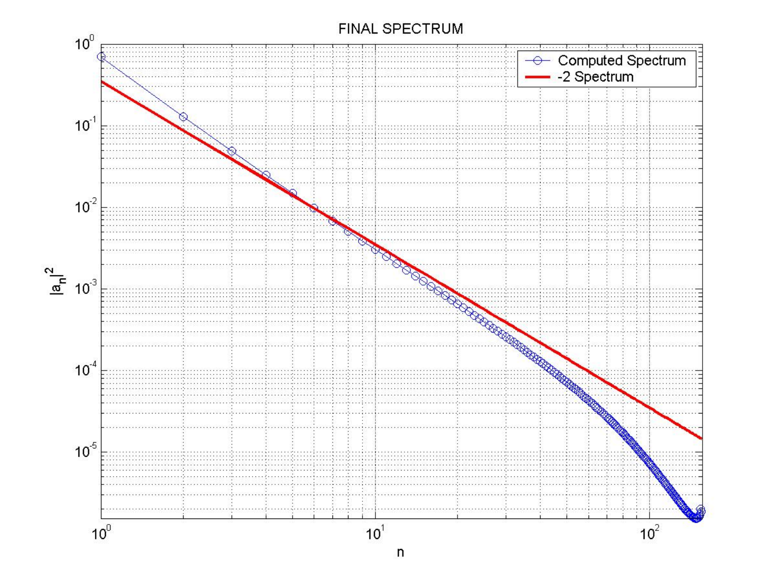
We may start with all the energy in the largest scale:

$$u(x,0) = U_1 \left(\frac{e^{ix} - e^{-ix}}{2i} \right) = U_1 \sin x \, ,$$

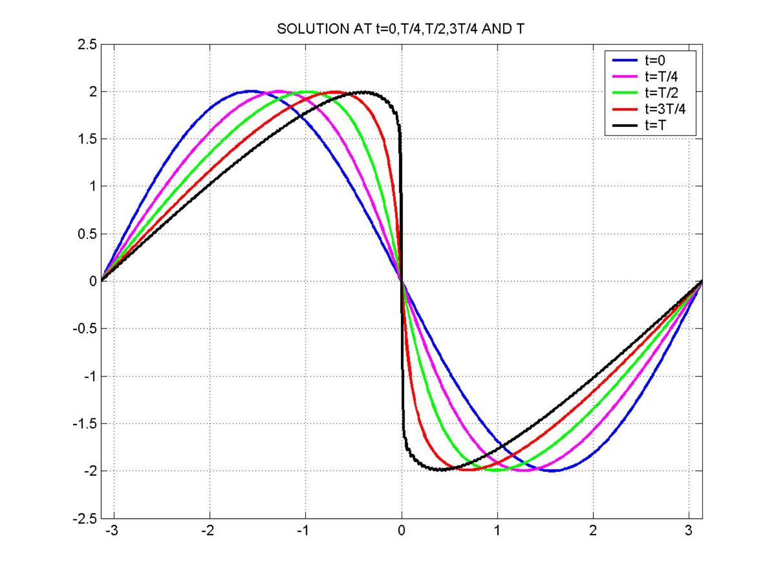
and the energy will quickly spread to other modes.



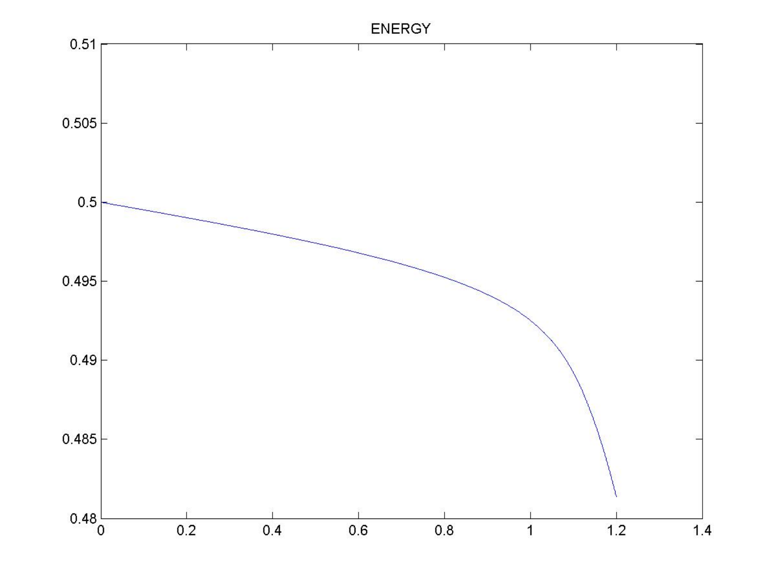
Initial conditions for Burgers' Equation.
Initial state is a pure sine-wave.



**Final spectrum for Burgers' Equation.
Energy has spread to all modes.**



Solution of Burgers' Equation. Shock has developed.
Initial state is a pure sine-wave.



Evolution of energy in time. Dissipation increases when energy reaches small scales.

End of Part 2